INTRODUCTORY TOPICS IN CELESTIAL MECHANICS

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Contents

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1. Central force fields

In this section we will discuss the main properties of central force fields. We refer to the books [3], [8], [12] and [15].

1.1. **Definitions and examples.**

Given a continuous map $f\colon (0,+\infty) \to \mathbb{R}$ we associate to f the continuous field

$$F\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d, \qquad F(x) = f(|x|)\frac{x}{|x|},$$

where $d = 2$ or $d = 3$. The field F is termed *central force field.*

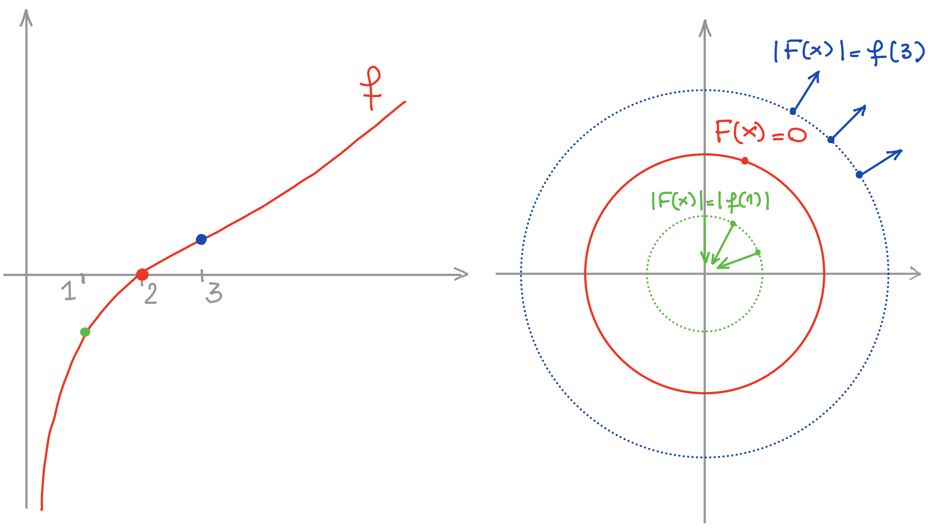

FIGURE 1. The changing-sign continuous function f generates a planar field F that is attractive on the ball of radius 2 and repulsive on the exterior of such ball. Points with distance 2 from the origin are equilibria.

Being $x = x(t)$ the position of a point particle with unitary mass at time t in the force field F, Newton's seconds law is the second order ordinary differential equation

$$\ddot{x} = F(x) = f(|x|)\frac{x}{|x|}. \tag{1}$$

In a central field, the force acting at x is always parallel to x: the force points towards the origin if $f(|x|)$ is negative; when $f(|x|)$ is positive, the force has the same direction of x. The field F is termed *attractive* if $f(r) < 0$ for any $r \in (0,+\infty)$, it is termed *repulsive* if $f(r) > 0$ for any $r \in (0,+\infty)$.

Example 1.1 (Newtonian gravitational field)**.** *Given a positive constant $\mu > 0$ and the function*

$$f(r) = -\frac{\mu}{r^2}$$

the corresponding attractive central force field is the Newtonian gravitational field

$$F(x) = -\frac{\mu}{|x|^3}x, \qquad x \in \mathbb{R}^3 \setminus \{0\}.$$

The system of second order differential equations describing the motion of a particle moving in a gravitational field generated by a mass fixed at the origin then is

$$\begin{cases} \ddot{x}_i = -\frac{\mu}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}x_i \\ i = 1, 2, 3. \end{cases}$$

The singular set $\{0\}$ is usually term collision set*.*

Example 1.2 (α-gravitational field, $\alpha > 0$)**.** *The Newtonian central force field can be generalized, considering different intensities of the attracting force. This can be done introducing a parameter $\alpha > 0$ and defining*

$$f_\alpha(r) = -\frac{\mu}{r^{1+\alpha}}.$$

The corresponding central force field then is

$$F_\alpha(x) = -\frac{\mu}{|x|^{2+\alpha}}x, \qquad x \in \mathbb{R}^3 \setminus \{0\}.$$

Of course, $\alpha = 1$ corresponds to the Newtonian case. In literature we refer to the case $\alpha \in (0, 2)$ as to the weak-force interaction*, while when $\alpha \geq 2$ we talk about* strong-force*. The role of this parameter is central in the study of the occurrence of collision in weak (or variational) solutions of this problem.*

A (classical) solution of a central force dynamical system (1) is a function $x : I \to \mathbb{R}^d$, $d = 2, 3$, for some interval $I \subseteq \mathbb{R}$, that admit second order derivative and such that (1) is satisfied for any $t \in I$.

Example 1.3 (Circular solutions for gravitational fields - Kepler 3^{rd} law)**.** *For any $\alpha > 0$ let us consider the circular planar trajectories*

$$x_{\rho,\omega}(t) = \rho\left(\cos(\omega t), \sin(\omega t), 0\right),$$

where ρ and ω are positive constants.
Since $\ddot{x}_{\phi,\omega} = -\omega^2 x_{\phi,\omega}$, it turns out that $x_{\phi,\omega}$ solves

$$\ddot{x}(t) = F_\alpha(x(t)), \quad t \in \mathbb{R}. \tag{2}$$

if and only if

$$\omega^2 = \frac{\mu}{\rho^{2+\alpha}} \text{ that is } \omega = \sqrt{\mu}\rho^{-\frac{2+\alpha}{2}}.$$

Hence, for any fixed $\alpha > 0$, we deduce the existence of a 1-parameter family of planar periodic circular solutions of the α-gravitational field. More precisely, for any $\rho > 0$, the circular motion centred at the origin, with radius ρ and period

$$T = T(\rho) = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\mu}}\rho^{\frac{2+\alpha}{2}}$$

solves (2). *Let us observe that when $\alpha = 1$ this fact is the content of Kepler 3^{rd} law for circular trajectories.*

1.2. **Basic properties and conserved quantities.** Let $x : I \subseteq \mathbb{R} \to \mathbb{R}^d$, $d = 2, 3$, for some interval $I \subset \mathbb{R}$, be a solution of (1), then x enjoys the following symmetry properties:

(P1) for any $c \in \mathbb{R}$ the function $x_c(t) := x(t + c)$ solves (1) *(translation invariance)*
(P2) $x^-(t) := x(-t)$ solves (1) *(time reversibility)*
(P3) for any $A \in O(d)$[1] $x_A(t) := Ax(t)$ solves (1) (isometry invariance).

Futhermore the following result holds.

Proposition 1.4. *A central force field is conservative, that is, there exists a C^1-function, termed* potential function, $U : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ *such that $F(x) = \nabla U(x)$. Furthermore if $x : I \subseteq \mathbb{R} \to \mathbb{R}^d$, $I \subset \mathbb{R}$, is a solution of* (1), *then the total energy*

$$h = \frac{1}{2}|\dot{x}(t)|^2 - U(x(t))$$

is constant for any $t \in I$.

Proof. Fix $r_0 > 0$ and define

$$U_{r_0}(x) := \int_{r_0}^{|x|} f(s)\,ds, \quad \text{for any } x \in \mathbb{R}^d \setminus \{0\}.$$

Then the i^{th} partial derivative of U is

$$\frac{\partial U_{r_0}}{\partial x_i}(x) = f(|x|)\frac{\partial}{\partial x_i}|x| = f(|x|)\frac{x_i}{|x|} = F_i(x), \quad i = 1, 2, 3.$$

□

Remark 1.5. *Since the (family of) potential introduced in the proof of Proposition 1.4 depends just on $|x|$, with a slight abuse of notation we define*

$$U_{r_0}(r) = \int_{r_0}^{r} f(s)\,ds, \qquad \text{for every } r > 0 \tag{3}$$

[1] Recall that $O(d) = \{A \in \mathcal{M}_d(\mathbb{R}) : A^T A = AA^T = I_d\}$

and we write equation (1) *as*

$$\ddot{x} = U'_{r_0}(|x|)\frac{x}{|x|}.$$

Example 1.6. *To the α-gravitational field introduced in Example 1.2 we can associate the family of potentials*

$$U_{r_0}(r) = \frac{\mu}{\alpha r^\alpha} + C_{\alpha,r_0}, \qquad C_{\alpha,r_0} = -\frac{\mu}{\alpha r_0^\alpha}.$$

In order to select one potential, a natural choice is to require the normalized condition

$$\lim_{r\to+\infty} U(r) = 0,$$

that corresponds to consider the potential

$$U(r) = \frac{\mu}{\alpha r^\alpha}.$$

The conserved total energy *of a solution x for the gravitational field, defined on $I \subset \mathbb{R}$, is*

$$h = \frac{1}{2}|\dot{x}(t)|^2 - \frac{\mu}{\alpha|x(t)|^\alpha}, \qquad t \in I.$$

Fixed the total energy $h \in \mathbb{R}$, since the kinetic part of the energy $\frac{1}{2}|\dot{x}(t)|$ is positive, solutions for the dynamical system (1) with total energy h are necessarily confined to the *Hill's region of level h*

$$\mathcal{H}_h := \left\{x \in \mathbb{R}^3 \setminus \{0\} : h \geq -U(x)\right\}.$$

Let us observe that when the boundary of $\mathcal{H}_h$ is not empty, a solution touch such a boundary when its velocity is 0. Furthermore, from Remark 1.5 it follows that Hill's regions for a central force field are radial sets.

Example 1.7. *Hill's regions associated to the α-gravitational field are*

$$\mathcal{H}_h := \left\{x \in \mathbb{R}^3 \setminus \{0\} : h \geq -\frac{\mu}{\alpha|x|^\alpha}\right\}.$$

When $h \geq 0$, since α and μ are positive, we immediately deduce that $\mathcal{H}_h = \mathbb{R}^3 \setminus \{0\}$, while when $h < 0$ orbits are confined to the ball

$$\mathcal{H}_h = \left\{x \in \mathbb{R}^3 \setminus \{0\} : |x|^\alpha \leq -\frac{\alpha}{h\mu}\right\} = B_{r(\alpha,h)}(0), \quad r(\alpha,h) = \left(-\frac{\alpha}{h\mu}\right)^{\frac{1}{\alpha}}.$$

Let us now consider the *angular momentum* associated to the position vector, $x = x(t)$, of a particle moving in a central force field, that is the vector

$$c(t) = x(t) \wedge \dot{x}(t).$$

The following result was discover by Kepler, observing the motion of Mars.

Proposition 1.8. *Given a solution of* (1) *defined on* $I \subseteq \mathbb{R}$*, there exists a vector* $c \in \mathbb{R}^3$ *such that*

$$c = x(t) \wedge \dot{x}(t), \quad \forall t \in I.$$

In particular, solutions of (1) *describe planar curves.*

Proof. Since in a central force field $x(t)$ and $\ddot{x}(t)$ are parallel, it turns out that

$$\dot{c}(t) = \dot{x}(t) \wedge \dot{x}(t) = x(t) \wedge \ddot{x}(t) = 0, \quad \forall t \in I,$$

and the thesis follows. □

Remark 1.9. *When* $c = 0$*, then the motion* x *is 1-dimensional: there exists a unitary vector* $v \in \mathbb{R}^3$ *and a function* $\lambda : I \subseteq \mathbb{R} \to \mathbb{R}$ *such that*

$$\frac{x(t)}{|x(t)|} \in v \quad \textit{and} \quad \dot{x}(t) = \lambda(t)x(t), \qquad \forall t \in I.$$

Now on, without loosing in generality, we will assume that the central force field F is planar, that is $F \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$.

1.3. **Sectorial velocity and Kepler's second law.** Given a continuous function $x \colon I \subseteq \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ let us introduce the continuous functions

$$r \colon I \subseteq \mathbb{R} \to (0, +\infty) \quad \text{and} \quad \vartheta \colon I \subseteq \mathbb{R} \to \mathbb{R}$$

such that

$$x(t) = r(t) \left(\cos \vartheta(t), \sin \vartheta(t)\right), \qquad \forall t \in I. \tag{4}$$

The function $r(t) = |x(t)|$ is uniquely determined, while $\vartheta(t)$ is unique up to $2k\pi$-translations, with $k \in \mathbb{Z}$. Assume now that that for some $t_0, t_1 \in I$ such that $t_0 < t_1$, there holds

- $\dot{\vartheta}(t) > 0$ for every $t \in (t_0, t_1)$, and
- $\vartheta(t_1) - \vartheta(t_0) < 2\pi$,

and consider the set (see Figure 2)

$$D := \{sx(t) : s \in [0, 1], t \in [t_0, t_1]\}.$$

Then the following result holds.

Proposition 1.10 (Kepler's second law for central force fields)**.** *Let* c *be the conserved angular momentum for a solution* $x \colon I \subseteq \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ *of* (1) *and let* t_0, t_1, r, ϑ, D *as above. Then*

$$\text{Area}(D) = \frac{1}{2}(t_1 - t_0)|c|.$$

Briefly, in equal time the radius vector sweeps out equal areas.

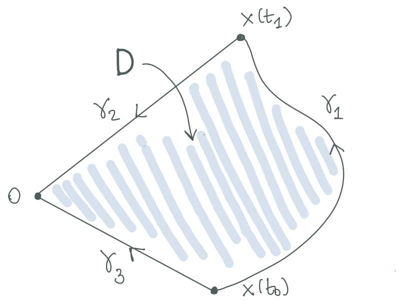

FIGURE 2. the area of the region D depend only on the time necessary to the particle to move from the initial point $x(t_0)$ to the final one $x(t_1)$ and on the constant quantity $|c|$.

Proof. We compute the area of the region D in Figure 2 by means of Gauss-Green Theorem[2]. Let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be the following counter-clockwise parametrization of the boundary ∂D in polar coordinates:

$$\begin{aligned}
\gamma_1(t) &= x(t) = r(t)\left(\cos\vartheta(t), \sin\vartheta(t)\right), \quad t \in [t_0, t_1] \\
\gamma_2(t) &= (1-t)r(t_1)\left(\cos\vartheta(t_1), \sin\vartheta(t_1)\right), \quad t \in [0,1] \\
\gamma_3(t) &= t\, r(t_0)\left(\cos\vartheta(t_0), \sin\vartheta(t_0)\right), \quad t \in [0,1].
\end{aligned}$$

The segments γ_2 and γ_3 do not give any contribution to the computation; hence the surface of D reduces to the integral on $\gamma_1(t) = x(t) = (x_1(t), x_2(t))$, that is

$$\text{Area}(D) = \int_{t_0}^{t_1} \left[x_1(t)\dot{x}_2(t) - \dot{x}_1(t)x_2(t)\right] dt = \int_{t_0}^{t_1} r^2(t)\dot{\vartheta}(t)\, dt.$$

We obtain the thesis as far as we remark that, since $\dot{\vartheta}(t) > 0$,

$$|c| = r^2(t)\dot{\vartheta}(t), \quad \forall t \in [t_0, t_1].$$

□

1.4. **The effective potential.** Given a solution of (1) in polar coordinate as in Eq. (4), we introduce the vectors

$$e_r(t) = \frac{x(t)}{|x(t)|} = (\cos\vartheta(t), \sin\vartheta(t)) \qquad e_\vartheta(t) = (-\sin\vartheta(t), \cos\vartheta(t)).$$

It turns out that

$$x(t) = r(t)e_r, \qquad \dot{x}(t) = \dot{r}(t)e_r + r(t)\dot{\theta}(t)e_\vartheta$$

and

$$\ddot{x}(t) = \Big(\ddot{r}(t) - r(t)\dot{\vartheta}^2(t)\Big)\, e_r + \Big(2\dot{r}(t)\dot{\theta}(t) + r(t)\ddot{\vartheta}(t)\Big)\, e_\vartheta.$$

[2]The area of a regular domain $D \subset \mathbb{R}^2$ whose positive boundary (i.e. the normal vector poits outside the region) is parametrized by the closed and regular curve $\gamma(t) = (x_1(t), x_2(t))$, $t \in [a,b]$ is

$$\text{Area}(D) = \frac{1}{2}\int_a^b \left[x_1(t)\dot{x}_2(t) - \dot{x}_1(t)x_2(t)\right] dt.$$

(if γ is the justapposition of a finite number of regular curves we just use the additivity of the integral).

By virtue of Proposition 1.4 and Remark 1.5, Eq. (1) reads

$$\ddot{x}(t) = U'(r)e_r(t).$$

Comparing with the expression of $\ddot{x}(t)$ already computed we obtain

$$\begin{cases} \ddot{r}(t) - r(t)\dot{\vartheta}^2(t) = U'(r) \\ 2\dot{r}(t)\dot{\theta}(t) + r(t)\ddot{\vartheta}(t) = 0. \end{cases}$$

Since $\dot{\vartheta}(t) = \frac{|c|}{r^2(t)}$, the first line of the system is the second order differential equation in the scalar variable $r(t)$

$$\ddot{r}(t) = \frac{|c|^2}{r^3(t)} + U'(r(t)).$$

This equation can be written as the one-dimensional conservative system

$$\ddot{r}(t) = V'(r(t)), \tag{5}$$

where

$$V(r) = U(r) - \frac{|c|^2}{2r^2}$$

is the *effective potential* of a central force field having potential U.
Summing up: fixed $|c| \geq 0$ we can (try to) solve the one-dimensional second order differential equation (5), then we replace the expression of $r(t)$ in $\dot{\vartheta}(t) = \frac{|c|}{r^2(t)}$ and we deduce

$$\vartheta(t) = \vartheta_0 + \int_{t_0}^{t} \frac{|c|}{r^2(s)}\, ds.$$

1.5. **The equation of orbits in polar coordinates and Bertrand's theorem.** The energy conservation law for Eq. (5) reads

$$h_r = \frac{1}{2}\dot{r}^2(t) - V(r), \qquad \text{for some } h_r \in \mathbb{R},$$

comparing this expression with the conservation of energy in the whole system in polar coordinates, we obtain

$$\begin{aligned} h &= \frac{1}{2}|\dot{x}(t)|^2 - U(|x(t)|) \\ &= \frac{1}{2}\dot{r}^2(t) + \frac{1}{2}r^2(t)\dot{\vartheta}^2(t) - U(r(t)) \\ &= \frac{1}{2}\dot{r}^2(t) - V(r(t)) = h_r. \end{aligned}$$

Hence

$$h = \frac{1}{2}\dot{r}^2(t) - V(r(t)).$$

Assuming $r(t)$ increases (strictly) on some interval $[t_0, t_1]$ we obtain a first order equation in the variable r:

$$\dot{r}(t) = \sqrt{2\,(h + V(r(t)))}.$$

By the assumed monotonicity, the map $r : [t_0, t_1] \to [r(t_0), r(t_1)]$ is invertible and we can write $t = t(r)$ with $t : [r(t_0), r(t_1)] \to [t_0, t_1]$. From the conservation of the angular momentum we compute

$$\dot{\vartheta}(t) = \frac{|c|}{r^2(t)} \qquad \Longrightarrow \qquad \frac{d\vartheta}{dr} = \frac{|c|/r^2}{\sqrt{2\,(h + V(r))}}$$

and we obtain the *equation of orbits in polar coordinates*

$$\vartheta(r) = \int_{r(t_0)}^{r(t_1)} \frac{|c|/r^2}{\sqrt{2\,(h + V(r))}}\, dr.$$

As already observed when we have introduced the notion of Hill's region, the energy relation possibly imposes a bound on the radial variable, indeed

$$-V(r) \leq \frac{1}{2}\dot{r}^2(t) - V(r) = h.$$

Inequality $-V(r) \leq h$ gives one (or more) annular region in the plane $0 \leq r_{\min} \leq r \leq r_{\max} \leq +\infty$. If $r_{\max} < +\infty$, then the motion is bounded and takes place inside the ring between $r_{\min}$ and $r_{\max}$. Points where $r = r_{\min}$ are called *pericentral*, points where and $r = r_{\max}$ are *apocentral*; each of the ray leading from the origin to a pericenter or to an apocenter is an axis of symmetry for the orbit. In general the orbit is not closed and the angle spanned by the piece of orbit between a pericenter and the consecutive apocenter is given by

$$\Theta = \int_{r_{\min}}^{r_{\max}} \frac{|c|/r^2}{\sqrt{2\,(h + V(r))}}\, dr.$$

Orbits are closed if and only if Θ is commensurable with 2π, that is $\Theta = 2\pi\frac{m}{n}$ for some $m, n \in \mathbb{N} \setminus 0$. If this condition is not satisfied then each orbit is everywhere dense in the annulus.

Theorem 1.11 (Bertrand's Theorem)**.** *There are only two cases in which all bounded orbits in a central force field are closed: the Newtonian gravitational field and the harmonic oscillator field, which are generated by the potentials*

$$U(r) = \frac{\mu}{r} \quad \textit{and} \quad U(r) = -kr^2, \quad \mu, k > 0.$$

The proof of this result can be found in Arnold's book [3] in Chapter 2.

1.6. **One-dimentional motions in an attractive central force field.** In this section we describe the motion in a central force field when the angular momentum is 0; in this case, as already observed in Remark 1.9, the motion is one-dimentional, hence there exists a unitary vector v such that

$$x(t) = r(t)v.$$

Eq. (1) then reads

$$\ddot{r}(t) = f(r(t))$$

and from the behaviour of its solutions we deduce the behaviour of $x(t)$. The following result can be applied to the Newtonian gravitational field and to its α-generalization.

Theorem 1.12. *Let $x(t) = r(t)v$ be a 1-dimensional solution of (1) and let (α, ω) its maximal definition interval. Assume that $f(r) < 0$ for any $r \in (0, +\infty)$. Then one of the following situations occur (see Figure 3):*

(i) *α, ω are both bounded, $\lim_{t\to\alpha^+} r(t) = \lim_{t\to\omega^-} r(t) = 0^+$ and there exists $t_0 \in (\alpha, \omega)$ such that $\dot{r}(t) > 0$ on (α, t_0), $\dot{r}(t_0) = 0$ and $\dot{r}(t) < 0$ on (α, t_0);*

(ii) *α is bounded, $\omega = +\infty$, $\lim_{t\to\alpha^+} r(t) = 0^+$, $\lim_{t\to+\infty} r(t) = +\infty$ and $\dot{r}(t) > 0$ on $(\alpha, +\infty)$;*

(iii) *$\alpha = -\infty$, ω is bounded, $\lim_{t\to-\infty} r(t) = +\infty$, $\lim_{t\to\omega^-} r(t) = 0^+$ and $\dot{r}(t) < 0$ on $(-\infty, \omega)$.*

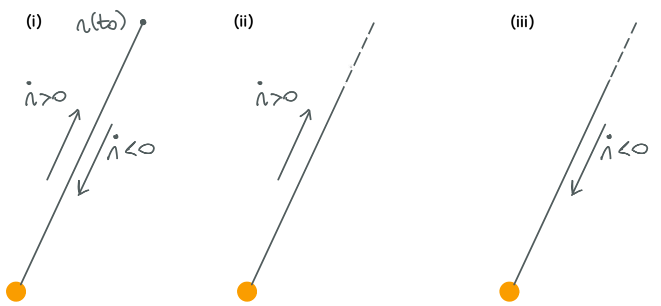

FIGURE 3. the behaviour of 1-dimensional orbits in a central force fields is limited to the three situations described in Theorem 1.12.

Proof. Since $x(t) = r(t)v$ solves (1) the 1-dimensional function r in of class C^2 on (α, ω). From the assumption on the sign of f, $\ddot{r}(t) = f(r(t)) < 0$, and $\dot{r}$ strictly decreases on (α, ω).

We then have to alternatives

(a) there exists a unique $t_0 \in (\alpha, \omega)$ such that $\dot{r}(t) > 0$ on (α, t_0), $\dot{r}(t_0) = 0$ and $\dot{r}(t) < 0$ on (α, t_0), or
(b) $\dot{r}(t)$ has constant sign on (α, ω).

We now claim (i) from alternative (a). Let us focus on $[t_0, \omega)$; let $\delta > 0$ be such that $t_0 + \delta < \omega$ and consider $t \in (t_0 + \delta, \omega)$, then (since $\dot{r}$ strictly decreases)

$$r(t) = r(t_0 + \delta) + \int_{t_0+\delta}^{t} \dot{r}(s)\, ds < r(t_0 + \delta) + \dot{r}(t_0 + \delta)(t - t_0 + \delta).$$

We infer that $\omega < +\infty$ arguing by contradiction; indeed assuming $\omega = +\infty$ we obtain, passing to the limit in the previous inequality

$$\lim_{t\to+\infty} r(t) < r(t_0 + \delta) + \dot{r}(t_0 + \delta) \lim_{t\to+\infty} (t - t_0 + \delta)$$

which forces $r(t) \to -\infty$ ($\dot{r}(t_0 + \delta)$ is strictly negative). This is clearly a contradiction since r is a positive quantity. In order to show that $r(t)$ tends to 0 as $t \to \omega^-$, we still argue by contradiction: r is monotone decreasing, hence it admits a limit as $t \to \omega^-$. If this limit is $l > 0$, then (α, ω) should not be the maximal definition interval of x.
We argue similarly on $(\alpha, t_0]$ in order to reach claim (i).

Assume now (b) and in particular that $\dot{r}(t) > 0$ on (α, ω). We want to claim (ii). Once more, $\dot{r}$ decreases strictly and admits a limit as $t \to \omega^-$. Assume by contradiction that $\omega < +\infty$ and take $t_0 \in (\alpha, \omega)$, then the quantity

$$r(\omega) = r(t_0) + \int_{t_0}^{\omega} \dot{r}(s)\, ds$$

is finite and (α, ω) should not be the maximal definition interval of x. Then $\omega = +\infty$. We now show that $r(t) \to +\infty$ as $t \to +\infty$. Such limit exists since $\dot{r} > 0$ so that r is monotone; assume that $r(t) \to l \in (0, +\infty)$ as $t \to +\infty$. Then there exists a sequence $t_n \to +\infty$ such that $\dot{r}(t_n) = 0$ and $\ddot{r}(t_n) \to f(l) < 0$. Let $\bar{t}$ such that $\ddot{r}(t) < \frac{f(l)}{2}$ for any $t > \bar{t}$; for any $t_n > \bar{t}$ there holds

$$\dot{r}(t_n) = \dot{r}(\bar{t}) + \int_{\bar{t}}^{t_n} \ddot{r}(s)\, ds < \dot{r}(\bar{t}) + (t_n - \bar{t})\frac{f(l)}{2} \to -\infty \quad \text{as } t_n \to +\infty,$$

in contradiction with $\dot{r} > 0$. We conclude that $r(t) \to +\infty$ as $t \to +\infty$. With similar arguments we conclude claim (ii). Claim (iii) is reached in a similar way. □

Example 1.13. *Let us understand the meaning of Theorem 1.12 in the 1-dimensional Kepler problem; assuming that $x(t) = r(t)v$, for some unitary vector v, is a solution on some interval (α, ω), then r solves*

$$\ddot{r}(t) = -\frac{\mu}{r^2(t)}, \qquad t \in (\alpha, \omega).$$

Of course $f(r) = -\mu/r^2 < 0$ *and the assumptions of Theorem 1.12 are satisfied. The result of Theorem 1.12 can also be deduced using the conservation of energy*

$$h = \frac{1}{2}\dot{r}^2(t) - \frac{\mu}{r(t)}$$

and the Hill's region for the problem which is determined by the inequality

$$-\frac{\mu}{r} \leq h.$$

We refer to Figure 4 where the graph of $g(r) = -U(r) = -\mu/r$ *is plotted. When* $h \geq 0$*, the sublevel*

$$g^{\leq h} = \{r \in (0, +\infty) : g(r) \leq h\}$$

is the whole $(0, +\infty)$*. In this case we have two possible different motions determined by the sign of* $\dot{r}$*, i.e.*

$$\dot{r}(t) = \sqrt{2h + 2\frac{\mu}{r(t)}} \qquad \text{or} \qquad \dot{r}(t) = -\sqrt{2h + 2\frac{\mu}{r(t)}}.$$

From the phase-plane (Figure 4) we understand that the first motion corresponds to situation (ii), while the second one to situation (iii). When $h \geq 0$ *we also deduce that* $\dot{r} \to \pm\sqrt{2h}$ *when* $r \to +\infty$*.*

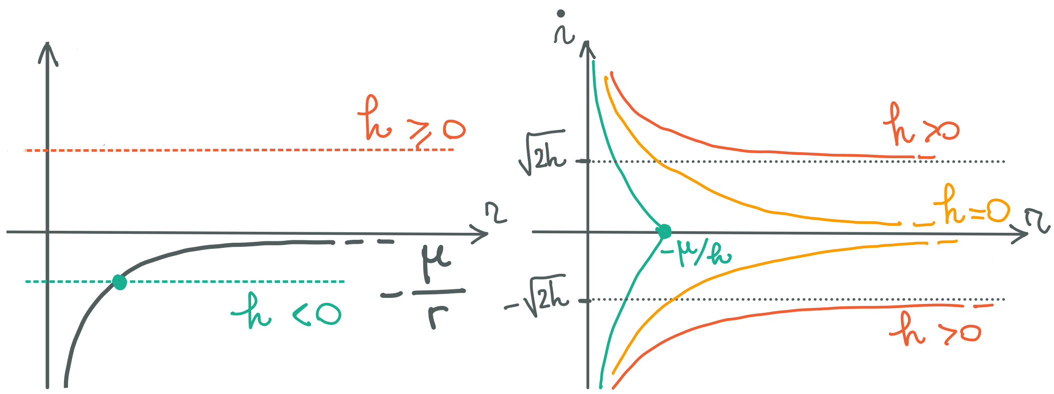

FIGURE 4. From the energy relation we can deduce the phase-plane analysis for 1-dimensional solutions of the Kepler problem.

When $h < 0$*, the sublevel* $g^{\leq h}$ *is the interval* $(0, -\mu/h)$*: motions are in this case bounded, there exists a unique instant* t_0 *such that* $\dot{r}(t_0) = 0$ *and (i) of Theorem 1.12 is satisfied. In this case the time to moove from* 0 *to* $-\mu/h$ *can* *be computed by separating variables in the first order differential equation*

$$\dot{r}(t) = \frac{dr}{dt} = \sqrt{\frac{2\mu}{r(t)} + 2h}$$

which gives the finite (elliptic) integral

$$t_0 - \alpha = \int_\alpha^{t_0} dt = \int_0^{-\mu/h} \frac{dr}{\sqrt{2\mu/r + 2h}} = \int_0^{-\mu/h} \frac{\sqrt{r}}{\sqrt{2(\mu + hr)}}\, dr.$$

When $h = 0$, still separating variables in the energy relation, we obtain (still choosing the branch with $\dot{r}(t) > 0$)

$$t - \alpha = \int_0^{r(t)} \frac{\sqrt{s}}{\sqrt{2\mu}} ds = \frac{2}{3}\frac{1}{\sqrt{2\mu}} r^{\frac{3}{2}}$$

hence

$$r(t) = C(t-\alpha)^{\frac{2}{3}}, \qquad \textit{where } C = \left(\frac{9}{2}\mu\right)^{\frac{1}{3}}.$$

2. Kepler's problem

In this section we will study the so called *Kepler problem*, that is the dynamical system

$$(K) \qquad \ddot{x}(t) = -\frac{\mu}{|x(t)|^2}\frac{x(t)}{|x(t)|},$$

describing the motion of a point particle with unitary mass moving under the action of an attractive force towards the origin. At the beginning of seventeenth century, much before the differential formulation of the gravitation law (K), Kepler formulated three laws that described the motion of a planer around the Sun, formulating them just by means of empirical observation. These laws described perfectly the observational data. Even today, with extremely precise data, these laws are a close first approximation to the truth. They also hold for various systems of satellites orbiting their primary.

Newton was the first one to explain these laws as a result of the laws of dynamics and gravitation (K). In this sense Kepler?s laws are a description of the solutions of a special case the gravitational problem of n point-masses, termed *bodies*: in this special situation all the masses but one are so small that they do not attract each other appreciably, but they are all attracted by the large mass.

We have already proved that if x is a solution for the Kepler problem then there are two quantities that are cnserved along the motion: the totalal energy $h = \frac{1}{2}|\dot{x}(t)|^2 - \frac{\mu}{|x(t)|}$ and the angular momentum $c = x(t) \wedge \dot{x}(t)$. The planarity of the motion follows from this second conservation. The contents of this section are mainly inspired from [8] and [12].

2.1. **Planar conics and Kepler's first law.** In the following result we propose a smart description of planar conic with a focal point ot the origin.

Proposition 2.1. *Any planar conic with a focal point at 0 consists of a set of point $x \in \mathbb{R}^2$ such that*

$$|x| + \langle e, x \rangle = k \tag{6}$$

for some $e \in \mathbb{R}^2$ and $k \in \mathbb{R}$. Furthermore, an equation of the form (6) *is a conic with a focal point at 0 when*

- $|e| < 1$ *and* $k > 0$, *in this case it is an ellipse;*
- $|e| = 1$ *and* $k > 0$, *in this case it is a parabola;*
- $|e| > 1$ *and* $k > 0$, *in this case it is a branch of hyperbola (the one closer to 0);*
- $|e| > 1$ *and* $k < 0$, *in this case it is a branch of hyperbola (the one far from 0).*

Proof. Let us start considering an (non degenerate) ellipse, $\mathcal{E}$ as in Figure 5; remark that A, O and x are vectors. For some constant $C > 0$ we have

$$x \in \mathcal{E} \quad \Longleftrightarrow \quad |x| + |A - x| = C \quad \Longleftrightarrow \quad |A - x|^2 = (C - |x|)^2$$

(observe that $C - |x| > 0$ so that the second *if and only if* holds). Hence

$$x \in \mathcal{E} \quad \Longleftrightarrow \quad |x| - \frac{1}{C}\langle A, x\rangle = \frac{C^2 - |A|^2}{2}$$

We conclude defining

$$e := -\frac{A}{C}, \qquad k := \frac{C^2 - |A|^2}{2}$$

and observing that $|e| < 1$ and $k > 0$ as far as $|A| < |x| + |A - x| = C$. Of course it could be that $|A| = |x| + |A - x| = C$, but in this case the ellipse reduces to two points, 0 and A).

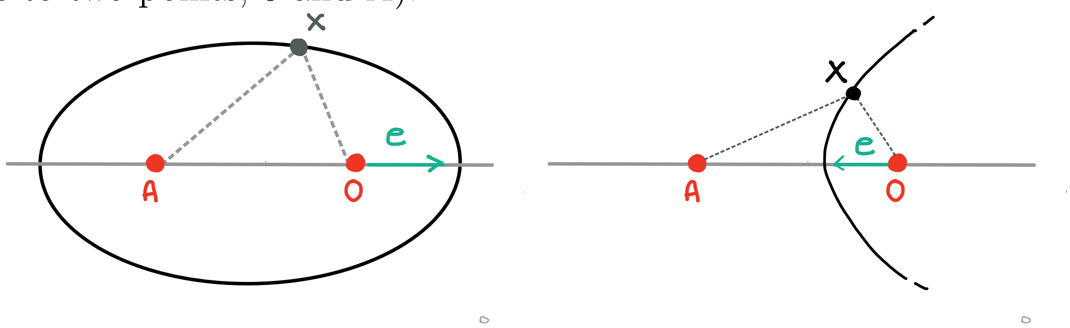

FIGURE 5. An ellipse with a focus at the origin can be described through the vector e, opposite to the second focus A. The branch of hyperbola with a focus at O and closer to it can be described via e, a scaling of A.

We now consider a parabola $\mathcal{P}$ with focal point at O, as in Figure 6; fixed a unitary vector v and a constant $c > 0$, the directrix L is the set of points $w \in \mathbb{R}^2$ such that $\langle v, w\rangle = c$. Then

$$x \in \mathcal{P} \quad \Longleftrightarrow \quad dist(x, L) = |x - 0| = |x|$$

Let L^0 be the half-plane determined by L and containing the origin, as in Figure 6; then $\mathcal{P} \cap (\mathbb{R}^2 \setminus L^0) = \emptyset$ and $\mathcal{P} \subset L^0$. Furthermore

$$x \in L^0 \quad \Longrightarrow \quad dist(x, L) = c - \langle x, v\rangle$$

hence

$$x \in \mathcal{P} \quad \Longleftrightarrow \quad |x| = c - \langle x, v\rangle$$

and we reach the claim with $e = v$ ($|e| = 1$) and $k = c$ (> 0). We conclude considering an hyperbola, $\mathcal{H}$ with focal points at A and at O. Let $\mathcal{H}_O$ be the branch of hyperbola closer to the origin. For some constant $C > 0$ we have

$$x \in \mathcal{H}_O \quad \Longleftrightarrow \quad |x - A| - |x| = C \quad \Longleftrightarrow \quad |A - x|^2 = (C + |x|)^2,$$

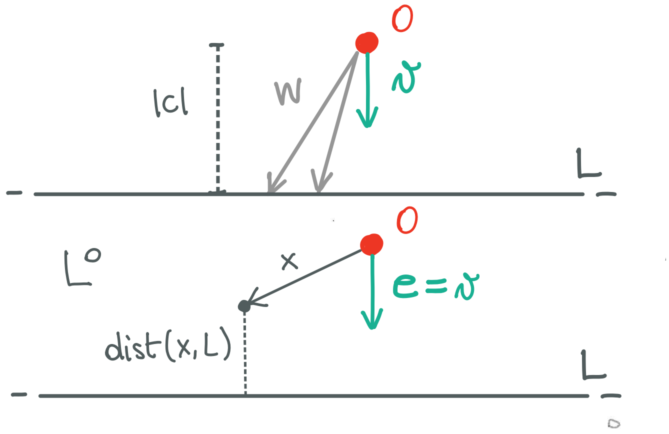

FIGURE 6. A parabola with a focus at the origin can be described through the vector $e = v$, and the directrix L.

hence

$$x \in \mathcal{H}_O \qquad \Longleftrightarrow \qquad |x| + \frac{1}{C}\langle A, x\rangle = \frac{|A|^2 - C^2}{2C}.$$

We reach the claim with $e = \frac{1}{C}A$ and $k = \frac{|A|^2-C^2}{2C}$. Rekark that in this case $|e| > 1$ and $k > 0$ both since $|A| > |A-x| - |x| = C$. Similarely we can consider the other branch of the hyperbola, $\mathcal{H}_A$, characterized by $|x| - |x - A| = C$. $\square$

Theorem 2.2 (Kepler's first law). *Let $x = x(t)$ be a solution for the Kepler equation and assume that ita angular momentum does not vanish. Then x moves on a conic with focal point at the origin.*

Proof. Recalling that for every u, v, $w \in \mathbb{R}^3$ there holds

$$(u \wedge v) \wedge w = \langle u, w\rangle v - \langle v, w\rangle u$$

we compute

$$\begin{aligned}\frac{d}{dt}\left(\frac{x}{|x|}\right) &= \frac{\dot{x}|x| - x\left\langle \frac{x}{|x|}, \dot{x}\right\rangle}{|x|^2} = \frac{\dot{x}\langle x, x\rangle - x\langle x, \dot{x}\rangle}{|x|^3} = \frac{(x \wedge \dot{x}) \wedge x}{|x|^3} \\ &= c \wedge \left(-\frac{1}{\mu}\ddot{x}\right) = -\frac{1}{\mu}\frac{d}{dt}(c \wedge \dot{x})\end{aligned}$$

and we deduce the existence of $v \in \mathbb{R}^2$ such that

$$\mu\left(\frac{x}{|x|} - v\right) = -c \wedge \dot{x}.$$

Projecting this equation on the direction x we obtain

$$\mu\left\langle \frac{x}{|x|} - v, x\right\rangle = -\langle c \wedge \dot{x}, , x\rangle,$$

hence (recalling that $\langle u \wedge v, w\rangle = \langle u, v \wedge w\rangle$)

$$\mu\left(|x| + \langle -v, x\rangle\right) = -\langle c, \dot{x} \wedge x\rangle = |c|^2,$$

which is Eq. (6) with $e = -v$ and $k = \frac{|c|^2}{\mu}$. □

Remark 2.3 (Classification of the keplerian motion with respect to the energy)**.** *Conservation of energy for the Kepler problem reads*

$$\frac{1}{2}|\dot{x}(t)|^2 - \frac{\mu}{|x(t)|} = h$$

From Eq. (17) *we have the equality*

$$\mu^2 \left| \frac{x}{|x|} + e \right|^2 = |c \wedge \dot{x}|^2.$$

hence, since c and $\dot{x}$ are orthogonal,

$$\mu^2 \left(1 + |e|^2 + \frac{2}{|x|}\langle x, e\rangle\right) = |c|^2|\dot{x}|^2.$$

Replacing $|\dot{x}(t)|^2 = \frac{2\mu}{|x(t)|} + 2h$ *and using once more Eq.* (6) *to deduce the quantity* $\frac{1}{|x|}\langle x, e\rangle$*, we obtain*

$$h = \frac{\mu^2}{2|c|^2}\left(|e|^2 - 1\right).$$

we then conclude that:

- $|e| < 1 \iff h < 0$ *(ellipse)*
- $|e| = 1 \iff h = 0$ *(parabola)*
- $|e| > 1 \iff h > 0$ *(branch of hyperbola)*

2.2. **Kepler's third law.** In order to prove Kepler's third Law, that is a relation between the period of a solution describing an ellipse and the major semi-axis of the ellipse, we need to guarantees that conics (and in particular ellipses) are completely spanned by solutions of the Kepler problem.

Theorem 2.4 (Global existence)**.** *Let $x = x(t)$ be a solution for the Kepler equation with $c \neq 0$. Then x is defined for every $t \in \mathbb{R}$.*

Proof. Kepler equation can be written as

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -\frac{\mu}{|x(t)|^3}x(t). \end{cases}$$

From the first Kepler law if x is a solution on $I \subseteq \mathbb{R}$ then $x(t)$ belong to a conic for every $t \in I$, hence x is bounded away from the origin, i.e.

$$\exists \rho > 0 \;:\; |x(t)| > \rho,\ \forall t \in I.$$

Then

$$\begin{cases} |\dot{x}(t)| = |y(t)|, \\ |\dot{y}(t)| \le \frac{\mu}{\rho^3}|x(t)|, \end{cases}$$

and the function $f(x,y) = (y, -\frac{\mu}{|x|^3}x)$ is sublinear in (x,y). Classical theory of o.d.e. guarantees that $I = \mathbb{R}$ (see for instance [13, Chapter 4, Theorem 1.6]). □

From the previous result immediately follows that a solution for the Kepler problem covers the entire conic on which it moves. Let us show this fact in the elliptic case. Let us write the trajectory in polar coordinates $x(t) = r(t)\,(\cos\vartheta(t), \sin\vartheta(t))$. Since $c = r^2(t)\dot{\vartheta}(t)\mathbf{k}$ then $\dot{\vartheta}(t)$ has constant sign and without restriction we assume that $\dot{\vartheta}(t) > 0$, for every $t \in \mathbb{R}$, and that

$$\dot{\vartheta}(t) = \frac{|c|}{r^2(t)} \quad \forall t \in \mathbb{R}.$$

Since x moves on an ellipse $\mathcal{E}$ and, for some $R > 0$, $\mathcal{E} \subset B_R(0)$, we have

$$\dot{\vartheta}(t) \ge \frac{|c|}{R^2} \quad \forall t \in \mathbb{R}.$$

Hence

$$\lim_{t\to\pm\infty} \vartheta(t) = \pm\infty,$$

ϑ is surjective on $\mathbb{R}$ and we can parametrise the ellipse $\mathcal{E}$ with the angle ϑ. To this aim, let $\varepsilon \in (0,1)$ and $\omega \in [0, 2\pi)$ be such that

$$e = \varepsilon(\cos\omega, \sin\omega)$$

hence $\langle x(t), e\rangle = \varepsilon r(t)\cos(\vartheta(t) - \omega)$ and Eq. (6) gives

$$r(t) = \frac{k}{1 + \varepsilon\cos(\vartheta(t) - \omega)} = \frac{|c|^2/\mu}{1 + \varepsilon\cos(\vartheta(t) - \omega)}, \tag{7}$$

where we have replaced $k = |c|^2/\mu$, as in the proof of the Kepler's first law. The map

$$\gamma \colon \mathbb{R} \to \mathbb{R}^2, \quad \vartheta \mapsto \frac{k}{1 + \varepsilon\cos(\vartheta(t) - \omega)}(\cos\vartheta(t), \sin\vartheta(t))$$

parametrizes the whole ellipse and, since it is 2π-periodic the particle passes an infinite number of times through every point of the ellipse.

Before proving the third Kepler's law let us determine the geometrical elements of an ellipse with one focus at the origin, eccentricity $|e| = \varepsilon$ and polar equation (7), for some $k > 0$. We refer to Figure 7. When $\vartheta = \omega$ then $x = P$, the pericenter of the ellipse, that is the point closer to the origin. When $\vartheta = \omega + \pi$

then $x = A$, the apocenter of the ellipse, that is the furthest point from the origin. Replacing in Eq. (7) these values for ϑ we have

$$|P - 0| = \frac{k}{1+\varepsilon} \qquad |A - 0| = \frac{k}{1-\varepsilon}$$

so that

$$a = \frac{k}{1-\varepsilon^2}.$$

Since

$$d := |C - O| = a - |P - 0| = \frac{\varepsilon k}{1-\varepsilon^2} = \varepsilon a$$

and

$$\begin{aligned} |B - 0| &= \frac{1}{2}\left(|B - 0| + |B - F|\right) \\ &= \frac{1}{2}\left(|P - 0| + |P - F|\right) = \frac{1}{2}\left(|P - 0| + |A - O|\right) = a \end{aligned}$$

we compute

$$b = \sqrt{1-\varepsilon^2}\, a \quad \text{and} \quad \varepsilon = \frac{|C - O|}{a}.$$

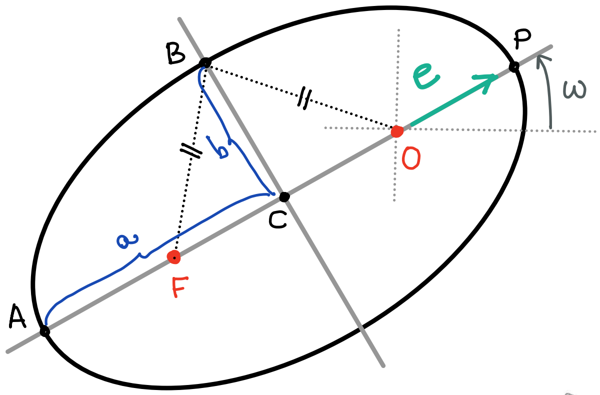

FIGURE 7. An ellipse with a focus at the origin and eccentricity $\varepsilon \in (0, 1)$.

We are now ready to prove Kepler's third law.

Theorem 2.5 (Kepler's third law)**.** *Let $x = x(t)$ be a solution for the Kepler equation with $c \neq 0$ and $h < 0$. Then x is periodic with period*

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}},$$

where a is the major semi-axes of the ellipse described by x.

Proof. We have already seen that it is not restrictive to assume that along the motion $\dot\vartheta(t) \geq C > 0$; hence there exists a unique $T > 0$ such that $\vartheta(T) = \vartheta(0) + 2\pi$. Such T is also a period for

$$x(t) = \frac{k}{1 + \varepsilon\cos(\vartheta(t) - \omega)}(\cos\vartheta(t), \sin\vartheta(t))$$

and finally also for

$$\dot x(t) = \dot r(t)e_r + r(t)\dot\vartheta(t)e_\vartheta = \dot r(t)e_r + \frac{|c|}{r(t)}e_\vartheta$$

Hence x, as a solution of the Kepler's problem, is periodic. Let us remark that T is the minimal period, indeed if $\tilde T$ is such that $x(\tilde T) = x(0)$ then for some $N \in \mathbb{N}$, $N \geq 1$

$$\vartheta(\tilde T) = \vartheta(0) + 2N\pi > \vartheta(0) + 2\pi = \vartheta(T).$$

Hence, since ϑ is strictly monotone, we infer $\tilde T > T$.

We now compute T. We term a and b respectively the major and the minor semi-axis of the ellipse $\mathcal{E}$ described by x. We have, still using Gauss-Green theorem (as at page 7) and $b = \sqrt{1-\varepsilon^2}\,a$,

$$\mathrm{Area}(\mathcal{E}) = \pi ab = \frac{1}{2}\int_0^T r^2(t)\dot\vartheta(t)\,dt = \frac{|c|}{2}T \implies T = \frac{2\pi ab}{|c|} = \frac{2\pi a^2\sqrt{1-\varepsilon^2}}{|c|}$$

Since $a = \frac{|c|^2}{\mu(1-\varepsilon^2)}$ we have the thesis. □

2.3. **Kepler's equation.** We now address to the following problem: fixed an initial position of the planet, can we determine the position of the particle at time t?

We focus on the case $h < 0$ and we look for a different parametrization of an ellipse $\mathcal{E}$, with focus at the origin and major semiaxis a. We refer to Figure 8 and we term *eccentric anomaly* the angle spanned by the vector $P' - O$ with respect to the $C - O$; we claim to write vector P as a function of u.

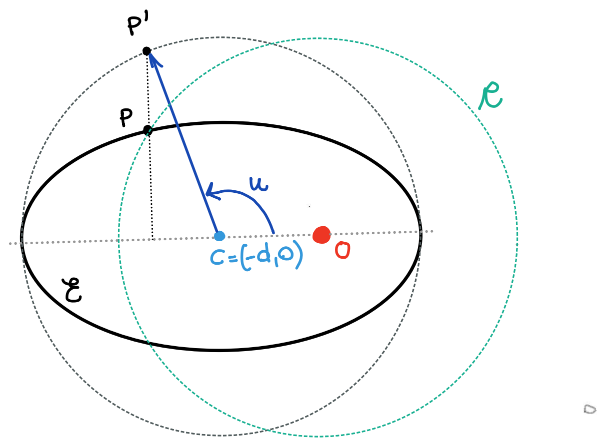

FIGURE 8. An ellipse with a focus at the origin and eccentricity $\varepsilon \in (0,1)$.

Let us consider the maps

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto (x+d,y) \quad \text{and} \quad L\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto \left(x, \frac{a}{b}y\right).$$

Their composition $T \circ L$ is a bijection between the ellipse $\mathcal{E}$ and the circle, $\mathcal{C}$, centered at the origin and with radius equal to a. We can then consider the inverse function

$$(T \circ L)^{-1}\colon \mathcal{C} \to \mathcal{E}, \quad (x,y) \mapsto \left(x-d, \frac{b}{a}y\right)$$

and, writing $(x,y) \in \mathcal{C}$ as $x = a\cos u$, $y = a \sin u$, we have

$$(T \circ L)^{-1}(a\cos u, a\sin u) = (a\cos u - d, b\sin u)$$

Being $d = \varepsilon a$ and $b = a\sqrt{1-\varepsilon^2}$ we obtain

$$x(t) = a\left(\cos u(t) - \varepsilon, \sqrt{1-\varepsilon^2}\sin u(t)\right)$$

and the first derivative

$$\dot{x}(t) = a\left(-\sin u(t), \sqrt{1-\varepsilon^2}\cos u(t)\right)\dot{u}(t).$$

Since $|c| = |x(t) \wedge \dot{x}(t)| = x_1(1)\dot{x}_2(t) - x_2(1)\dot{x}_1(t)$ is a conserved quantity we have the first order o.d.e. in the function $u = u(t)$

$$\dot{u}(t) = \frac{|c|}{a^2\sqrt{1-\varepsilon^2}\,(1-\varepsilon\cos u(t))}.$$

Separating variables and assuming that $u(t_0) = 0$, that is the body os at the pericenter at $t = t_0$, we obtain

$$u(t) - \varepsilon \sin u(t) = \frac{|c|}{a^2\sqrt{1-\varepsilon^2}}(t-t_0).$$

Recalling that the major semi-axis is $a = k/(1-\varepsilon^2)$ and $k = |c|^2/\mu$, we obtain the so called *Kepler equation*

$$u(t) - \varepsilon \sin u(t) = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}(t-t_0). \tag{8}$$

So we haven't really found a solution for our initial problem, but Eq. (8) answers to the question: at what time a particle will be at a certain position (described by u) on the ellipse $\mathcal{E}$?

2.4. **Bessel's function and Kepler's equation.** Let us go back to our initial problem: fixed an initial position of the planet, can we determine the position of the particle at time t? An approximate solution can be computed as long as the eccentricity is small. Indeed, the inversion of equation (8) provides u as a function of $\tilde{\mu}t$ (here, by the sake of simplicity, we assume that $t_0 = 0$), with $\tilde{\mu} = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}$ as a series in the eccentricity ε. In order to do that, let us obtain from (8)

$$\begin{aligned} u &= \tilde{\mu}t + \varepsilon \sin u = \tilde{\mu}t + \varepsilon \sin(\tilde{\mu}t + \varepsilon \sin u) \\ &= \tilde{\mu}t + \varepsilon \sin[\tilde{\mu}t + \varepsilon \sin(\tilde{\mu}t + \varepsilon \sin u)] \end{aligned}$$

and approximate it with a polynomial in ε as $\varepsilon \to 0$

$$u(t) = \tilde{\mu}t + \varepsilon \sin(\tilde{\mu}t) + \frac{1}{2}\varepsilon^2 \sin(2\tilde{\mu}t) + \frac{1}{8}\varepsilon^3 \left(3\sin(3\tilde{\mu}t) - \sin(\tilde{\mu}t)\right) + O(\varepsilon^4).$$

With this strategy we can of course approximate with a desired accuracy, but just for small eccentricities.

On the other hand, when ε is not so small, we should find the inverse relation of (8). Let us start fixing $\varepsilon \in (0,1)$ and defining the function $f_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ as

$$f_\varepsilon(u) = u - \varepsilon \sin u.$$

This function satisfies

$$f_\varepsilon(u + 2\pi) = f_\varepsilon(u) + 2\pi, \qquad f_\varepsilon(-u) = -f_\varepsilon(u),$$

and, since $f'(u) = 1 - \varepsilon \cos u > 0$, for every $u \in \mathbb{R}$, f_ε is $C^\infty(\mathbb{R})$ and invertible. Let $K_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ be the inverse function, that is

$$\zeta = f_\varepsilon(K_\varepsilon(\zeta)) \qquad \text{and} \qquad u = K_\varepsilon(f_\varepsilon(u)) \quad \text{for every } u, \zeta \in \mathbb{R}$$

The evaluation of the inverse function K_ϵ corresponds to the resolution of Kepler equation (8) and actually ζ is proportional to the time for the perihelion, that is

$$\zeta = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}(t - t_0).$$

Furthermore K_ε inherits the properties of f_ε, that is

$$K_\varepsilon(\zeta + 2\pi) = K_\varepsilon(\zeta) + 2\pi, \qquad K_\varepsilon(-\zeta) = -K_\varepsilon(\zeta).$$

We can then define the odd, C^∞ and 2π-periodic function

$$h(\zeta) = K_\varepsilon(\zeta) - \zeta,$$

which can be written by means of Fourier expansions

$$h(\zeta) = \sum_{n=1}^{+\infty} b_n \sin(n\zeta), \ \zeta \in \mathbb{R} \ \text{ where } b_n = \frac{2}{\pi}\int_0^\pi h(\zeta)\sin(n\zeta)\, d\zeta.$$

Replacing now the expression of h in b_n and integrating by parts we obtain

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi [K_\varepsilon(\zeta)-\zeta]\sin(n\zeta)\,d\zeta \\ &= \frac{2}{n\pi}\Big[(\zeta-K_\varepsilon(\zeta))\cos(n\zeta)\Big]_0^\pi + \frac{2}{n\pi}\int_0^\pi [K_\varepsilon'(\zeta)-1]\cos(n\zeta)\,d\zeta \\ &= \frac{2}{n\pi}\int_0^\pi K_\varepsilon'(\zeta)\cos(n\zeta)\,d\zeta, \end{aligned}$$

being $K_\varepsilon(0)=0$, $K_\varepsilon(\pi)=\pi$ and $\int_0^\pi \cos(n\zeta)\,d\zeta=0$. Let now implement the (admissible) variable change in the integral $u=K_\varepsilon(\zeta)$ that means $\zeta=f_\varepsilon(u)=u-\varepsilon\sin u$ and hence

$$b_n = \frac{2}{n\pi}\int_0^\pi \cos[n(u-\varepsilon\sin u)]\,du.$$

We now use the results in Appendix A, in particular Proposition A.1, in order to write

$$b_n = \frac{2}{n}J_n(n\varepsilon)$$

obtaining

$$K_\varepsilon(\zeta) = \zeta + \sum_{n=1}^{+\infty}\frac{2}{n}J_n(n\varepsilon)\sin(n\zeta),\ \zeta\in\mathbb{R},$$

that is

$$u(t) = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}(t-t_0) + \sum_{n=1}^{+\infty}\frac{2}{n}J_n(n\varepsilon)\sin\left(n\frac{\sqrt{\mu}}{a^{\frac{3}{2}}}(t-t_0)\right),\ \zeta\in\mathbb{R}, \tag{9}$$

By means of this formula we can write the explicit parametrization of an elliptic orbit with major semi-axis a, eccentricity ε with pass at its perihelion at time t_0

$$x_1(t) = a(\cos u(t)-\varepsilon) \qquad x_2(t) = a\sqrt{1-\varepsilon^2}\sin u(t)$$

where u has the explicit form (9).

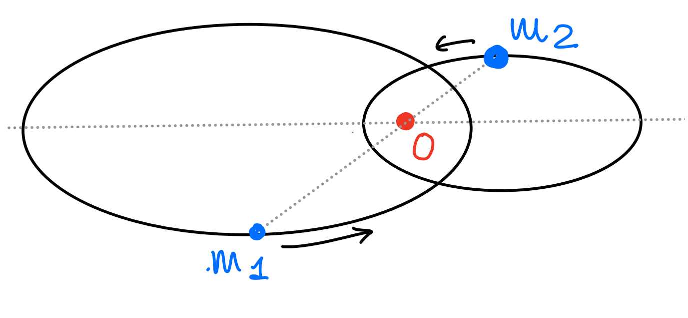

FIGURE 9. Elliptic trajectories for the two body problem with $h<0$ and $m_1<m_2$.

2.5. **The 2-body problem.** We now consider two masses, m_1 and m_2, that move on trajectories $x_1(t)$ and $x_2(t)$ under their mutual gravitational attraction

$$\begin{cases} \ddot{x}_1(t) & = Gm_2 \frac{x_2(t)-x_1(t)}{|x_2(t)-x_1(t)|^3} \\ \ddot{x}_2(t) & = Gm_1 \frac{x_1(t)-x_2(t)}{|x_2(t)-x_1(t)|^3} \end{cases}$$

and we immediately remark that the center of mass $g = m_1x_1(t) + m_2x_2(t)$ moves with uniform velocity. We can then fix it, staying in the inertial frame centered at g, i.e. assuming

$$x_2(t) = -\frac{m_1}{m_2}x_1(t). \tag{10}$$

The vector $x_2(t) - x_1(t)$ can then be written both in terms of $x_1(t)$ and $x_2(t)$, indeed

$$x_2(t) - x_1(t) = -\frac{m_1 + m_2}{m_2}x_1(t) = \frac{m_1 + m_2}{m_1}x_2(t)$$

and the equation of motion are simply Keplerian equations for each one of the bodies

$$\begin{cases} \ddot{x}_1(t) & = -G\mu_1 \frac{x_1(t)}{|x_1(t)|^3} \\ \ddot{x}_2(t) & = -G\mu_2 \frac{x_2(t)}{|x_2(t)|^3} \end{cases}$$

with $\mu_1 = \frac{m_2^3}{(m_1+m_2)^2}$ and $\mu_2 = \frac{m_1^3}{(m_1+m_2)^2}$. We then conclude that in the inertial frame centered in the baricenter of the masses, the two bodies move on conics linked by the relation (10) (see Figure 9). In particular their trajectories are coplanar.

3. The N-body problem

The N-body problem consists in the study of the motion of N point particles (the *bodies*) in the d-dimensional space, when on each particle act just the gravitational forces induced by the other $N-1$.
In order to formalize this dynamical system, let us introduce the *masses* of the bodies

$$m_1, \dots, m_N > 0$$

and their positions $x_1(t), \dots, x_N(t)$, which form the vector

$$x(t) = (x_1(t), \dots, x_N(t)) \in \mathbb{R}^{dN}.$$

The gravitational force that the mass m_j acts on the mass m_i is

$$F_{ij}(x) = m_i m_j G \frac{x_j - x_i}{|x_j - x_i|^3}$$

where G is the gravitational constant. Without loosing in generality we now on assume that $G = 1$. The total force acting on the i-th mass then is

$$F_i = \sum_{j \neq i, j=1}^{n} F_{ij}$$

and Newton's law reads

$$m_i \ddot{x}_i(t) = F_i.$$

Of course, in general F_i is not a central force field. Dividing both sides of the previous equation by m_i, we obtain the N-body dynamical system

$$\begin{cases} \ddot{x}_i(t) = \displaystyle\sum_{j \neq i, j=1}^{n} m_j \frac{x_j - x_i}{|x_j - x_i|^3} \\ i = 1, \dots, n \end{cases} \tag{11}$$

which is defined when $x \in \mathbb{R}^{dN} \setminus \Delta$ where Δ is the *collision set*

$$\Delta = \left\{ x \in \mathbb{R}^{dN} : x_i = x_j \text{ for some } i \neq j \right\}.$$

From now, we consider $d = 3$. Introducing the force vector

$$F(x) = (F_1(x), \dots, F_N(x)) \in \mathbb{R}^{3N}$$

and the diagonal-block matrix of dimension $3N \times 3N$

$$M = \mathrm{diag}\,(m_1 I_3, \dots, m_N I_3)$$

we can write system (11) in the more compact form

$$M \ddot{x}(t) = F(x(t)). \tag{12}$$

Lemma 3.1. *The dynamical system* (11) *is conservative and the* C^1 *scalar function*

$$V(x) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|} \quad x \in \mathbb{R}^{3N} \setminus \Delta \tag{13}$$

is such that (11) *reads*

$$M\ddot{x}(t) = \nabla V(x(t)). \tag{14}$$

Proof. For any $k = 1, \dots, n$, we have

$$\frac{\partial}{\partial x_k} V(x) = \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \frac{m_i m_k}{|x_i - x_k|} \right] = \sum_{i \neq k} m_i m_k \frac{x_i - x_k}{|x_i - x_k|^3} = \sum_{i \neq k} F_{ki} = F_k.$$

□

3.1. **First integrals.** From Lemma 3.1 follows immediately that as far as $x : I \subset \mathbb{R} \to \mathbb{R}^{3N}$ is a solution of (11) then the *total energy* is conserved along the motion x, that is

$$\frac{1}{2}\langle M\dot{x}(t), \dot{x}(t)\rangle - V(x(t)) = h, \qquad \text{for any } t \in I \tag{15}$$

for some constant $h \in \mathbb{R}$. The function

$$K(p) := \frac{1}{2}\langle Mp, p\rangle, \qquad p \in \mathbb{R}^{3N}$$

is the *kinetic energy*, while the *potential energy* is $-V(q)$, $q \in \mathbb{R}^{3N} \setminus \Delta$. In order to verify Eq. (15) we compute

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\langle M\dot{x}(t), \dot{x}(t)\rangle - V(x(t))\right] \\
&= \frac{1}{2}\langle M\ddot{x}(t), \dot{x}(t)\rangle + \frac{1}{2}\langle M\dot{x}(t), \ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle \\
&= \langle M\ddot{x}(t), \dot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle \\
&= \langle M\ddot{x}(t) - F(x(t)), \dot{x}(t)\rangle
\end{aligned}$$

and the last term vanishes, since x solves system (14).

As in the 2-boy problem, the centre of mass moves with constant velocity, indeed

$$\sum_{i=1}^{N} m_i \ddot{x}_i(t) = \sum_{i=1}^{N} F_i = \sum_{i \neq j} F_{ij} = \sum_{i<j} (F_{ij} - F_{ji}) = 0$$

hence it not restrictive to work in the inertial frame that moves with the centre of mass, or equivalently, to assume that the centre of mass is fixed at the origin. Up to now we have then found seven integrals of motions. As for central force fields, also the angular momentum is conserved. More precisely, let us introduce the *total angular momentum* for a solution $x(t)$ $\sum_{i=1}^n [x_i(t) \wedge \dot{x}_i(t)]$. It turns out that there exists a constant vector $c \in \mathbb{R}^3$ such that

$$c = \sum_{i=1}^n [x_i(t) \wedge \dot{x}_i(t)], \qquad \text{for any } t.$$

Indeed

$$\begin{aligned}
\frac{d}{dt} \sum_{i=1}^n [x_i(t) \wedge \dot{x}_i(t)] &= \sum_{i=1}^n [x_i(t) \wedge \ddot{x}_i(t)] \\
&= \sum_{i=1}^n \left[x_i(t) \wedge \left(\sum_{j \neq i} m_i m_j \frac{x_j(t) - x_i(t)}{|x_i(t) - x_j(t)|^3} \right) \right] \\
&= \sum_{i=1}^n \left(\sum_{j \neq i} m_i m_j \frac{x_i(t) \wedge x_j(t)}{|x_i(t) - x_j(t)|^3} \right) = 0
\end{aligned}$$

since in the last line appears all pairs of opposite terms. Hence we have found ten first integrals. Let us observe that as far as we take $N \geq 3$, system (14) has $K \geq 18$ degrees of freedom. On the (non) integrability of the N-body problem we suggest for instance [5, 14, 18, 19, 6].

3.2. **Special soutions.** In this paragraph we examinate some special classes of solution for the N-body problem, and precisely we will deal with:

(a) Constant solutions or equilibrium point for the system. These are solutions of (14) in the form

$$x(t) = \bar{x} \in \mathbb{R}^{3N} \setminus \Delta, \quad \text{for every } t \in I \subseteq \mathbb{R}.$$

(b) Homographic solutions. We investigate the existence of

$$\lambda \colon I \to (0, +\infty), \quad A \colon I \to SO(3)^3 \text{ and } \bar{x} \in \mathbb{R}^{3N} \setminus \Delta$$

such that $x(t) = \lambda(t) A(t) \bar{x}$ solves (14), where the matrix $A(t)$ acts on each component

$$A(t)\bar{x} = (A(t)\bar{x}_1, \dots, A(t)\bar{x}_N).$$

(b1) Homothetic solutions. When, in situation (b), the matrix function A is constantly equal to the identity, the motion we are deal with has the form $x(t) = \lambda(t)\bar{x}$, for some

$$\lambda \colon I \to (0, +\infty) \quad \text{and } \bar{x} \in \mathbb{R}^{3N} \setminus \Delta.$$

[3] $SO(3)$ is the set of 33×3 matrices with determinant equal 1 (rotations in $\mathbb{R}^3$). When $A \in SO(3)$, there exists a vector $w \in \mathbb{R}^3$ such that $Aw = w$ and A acts as a rotation on planes orthogonal to w.

(b2) Relative equilibria. When, in situation (b), the scalar function λ is constant, the motion we are deal with has, up to a constant, the form $x(t) = A(t)\bar{x}$, for some

$$A\colon I \to SO(3) \quad \text{ and } \bar{x} \in \mathbb{R}^{3N} \setminus \Delta.$$

This classical result is useful to deal with solutions of type (a).

Euler's Theorem. *Let $p \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^d$ be a conic set and $f\colon \Omega \to \mathbb{R}$ be a differentiable function and assume that f is p-homogeneous, that is*

$$f(\lambda x) = \lambda^p f(x), \quad \text{for any } x \in \Omega \text{ and } \lambda > 0. \tag{16}$$

Then

$$\langle \nabla f(x), x\rangle = pf(x), \quad \text{for any } x \in \Omega.$$

Proof. We obtain the thesis differentiating with respect to λ equality (16) and evalueting the result at $\lambda = 1$. □

Corollary 3.2. *System* (14) *does not admit any constant solution.*

Proof. By contradiction, assume that there exists a sulution of the form $x(t) = \bar{x}$ for some $\bar{x} \in \mathbb{R}^{3N} \setminus \Delta$. Then $M\ddot{x}(t) = 0$, hence $\nabla V(\bar{x}) = 0$ and $\langle \nabla V(\bar{x}), \bar{x}\rangle = 0$. Since V is -1-homogeneous we obtain, applying Euler's Theorem, $-V(\bar{x})$. This is in contradiction with the definition of V given in Eq. (13). □

Let us now investigate the existence of solutions of type (b1). We replace $x(t) = \lambda(t)\bar{x}$ in (14) and we obtain

$$M(\ddot{\lambda}(t)\bar{x}) = \nabla V(\lambda(t)\bar{x})$$

hence

$$\ddot{\lambda}(t) M\bar{x} = [\lambda(t)]^{-2}\nabla V(\bar{x}). \tag{17}$$

In order to find an equation for λ, let us now project both terms of this equation in the direction of $\bar{x}$, obtaining

$$\ddot{\lambda}(t)\langle M\bar{x}, \bar{x}\rangle = [\lambda(t)]^{-2}\langle \nabla V(\bar{x}), \bar{x}\rangle.$$

Introducing the *moment of inertial* of a configuration $x \in \mathbb{R}^{3N}$

$$I(x) := \frac{1}{2}\langle Mx, x\rangle \tag{18}$$

and using Euler's Theorem, we deduce that λ satisfies the one-dimensional Kepler equation

$$\ddot{\lambda}(t) = -\frac{\mu}{[\lambda(t)]^2}, \qquad \text{where } \mu = \frac{V(\bar{x})}{2I(\bar{x})}. \tag{λ}$$

Remark 3.3. *It is not restrictive to assume $I(\bar{x}) = 1$; indeed, if this is not the case and $I(\bar{x}) \neq 1$ we can define*

$$\tilde{x} = \frac{\bar{x}}{I(\bar{x})}, \qquad \tilde{\lambda}(t) = I(\bar{x})\lambda(t),$$

and consider the homothetic motion $x(t) = \tilde{\lambda}(t)\tilde{x}$.

In virtue of the previous remark, we assume $I(\bar{x}) = 1$, and replace the expression of $\ddot{\lambda}(t)$ in (17) in order to obtain

$$M\bar{x} = -\frac{1}{\mu}\nabla V(\bar{x}).$$

Actually, from this equation we understand that the position vector of each body in a central configuration is opposite to the one of the force acting on it with a common scale factor (equal to $-\mu$); furthermore the previous equation can be written as

$$(\bar{x}) \qquad \nabla V(x)|_{x=\bar{x}} - \mu \nabla I(x)|_{x=\bar{x}},$$

and the following result follows.

Proposition 3.4. *The function $x(t) = \lambda(t)\bar{x}$ (as in (b1)) solves Eq.* (14) *on $I \subseteq \mathbb{R}$ if and only if*

(i) $\lambda \colon I \to (0, +\infty)$ *solves the one-dimensional Kepler problem* (λ) *and*
(ii) $\bar{x}$ *is a critical point of the potential V constrained to the* inertia ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^{3N} \colon I(x) = 1\}$$

Definition 3.5. *A configuration $\bar{x} \in \mathbb{R}^{3N} \setminus \Delta$ that is a critical point of the potential V constrained to the inertia ellipsoid $\mathcal{E}$, $V|_{\mathcal{E}}$, is termed* central configuration.

We will investigate the problem of searching central configuration in Section 3.3, while here we give a result for motion of kind (b2).

Theorem 3.6. *If the function $x(t) = A(t)\bar{x}$ (as in (b2)) solves Eq.* (14) *on $I \subseteq \mathbb{R}$ then*

(i) *there exists $w \in \mathbb{R}^3$ such that $A(t)w = w$, for any $t \in I$ and $A(t)$ is a uniform rotation with rotation axes w;*
(ii) $\bar{x}$ *is a planar central configuration and its plane is orthogonal to w.*

In particular, x is a planar motion.

The proof of this results follows by a sequence of lemmata. First of all, as in the homothetic case, let us replace $x(t) = A(t)\bar{x}$ in (14) and obtain

$$M\ddot{A}(t)\bar{x} = \nabla V(A(t)\bar{x}) = A(t)\nabla V(\bar{x})$$

or, equivalently,

$$A^{-1}(t)\ddot{A}(t)M\bar{x} = \nabla V(\bar{x}) \tag{19}$$

Let us define the matrix

$$W(t) = A^{-1}(t)\dot{A}(t), \qquad t \in I;$$

Lemma 3.7. *The matrix $W(t)$ satisfies the following properties, for every $t \in I$:*
$(W1)$ $W(t)$ is antisymmetric;
$(W2)$ $\dot{W}(t) + W^2(t) = A^{-1}(t)\ddot{A}(t)$.

Proof. In order to prove $(W1)$, let us make the following computation

$$\begin{aligned} A(t)A^T(t) = I_3 \quad &\Longrightarrow \quad \frac{d}{dt}[A(t)A^T(t)] = 0_3 \\ &\Longrightarrow \quad \dot{A}(t)A^T(t) + A(t)\dot{A}^T(t) = 0_3 \\ &\Longrightarrow \quad A^{-1}(t)\dot{A}(t) = -\dot{A}^T(t)[A^T(t)]^{-1} \\ &\Longrightarrow \quad W(t) = A^{-1}(t)\dot{A}(t) = -\left[\dot{A}(t)A^{-1}(t)\right]^T = -W^T(t). \end{aligned}$$

With a similar computation we have

$$\begin{aligned} \frac{d}{dt}[A^{-1}(t)A(t)] = 0_3 \quad &\Longrightarrow \quad \dot{A^{-1}}(t)\dot{A}(t) + A^{-1}(t)\ddot{A}(t) = 0_3 \\ &\Longrightarrow \quad \dot{A^{-1}}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t) \end{aligned}$$

Concerning $(W2)$ we compute

$$\begin{aligned} \dot{W}(t) &= \dot{A^{-1}}(t)\dot{A}(t) + A^{-1}(t)\ddot{A}(t) \\ &= -A^{-1}(t)\dot{A}(t)A^{-1}(t)\dot{A}(t) + A^{-1}(t)\ddot{A}(t) = -W^2(t) + A^{-1}(t)\ddot{A}(t). \end{aligned}$$

□

From property $(W1)$ we deduce the existence of a vector function $w(t) = (w_1(t), w_2(t), w_3(t))$, $t \in I$, such that

$$W(t) = \begin{bmatrix} 0 & -w_3(t) & w_2(t) \\ w_3(t) & 0 & -w_1(t) \\ -w_2(t) & w_1(t) & 0 \end{bmatrix}.$$

Lemma 3.8. *There exists $w \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ such that $w(t) = w$, $\forall t \in I$.*

Sketch of the proof After some geometric arguments (see [11, pp.35-37]), Eq. (19) implies that

$$\forall x \in \mathbb{R}^3,\ \exists y \in \mathbb{R}^3 :\ A^{-1}(t)\ddot{A}(t)x = y\ \forall t \in I,$$

hence, by means of $(W2)$ also $\dot{W}(t)+W^2(t)$ is constant and from straightforward computations the thesis follows. □

Without loosing in generality we can assume that

$$\exists k \in \mathbb{R} \setminus \{0\} \ : \ w(t) = (0,0,k) \, \forall t \in I.$$

Lemma 3.9. *Each element of the central configuration* $\bar{x} = (\bar{x}_1, \dots, \bar{x}_N) \in \mathbb{R}^{3N}$*, belongs to the plane,* $Z := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \, : \, x_3 = 0\}$*, orthogonal to* w*.*

Proof. From Lemma 3.8 and assertion $(W1)$ of Lemma 3.7 we deduce that

$$A^{-1}(t)\ddot{A}(t) = \begin{bmatrix} -k^2 & 0 & 0 \\ 0 & -k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \forall t \in I,$$

hence, $A^{-1}(t)\ddot{A}(t)\bar{x} \in Z$ for every $t \in I$. From Eq. (19) we deduce that $M^{-1}\nabla V(\bar{x}) \in Z$, that is

$$\frac{1}{m_i}\nabla_{x_i} V(\bar{x}) = \sum_{j \neq i} m_j \frac{\bar{x}_j - \bar{x}_i}{|\bar{x}_j - \bar{x}_i|^3} \in Z, \quad \forall i = 1, \dots, N.$$

Let now $\ell \in \{1, \dots, N\}$ be such that

$$\max_{i=1,\dots,N} \bar{x}_{i3} = \bar{x}_{\ell 3}$$

and consider the third component of the previous equation when $i = \ell$

$$\sum_{j \neq \ell} m_j \frac{\bar{x}_{j3} - \bar{x}_{\ell 3}}{|\bar{x}_{j3} - \bar{x}_{\ell 3}|^3} = 0.$$

From the choice of the index ℓ, this choice forces

$$\bar{x}_{j3} - \bar{x}_{\ell 3} = 0,$$

for any choice of index j. This concludes the proof. □

We are ready to prove the main result concerning motions of type (b2).

Proof of Theorem 3.6. Since $W = A^{-1}(t)\dot{A}(t)$ for any $t \in I$, the matrix A solves

$$\dot{A}(t) = WA(t), \qquad \forall t \in I.$$

In order to have a Cauchy problem that fits with our construction, we fix an initial condition $A(0) = B$ with $B \in SO(3)$ and $Bw = w$. Indeed $\bar{x} \in Z$ and we want our motion $x(t) = A(t)\bar{x}$ to start from a configuration that is (a rotation of) $\bar{x}$. It is well known that the unique solution of such Cauchy problem is the matrix

$$A(t) = Be^{Wt} = B \begin{bmatrix} \cos(kt) & \sin(kt) & 0 \\ \sin(kt) & \cos(kt) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

By means of Theorem 3.6 we can consider

$$\bar{x} = (\bar{x}_1, \dots, \bar{x}_N) \quad \text{and} \quad A(t) = e^{ib}e^{ikt},$$

for some $b, k \in \mathbb{R}$. Multiplying both sides of Eq. (19) by $\bar{x}$ we obtain

$$-2k^2 = -\nabla V(\bar{x}), \qquad \text{hence } k^2 = \frac{1}{2}V(\bar{x}).$$

Summing-up, we have shown that: *given a planar central configuration $\bar{x}$, there are two solutions associated to it. Such solutions are uniform rotations around the centre of mass $\bar{x}$ (the origin) and have angular velocity equal to*

$$k = \pm\frac{\sqrt{2V(\bar{x})}}{2}.$$

This motions are stationary in a rotating frame centered at the origin, for this reason are termed *relative equilibrium motions.*

Let us conclude this section with a result on solutions of kind (b), namely, homographic solutions.

Theorem 3.10. *Assume that the homographic function $x(t) = \lambda(t)A(t)\bar{x}$ (as in (b)) solves Eq. (14) on $I \subseteq \mathbb{R}$ and that the matrix-function A is not the identity matrix at any time. Then the motion x is planar.*

By means of the previous result we can assume that $\bar{x} \in \mathbb{R}^{2N} \equiv \mathbb{C}^N$ and the existence of a function $\varphi\colon I \subseteq \mathbb{R} \to \mathbb{C} \setminus \{0\}$ such that

$$\varphi(t)x = \lambda(t)A(t)x, \qquad \forall x \in \mathbb{R}^2,\ \forall t \in I.$$

An homographic solution has then the form $x(t) = \varphi(t)\bar{x}$ and the following equation has to be satisfied

$$\ddot{\varphi}(t)M\bar{x} = \frac{\varphi(t)}{|\varphi(t)|^3}\nabla V(\bar{x}). \tag{20}$$

Arguing as at p. 28 we find that the function φ has to satisfy the planar two-dimensional Kepler problem

$$\ddot{\varphi}(t) = -\mu\frac{\varphi(t)}{|\varphi(t)|^3}, \qquad \text{with } \mu = \frac{V(\bar{x})}{2I(\bar{x})},$$

while $\bar{x}$ is necessarily a planar central configuration.

3.3. **The search of central configurations.** We have already defined a central configuration (see Definition 3.5 at p. 29) as a solution $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N) \in \mathbb{R}^{3N}$ of equation $(\bar{x})$, namely a solution of the *central configuration equation*

$$M\bar{x} = -\frac{2I(\bar{x})}{V(\bar{x})}\nabla V(\bar{x}).$$

Let us remark that given a solution $\bar{x}$ and any $a \in \mathbb{R}$ and $A \in SO(3)$, then also $a\bar{x} = (a\bar{x}_1, \ldots, a\bar{x}_N)$ and $A\bar{x} = (A\bar{x}_1, \ldots, A\bar{x}_N)$ are solutions; hence it is not restrictive to assume that $I(\bar{x}) = 1$ and understand the central configuration equation and a constrained optimization problem. A central configuration is a critical point of V on the inertial ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^{3N} \colon I(x) = 1\}.$$

The next result ensures that, for any distribution of the masses, a central configuration always exists.

Lemma 3.11. *Given $m_1, \dots, m_N > 0$, there exists at least one central configuration for the N-body problem with this distribution of the masses.*

Proof. We will prove that the constrained function $V_{|\mathcal{E}}$ admits a global minimizer. We first observe that $\mathcal{E} \cap \Delta \neq \emptyset$ and

$$\lim_{dist(x,\Delta)\to 0} V_{|\mathcal{E}}(x) = +\infty;$$

We understand V defined also on such points, that is we define $V(x) = +\infty$, whenever $x \in \mathcal{E} \cap \Delta$. Furthermore there exist points in $\mathcal{E}$ which are not in Δ, let then $\hat{x} \in \mathcal{E}$, $\hat{x} \notin \Delta$, and $k := 2V(\hat{x})$. Consider the set

$$V^{>k} := \{x \in \mathcal{E} \colon V(x) > k\}.$$

Of course $\Delta \cap \mathcal{E} \subset V^{>k}$ and the set

$$V^{\leq k} := \mathcal{E} \setminus V^{>k} = \{x \in \mathcal{E} \colon V(x) \leq k\}$$

is closed and bounded. Hence that exists $x_0 \in V^{\leq k}$ such that

$$\min_{x \in V^{\leq k}} V(x) = V(x_0)$$

Since $\hat{x} \in V^{\leq k}$ and $V(\hat{x}) = k/2 < 2k$, then $V(x_0) \leq k/2$ and x_0 does not belong to the boundary of $V^{\leq k}$. We conclude that x_0 is a critical point of $V_{|\mathcal{E}}$, hence a central configuration. □

The next result guarantees a certain number of central configurations in which all the bodies stay on the same line.

Moulton's Theorem on collinear central configurations. *Given $N \geq 3$ and N positive masses, $m_1, \dots, m_N$, there exist $N!$ central configurations for the 1-dimensional N-body problem, one for each permutation of the indexes.*

Proof. We prove the result when $N = 3$ (in this case the present theorem has been proved By Euler in 1767 in [7]).
Let $x_1, x_2, x_3 \in \mathbb{R}$ be the positions of the three bodies on a common line and consider the inertia ellipsoid

$$\mathcal{E} := \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ m_1 x_1^2 + m_2 x_2^2 + m_3 x_i^3 = 2\big\}.$$

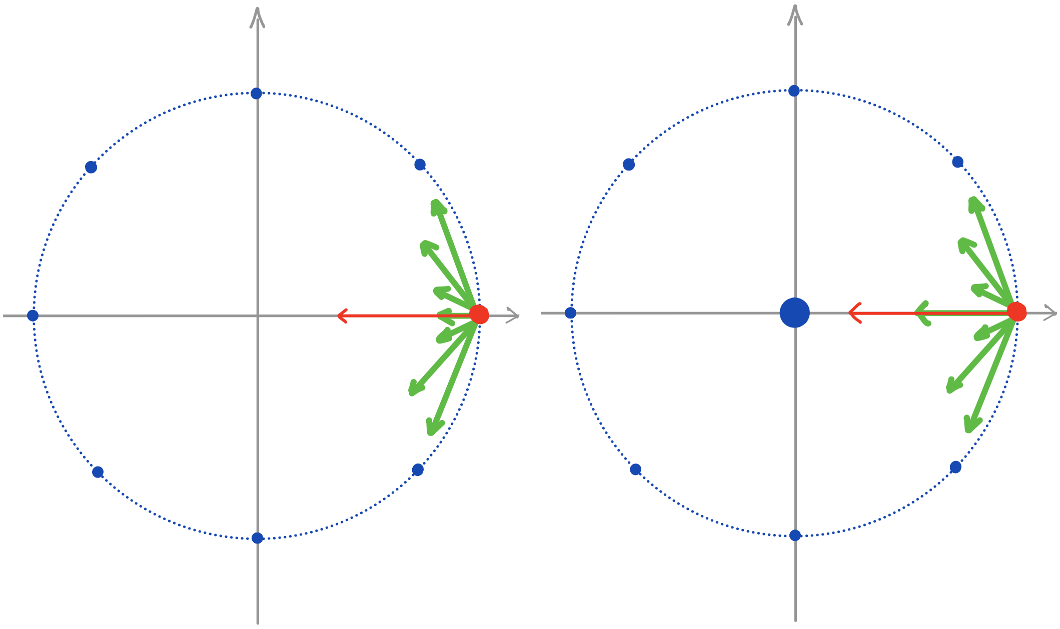

FIGURE 10. Left: a central configuration for the 8-body problem with equal masses. The force acting on a body (red arrow) is opposite to its position. Right: a big mass is in the centre of the octagon, the situation remains the same.

As we already know, without loosing in generality, we can assume that the three bodies has the centre of mass at the origin, this means that (x_1, x_2, x_3) belong to the plane

$$G := \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ m_1x_1 + m_2x_2 + m_3x_3 = 0\}.$$

Of course, $(0, 0, 0) \in G$ and $\mathcal{E} \cap G$ is homeomorphic to $\mathbb{S}^1$. Let us now consider, for any $i, j \in \{1, 2, 3\}$ with $i < j$, the sets

$$\Delta_{ij} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_i = x_j\}$$

which are actually three planes containing the origin, distinct from G (indeed G cointains also non collisional elements). We can then define three couples of points (see Figure 11)

$$\{P_{ij}, P'_{ij}\} = \Delta_{ij} \cap (G \cap \mathcal{E}), \qquad i, j \in \{1, 2, 3\},\ i < j.$$

Let us observe that P_{ij} and P'_{ij} are antipodal with respect to the origin and

$$(G \cap \mathcal{E}) \setminus \bigcup_{i<j} \{P_{ij}, P'_{ij}\}$$

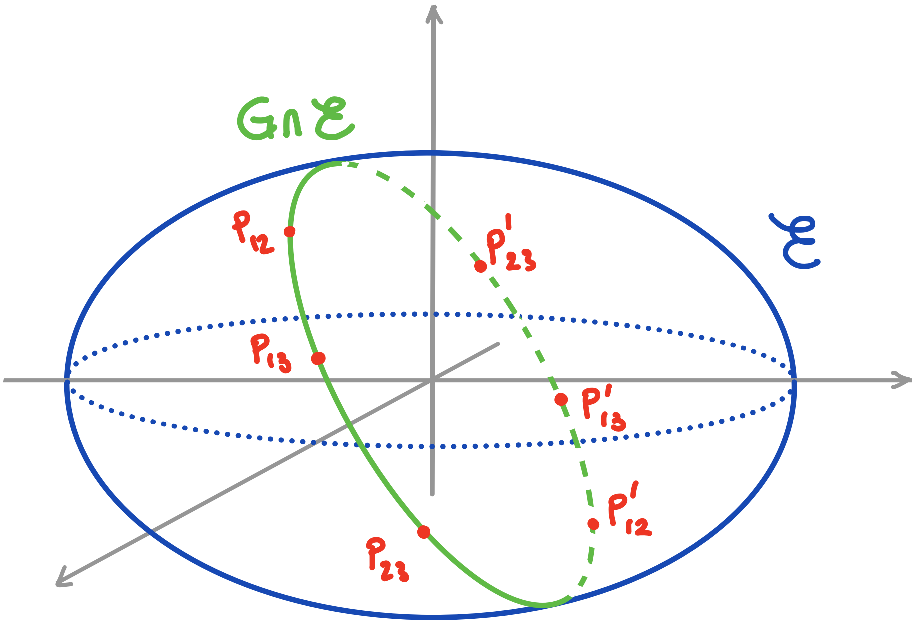

FIGURE 11. Construction of the proof of Multon's Theorem when $N = 3$.

has six connected components. Arguing as in the proof of Lemma 3.11, we deduce the existence of a central configuration in each component. We conclude the proof showing that such central configuration is unique. In order to do that, we prove that the hessian of the function $V_{|G\cap\mathcal{E}}$ is positive definite on any central configuration, hence any critical point must be a minimizer and we deduce its uniqueness.
In order to compute the hessian of $V_{|G\cap\mathcal{E}}$ at a central configuration, we define the auxiliary function

$$f(x) := \sqrt{I(x)}V(x), \qquad x \in (\mathbb{R}^3)^3.$$

Then for any $x \in \mathcal{E}$ we have that $f(x) = V(x) = V_{|\mathcal{E}}(x)$. We now compute

$$\begin{aligned}\nabla f(x) \cdot v &= \frac{d}{dt} f(x + tv)_{|t=0} \\ &= \frac{V(x)}{2\sqrt{I(x)}} \nabla I(x) \cdot v + \sqrt{I(x)} \nabla V(x) \cdot v\end{aligned}$$

and

$$\begin{aligned}
Hf(x)[v,v] &= \frac{d}{dt}[\nabla f(x+tv)\cdot v]_{|t=0} \\
&= -\frac{V(x)}{4[I(x)]^{\frac{3}{2}}}(\nabla I(x)\cdot v)^2 + \frac{V(x)}{2\sqrt{I(x)}}HI(x)[v,v] \\
&+ \frac{\nabla V(x)\cdot v}{\sqrt{I(x)}}\nabla I(x)\cdot v + \sqrt{I(x)}HV(x)[v,v].
\end{aligned}$$

Choosing $\hat{x}\in\mathcal{E}$ (hence $I(\bar{x})=1$) and $v\in T_{\bar{x}}\mathcal{E}$ (hence $\nabla I(\hat{x})\cdot v=0$) we have

$$HV_{|\mathcal{E}}(\hat{x})[v,v] = Hf(\hat{x})[v,v] = \frac{1}{2}(Mv\cdot v)V(\hat{x}) + HV(\hat{x})[v,v]$$

and, since $(Mv\cdot v)V(\hat{x})>0$, we conclude the proof showing that

$$HV(\hat{x})[v,v]\geq 0, \forall v\in T_{\hat{x}}\mathcal{E}.$$

We first observe that for any $x\in\mathbb{R}^n$, $v\in T_x\mathbb{R}^n\equiv\mathbb{R}^n$ it holds

$$\begin{aligned}
\nabla V(x)\cdot v = \sum_{i=1}^{n}\nabla_{x_i}V(x)v_i &= \sum_{i=1}^{n}\left(\sum_{j\neq i} m_i m_j \frac{x_j-x_i}{|x_j-x_i|^3}\right)v_i \\
&= \sum_{i<j;\,i,j=1}^{n} m_i m_j \frac{x_j-x_i}{|x_j-x_i|^3}(v_i-v_j) \\
&= -\sum_{i<j;\,i,j=1}^{n} m_i m_j \frac{x_i-x_j}{|x_i-x_j|^3}(v_i-v_j),
\end{aligned}$$

hence for any $x,v,w\in\mathbb{R}^n$

$$\begin{aligned}
HV(x)[v,w] &= \frac{d}{d\varepsilon}\left[\nabla V(x+\varepsilon w)\cdot v\right]_{|\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left[-\sum_{i<j;\,i,j=1}^{n} m_i m_j \frac{x_i-x_j+\varepsilon(w_i-w_j)}{|x_i-x_j+\varepsilon(w_i-w_j)|^3}(v_i-v_j)\right]_{|\varepsilon=0} \\
&= -\sum_{i<j;\,i,j=1}^{n} m_i m_j \frac{w_i-w_j}{|x_i-x_j|^3}(v_i-v_j) \\
&\quad + 3\sum_{i<j;\,i,j=1}^{n} m_i m_j \frac{(x_i-x_j)(v_i-v_j)}{|x_i-x_j|^5}(x_i-x_j)(w_i-w_j) \\
&= 2\sum_{i<j;\,i,j=1}^{n} m_i m_j \frac{w_i-w_j}{|x_i-x_j|^3}(v_i-v_j)
\end{aligned}$$

and the required inequality is proved. ◻

We conclude the section with a simple and interesting result due to J.L. Lagrange (see [10]) concerning the 3-body problem. In this special case the 3 bodies are necessarily on the same plane, hence the central configuration equation has 6 unknown. Assuming that the centre of mass is at the origin, then the left unknown are 4; identifying two configuration when the first a rotation of the second, the left unknown are just 3. It turns out that the mutual distances

$$\rho_{12} = |x_1 - x_2|, \qquad \rho_{13} = |x_1 - x_3|, \qquad \rho_{23} = |x_2 - x_3|$$

can be chosen as a set of coordinates for the planar central configurations equations and we write both the potential V and the moment of inertia I as a function of $\rho = (\rho_{12}, \rho_{13}, \rho_{23})$. With a slight abuse of notation we obtain

$$V(\rho) = \sum_{i<j;\, i,j=1}^{3} \frac{m_i m_j}{\rho_{ij}},$$

and, introducing $M = m_1 + m_2 + m_3$,

$$I(\rho) = \frac{1}{2M}(m_1 m_2 \rho_{12}^2 + m_1 m_3 \rho_{13}^2 + m_2 m_3 \rho_{23}^2).$$

To justify the last identity we observe that

$$\begin{aligned} m_1 m_2 \rho_{12}^2 + m_1 m_3 \rho_{13}^2 + m_2 m_3 \rho_{23}^2 &= \sum_{i<j} m_i m_j |x_i - x_j|^2 \\ &= \sum_{i<j} m_i m_j (|x_i|^2 + |x_j|^2 - 2x_i \cdot x_j) \\ &= 2MI(x) - \sum_{i=1}^{3}(m_i x_i) \sum_{j=1}^{3}(m_j x_j) \\ &= 2MI(x). \end{aligned}$$

Theorem 3.12. *When $N = 3$ and for any choice of $m_1, m_2, m_3 > 0$, any non-collinear central configuration is a regular triangle.*

Proof. In virtue of what we have observed just before the statement of this theorem, in order to determine a central configuration for the planar three body problem we need to find $\rho \in (\mathbb{R}^+)^3$ and $\lambda \neq 0$ such that

$$\nabla V(\rho) = \lambda \nabla I(\rho),$$

which means

$$-\frac{m_i m_j}{\rho_{ij}^2} = \lambda \frac{1}{M} m_i m_j \rho_{ij}, \qquad i, j = 1, \ldots, 3.$$

This implies $\rho_{12} = \rho_{13} = \rho_{23} = \left(-\frac{M}{\lambda}\right)^{\frac{1}{3}}$, which is the thesis. ☐

Corollary 3.13. *When $N = 3$ and for any choice of the masses, there exist exactly (up to rotations) five central configurations for the 3-body problem centered at the origin: three of them are collinear and the other two form a regular triangle.*

$\boxed{\texttt{** TO DO **}}$ *Figura con i due triangoli e i moti omografici .*

Remark 3.14. *With a proof similar to the one of Theorem 3.12 one can show that when $N = 4$ the only non planar central configuration is the tetrahedron. In spite of the simplicity of these proofs, there are many open problems in the search of central configurations (see for instance* [17, Problem 6] *and* [16]*). The most struggling one seems to be:* is the number of central configurations finite, up to symmetry? *The phrase* up to symmetry *is important since the set of central configurations is invariant under rotations, translations and dilations. This question has a positive answer when $N = 3$ and we will prove it in the next pages. When $N = 4$, it has been recently proved with a computer assisted proof in* [9]*; when $N = 5$ and just in the planar case, it is the subject of the paper* [1]*.*

Remark 3.15. *The role of central configuration is not confided to the study of homothetic and homografic motions. Central configurations are also important for the study of collisions in the N-body problem. Homothetic motions are examples of total-collision orbits; at a certain moment, all of the bodies collide at the center of mass. Although there exist other nonhomothetic total-collision orbits and all such orbits approach central configurations at collision. More precisely, if the collapsing configuration is blown-up, say to have moment of inertia 1, then the rescaled configuration approaches the set of central configurations. For example, in the three-body problem, Siegel showed that every triple collision solution has asymptotic shape either an equilateral triangle or one of Euler's collinear central configuration.*

4. The restricted 3-body problem

In the restricted three-body problem, a body of negligible mass (the *planetoid*) moves under the influence of two massive bodies (the *primaries*) whose masses are m_1 and m_2. Having negligible mass, the force that the planetoid exerts on the primaries may be neglected, and the system can be analysed and therefore be described in terms of a 2-body motion. If m_1, m_2 are the masses of the primaries and $x_1(t)$, $x_2(t)$ their positions at time t we then have

$$m_1 x_1(t) = -m_2 x_2(t).$$

The restricted three-body problem is easier to analyze theoretically than the full problem. It is of practical interest as well since it accurately describes many real-world problems. For these reasons, it has occupied an important role in the historical development of the three-body problem. In the restricted 3-body problem we are then left to understand the motion of the planetoid under the gravitational influence of the two primaries, this means to study the differential equations given by the Newton's law

$$\ddot{x}_3 = \frac{m_1}{|x_1(t) - x_3|^3}(x_1(t) - x_3) + \frac{m_2}{|x_2(t) - x_3|^3}(x_2(t) - x_3), \tag{21}$$

where we have scaled $G = 1$. Let us remark that (21) is a non-autonomous second order ordinary differential equation.

A usual assumption is that the two-body motion consists of circular orbits around the center of mass, and the planetoid is assumed to move in the plane defined by the circular orbits. We will term this problem the *restricted circular planar 3-body problem* (RCP3BP). We furthermore make the not restrictive assumptions $m_1 + m_2 = 1$, or equivalently that

$$\text{there exists } \mu \in \left(0, \frac{1}{2}\right) \text{ such that } m_1 = 1 - \mu,\ m_2 = \mu.$$

With these assumptions and fixing the center of mass of the two primaries at the origin, we can write

$$x_1(t) = -\mu e^{it}, \qquad x_2(t) = (1-\mu)e^{it},$$

and Equation (21) reads

$$\ddot{x}_3 = \frac{1-\mu}{|-\mu e^{it} - x_3|^3}(-\mu e^{it} - x_3) + \frac{\mu}{|(1-\mu)e^{it} - x_3|^3}\left((1-\mu)e^{it} - x_3\right). \tag{22}$$

We now pass to the rotating system where the two primaries are fixed at

$$P_1 = (-\mu, 0), \qquad P_2 = (1-\mu, 0),$$

and

$$x_1(t) = R(t)\begin{pmatrix} -\mu \\ 0 \end{pmatrix}, \quad x_2(t) = R(t)\begin{pmatrix} 1-\mu \\ 0 \end{pmatrix}, \quad x_3(t) = R(t)z(t),$$

where

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

A stationary solution for the planetoid in the rotating frame corresponds to a circular motion in the inertial one.

The equation of motion of the planetoid in the rotating frame can be computed obtaining

$$\ddot{z} + 2K\dot{z} = \nabla\phi(z). \tag{23}$$

where $K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$\phi(z) = \frac{1}{2}|z|^2 + \frac{1-\mu}{|P_1 - z|} + \frac{\mu}{|P_2 - z|}.$$

The following quantity, the *Jacobi's integral*, is constant along solutions

$$J(t) = 2\phi(z(t)) - |\dot{z}(t)|^2. \tag{24}$$

indeed

$$\frac{d}{dt}J(t) = 2\langle \nabla\phi(z), \dot{z}\rangle - 2\langle \dot{z}, \ddot{z}\rangle = 4\langle \dot{z}, K\dot{z}\rangle = 0.$$

Furthermore, J allows to define the *Hill's region*

$$H_c = \left\{ z \in \mathbb{R}^2 : \phi(z(t)) \geq \frac{J(t)}{2} \right\}. \tag{25}$$

4.1. **Lagrangian equilibrium points.** Let us now determine equilibria for Eq.(23) imposing $\nabla\phi(z) = 0$[4], that is

$$z = \frac{1-\mu}{|z - P_1|^3}(z - P_1) + \frac{\mu}{|z - P_2|^3}(z - P_2). \tag{26}$$

Writing this equation in components we have

$$\begin{cases} x = \dfrac{1-\mu}{\rho_1^3}(x+\mu) + \dfrac{\mu}{\rho_2^3}(x+\mu-1), \\ y = \dfrac{1-\mu}{\rho_1^3}y + \dfrac{\mu}{\rho_2^3}y. \end{cases} \tag{27}$$

where $z = (x, y)$, $P_1 = (-\mu, 0)$, and $\rho_i = |z - P_i|, i = 1, 2$.

Let us remark that $y = 0$ solves the second equation; hence replacing in the first one we get $x = h(x)$, with

$$h(x) = \frac{1-\mu}{\rho_1^3(x)}(x+\mu) + \frac{\mu}{\rho_2^3(x)}(x+\mu-1), \tag{28}$$

---

[4] Indeed writing the Eq.(23) as a first order system

$$\dot{z} = w, \; \dot{w} = -2K\dot{z} + \nabla\phi(z)$$

its critical or stationary points satisfy $w = 0$ and $\nabla\phi(z) = 0$.

where, since $y = 0$,

$$\rho_1(x) = |x + \mu|, \;\; \rho_2(x) = |x + \mu - 1|.$$

From a qualitative study of function h (see Figure 12) we infer the existence of three solutions, for any value of the parameter μ: L_1, where the third body stays between the other two, L_2 ed L_3. This point are termed *collinear equilibria*, indeed the third body stay on the line generated by the two primaries.

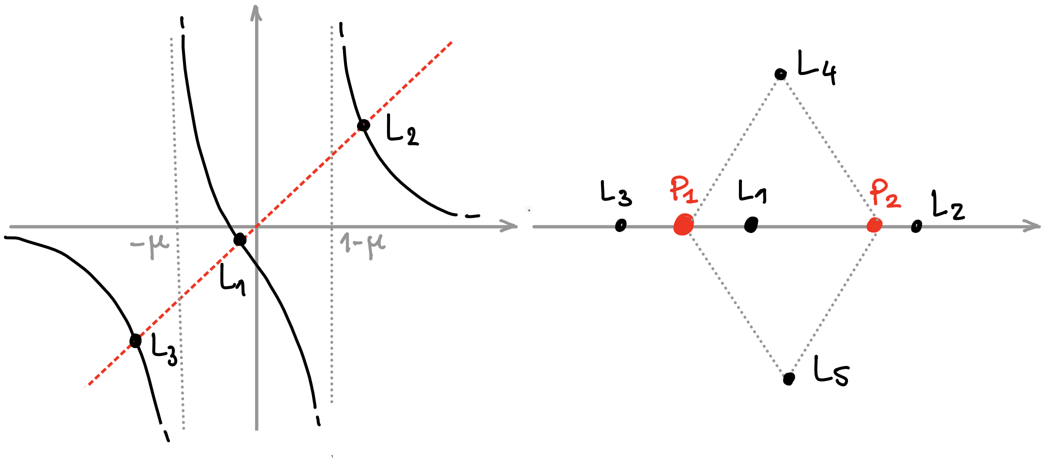

FIGURE 12. At the left side the qualitative graph of function h. At the right side, the 5 equilibria for the RCP3BP: in L_1, L_2 and L_3 the 3 body are in a collinear configuration. In L_4 and L_5 the 3 bodies stay on the vertices of a regular triangle.

Let us now assume that $y \neq 0$. From Eq. (26) we infer

$$\left(-\frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3}\right) z = -\frac{1-\mu}{\rho_1^3} P_1 - \frac{\mu}{\rho_2^3} P_2. \tag{29}$$

Since $y \neq 0$, this equation is equivalent to

$$\begin{cases} 1 - \left(\dfrac{1-\mu}{\rho_1^3} + \dfrac{\mu}{\rho_2^3}\right) = 0; \\ -\dfrac{1-\mu}{\rho_1^3}\mu - \dfrac{\mu}{\rho_2^3}(1-\mu) = 0. \end{cases} \tag{30}$$

From the second equation we obtain $\rho_1 = \rho_2 = \rho$; replacing in the first one we have

$$1 - \frac{1}{\rho^3} = 0 \iff \rho = 1. \tag{31}$$

We conclude that when $y \neq 0$ we obtain two new equilibria in which $|z - P_1| = |z - P_2| = 1$. Since $|P_1 - P_2| = 1$, in these equilibria, termed L_4 and L_5, the third body forms a regular triangle with the primaries. L_4 and L_5 are termed *triangular equilibria*.

We then conclude that Eq. (23) admits five equilium points (see Figure 12).

4.2. **Stability via linearization.** Let us now study the stability of the five Lagrangian points via linearization (see the appendix for a short review). We write Eq. (23) as a first order system

$$\begin{cases} \dot{z} = w, \\ \dot{w} = -2Kw + \nabla\phi(z), \end{cases}$$

we define $u = (z, w)$ and we linearize the system at L_j that is at $u_j = (L_j, 0)$, $j = 1, \ldots, 5$, obtaining

$$\dot{u} = A_j u \qquad \text{where} \quad A_j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_j & b_j & 0 & 2 \\ b_j & c_J & -2 & 0 \end{pmatrix}.$$

The 2×2 matrix dawn-left is the hessian of ϕ at u_j, $\phi''(L_j)$, that is

$$a_j = \phi_{xx}(u_j),\ b_j = \phi_{xy}(u_j),\ c_j = \phi_{yy}(u_j).$$

In order to the test applicability of Hartmann-Grobmann Theorem we compute the eigenvalues of A_j, which indeed solves

$$\lambda^4 - (a_j + c_j - 4)\,\lambda^2 + \left(a_j c_j - b_j^2\right) = 0. \tag{32}$$

When λ is a solution of the previous equation, also $-\lambda$, $\overline{\lambda}$[5] and $-\overline{\lambda}$ solve it; for this reason asymptotic stability is never reached. Furthermore, to avoid instability you need four eigenvalues with vanishing real part.

In order to determine solutions of (32), we compute the terms of the hessian of ϕ. Since $\nabla\phi(z) = z - (1-\mu)\frac{z-P_1}{|z-P_1|^3} - \mu\frac{z-P_2}{|z-P_2|^3}$, we obtain

$$\begin{aligned} \phi''(z) = Id_2 - (1-\mu)\frac{Id_2}{|z-P_1|^3} + 3\,(1-\mu)\frac{(z-P_1)\otimes(z-P_1)}{|z-P_1|^5} \\ -\mu\frac{Id_2}{|z-P_2|^3} + 3\mu\frac{(z-P_2)\otimes(z-P_2)}{|z-P_2|^5}, \end{aligned}$$

[5] $\bar{\lambda}$ is the complex-conjugate to λ.

where $w \otimes w = \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix}$. Since $z - P_1 = \begin{pmatrix} x+\mu \\ y \end{pmatrix}$ and $z - P_2 = \begin{pmatrix} x+\mu-1 \\ y \end{pmatrix}$, we have

$$\begin{aligned}
&\phi_{xx} = 1 - \frac{1-\mu}{\rho_1^3} + 3\left(1-\mu\right)\frac{\left(x+\mu\right)^2}{\rho_1^5} - \frac{\mu}{\rho_2^3} + 3\mu\frac{\left(x+\mu-1\right)^2}{\rho_2^5}, \\
&\phi_{xy} = \phi_{yx} = 3\left(1-\mu\right)\frac{\left(x+\mu\right)y}{\rho_2^5}, \\
&\phi_{yy} = 1 - \frac{1-\mu}{\rho_1^3} + 3\left(1-\mu\right)\frac{y^2}{\rho_1^5} - \frac{\mu}{\rho_2^3} + 3\mu\frac{y^2}{\rho_2^5}.
\end{aligned} \tag{33}$$

Collinear equilibria L_1, L_2 *and* L_3. In this case $y = 0$ and

$$\rho_1^2 = \left(x+\mu\right)^2 \qquad \rho_2^2 = \left(x+\mu-1\right)^2.$$

Replacing in (33) we get, for $j = 1, 2, 3$,

$$\begin{cases} a_j = 1 + 2\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) > 0; \\ b_j = 0; \\ c_j = 1 - \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right). \end{cases} \tag{34}$$

It turns out, by some smart geometric remarks, that $c_j < 0$, for $j = 1, 2, 3$, hence $a_j c_j < 0$ and the quanties

$$\lambda_\pm^2 = \frac{\left(a_j + c_j - 4\right) \pm \sqrt{\left(a_j + c_j - 4\right)^2 - 4a_j c_j}}{2}. \tag{35}$$

are real. Concerning collinear equilibria we can then conclude that

- since $\lambda_+^2 > 0$, then two eigenvalues are real with opposite sign
- since $\lambda_-^2 < 0$, then two eigenvalues are purely imaginary numbers.

From the first statement we infer that L_1, L_2 *and* L_3 *are unstable* for any choice of the parameter μ.

Collinear equilibria L_4 *and* L_5. In this case $\rho_1 = \rho_2 = 1$ and

$$x + \mu = x + \mu - 1 = \frac{1}{2}, \quad y = \frac{\sqrt{3}}{2}.$$

hence, replacing in (33) we obtain

$$a_j = \frac{3}{4}, \quad b_j = \frac{3\sqrt{2}}{4}\left(2\mu - 1\right), \quad c_j = \frac{9}{4}. \tag{36}$$

Equation (32) reads

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu\left(1-\mu\right) = 0,$$

hence

$$\lambda_\pm^2 = \frac{-1 \pm \sqrt{1-27\mu\,(1-\mu)}}{2}.$$

Studying the function

$$f(\mu) = 1 - 27\mu(1-\mu) \quad \text{on } \left[0, \frac{1}{2}\right]$$

it turns out that $f(\mu) > 0$ on $[0, \mu^*)$, while $f(\mu) < 0$ on $(\mu^*, 1/2]$, where $\mu^* \sim 0.0385$, hence we conclude that

- if $\mu \in [0, \mu^*]$, then $\lambda_\pm^2$ are real and negative. In this case the four eigenvalues are purely imaginary, hence the linearization method does not give any information about the stability of L_4 and L_5;
- if $\mu \in (\mu^*, 1/2]$, then $\lambda_\pm^2 \in \mathbb{C} \setminus \mathbb{R}$, hence the four eigenvalues are distinct complex numbers; two of them have positive real part and the other two have negative real part. We infer that in this case L_4 L_5 are unstable.

4.3. **Small oscillation near Lagrangian points: an application of the Lyapunov theorem.** In this paragraph we will investigate the presence of periodic solutions for (23) near a Lagrangian point when the linearized system admits pairs of purely complex eigenvalues. In order to do that we will use Lyaponov's center theorem (a reference for this part is [2]).

4.3.1. *Lyapunov center theorem.* Assume that a nonlinear system

$$\dot{u}(t) = f(u(t)), \quad f \in C^2(\mathbb{R}^n, \mathbb{R}^n), \tag{37}$$

admits $p = 0$ as a singular points, i.e. $f(0) = 0$. Assume now that the jacobian matrix

$$A = f'(0) = J_f(0)$$

has a pair of purely complex eigenvalues $\pm i\omega_0$, for some $\omega_0 \in \mathbb{R}$: this is a necessarely but not sufficient condition to have periodic solution near 0 in the nonlinear system (see the appendix). In order to make this condition sufficient we need some further auumptions: this is the content of Lyapunov theorem.

Assume that Eq.(37) admits a first integral[6] $b : \mathbb{R}^n \to \mathbb{R}$ at least of class C^2 and let us now merge the dynamical system into the one-parameter family

$$\dot{u}(t) = \psi(\mu, u(t)), \qquad \psi(\mu, u) = f(u) + \mu \nabla b(u), \quad \mu \in \mathbb{R}. \tag{38}$$

Assuming b of class at least C^2, we compute the jacobian matrix at $u = 0$

$$A_\mu = \frac{\partial}{\partial u}\psi(\mu, u)|_{u=0} = [f'(u) + \mu b''(u)]_{u=0} = f'(0) + \mu B = A + \mu B.$$

[6] A first integral for (37) is a non-constant function $b \in C^1(\mathbb{R}^n, \mathbb{R})$ such that if u is a solution for Eq.(37) defined on the interval $I \subset \mathbb{R}$, then there exists $c \in \mathbb{R}$ such that $b(u(t)) = c$, for every $t \in \mathbb{R}$.

Where $b''(u)$ is the hessian of b at u and $B = b''(0)$. Of course $A_0 = A = f'(0)$. Let now $\lambda(\mu) = (\lambda_1(\mu), ..., \lambda_n(\mu))$ be the n-uple of eigenvalues of A_μ with

$$\lambda(\mu) = \alpha(\mu) + i\beta(\mu) \ , \ \forall \mu \in \mathbb{R}, \tag{39}$$

Since ϕ is of class C^1 it turns out that α, β are both of class $C^1(\mathbb{R}, \mathbb{R}^n)$. We consider the following assumptions on the eigenvalues of A_μ

(a) $A = A_0$ is a non-singular matrix that admits a pair of simple and purely imaginary eigenvalues of the form $\pm i\omega_0$, with $\omega_0 \in \mathbb{R}$;
(b) $ik\omega_0$ is not an eigenvalue for A, $\forall k \in \mathbb{Z}$, $k \neq \pm 1$;
(c) $\alpha'(0) \neq 0$.

Under this assumptions we claim to prove the existence of a family of periodic solutions of (37) that approaches (in some sense) to the solution $u = 0$. More precisely, we will prove the presence of *small oscillations* near $u = 0$ in the following sense.

Definition 4.1. *System* (37) *admits* small oscillations *near* $u = 0$ *if there exist* $r > 0$, $\omega : (-r, r) \longrightarrow \mathbb{R}^+$ *of class* C^1 *and a family of functions* $(u_s)_s$, $s \in (-r, r)$, $u_s : \mathbb{R} \to \mathbb{R}^n$, *such that:*

- *for every* $s \in (-r, r)$, u_s *is a periodic and non-constant solution of* (37) *with period* $T_s = \frac{2\pi}{\omega(s)}$;
- $\omega(s) \longrightarrow \omega_0$ *if* $s \longrightarrow 0$ *(this means, since* ω *is continuous, to require that* $\omega(0) = \omega_0$*);*
- $\max_{t\in[0,T_s]} \|u_s(t)\| \longrightarrow 0$ *if* $s \longrightarrow 0$.

Here the statement of Lyapunov theorem.

Lyapunov's Center Theorem. *Consider the dynamical system* (37)*, assume that* $f(0) = 0$ *and assumptions (a) e (b) on the jacobian matrix* A*. Furthermore assume the existence of a first integral* $b \in C^2(\mathbb{R}^n, \mathbb{R})$ *such that* $b''(0)$ *is not singular. Then* (37) *admits small oscillations near* $u = 0$.

In order to prove Lyapunov theorem we need to preliminary Lemmata. The first one is a bifurcation result due to Hopf for which we will not provide a proof.

Lemma 4.2 (Hopf's Theorem)**.** *Consider the family of dynamical systems* (38)*, where* $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $b \in C^3(\mathbb{R}^n, \mathbb{R})$ *and* $\phi(\mu, 0) = 0$ *for every* $\mu \in \mathbb{R}$*. If assumptions (a),(b) and (c) on* $A = f'(0)$ *are satisfied, then:*
there exist $r > 0$ *and* $s_0 \in (-r, r)$*, two functions* $\omega : (-r, r) \longrightarrow \mathbb{R}$ *and* $\mu : (-r, r) \longrightarrow \mathbb{R}$ *both of class* $C^1(-r, r)$*, with* $\omega(s) > 0$ *for every* s*, and there exists a family of periodic functions* $(u_s)_s$ $s \in (-r, r)$*, where* $u_s : \mathbb{R} \longrightarrow \mathbb{R}^n$*, such that:*

i) *for every* s, u_s *solves* (38) *with* $\mu = \mu(s)$;
ii) $\mu(s) \longrightarrow 0$ *and* $\omega(s) \longrightarrow \omega_0$ *if* $s \longrightarrow s_0$;

iii) *the functions* u_s *have period* $T_s = \frac{2\pi}{\omega(s)}$;
iv) $\max_{t\in[0,T_s]} \|u_s(t)\| \longrightarrow 0$ *if* $s \longrightarrow s_0$.

The second lemma allows to determine periodic solutions for (37) as periodic solutions for (38) for some μ.

Lemma 4.3. *If u is a T-periodic solution of* (38)*, then u is a T-periodic solution for* (37)

Proof. If $\mu = 0$, then the result is trivial. Let $u = u(t)$ be a T-periodic solution of (38) for some $\mu \neq 0$. Let $\beta(t) = b(u(t))$, then

$$\dot\beta(t) = \frac{d}{dt}b(u(t)) = \nabla b(u(t))\cdot u'(t) = \nabla b(u(t))\cdot f(u(t)) + \mu \mid \nabla b(u(t)) \mid^2 . \tag{40}$$

For every $\xi \in \mathbb{R}^n$, since f is of class C^2, the Cauchy problem

$$\dot u(t) = f(u), \qquad u(0) = \xi$$

admits a unique solution, $u_\xi(t)$. Since b is a first integral we have

$$0 = \frac{d}{dt}(b(u_\xi(t))) = \nabla b(u_\xi(t))\cdot \dot u_\xi(t) = f(u_\xi(t))\cdot\nabla b(u_\xi(t)),$$

for every t in the domain of u_ξ. In particular if $t = 0$ we have

$$f(\xi)\cdot\nabla b(\xi) = 0. \tag{41}$$

Since the choice of ξ is arbitrary, Eq. (40) reads

$$\dot\beta(t) = \mu \mid \nabla b(u(t)) \mid^2,$$

hence the fuction β is monotone; for instance, if $\mu > 0$, then $\beta(t)$ is not-decreasing. Since u is T-periodic, then $\beta(0) = b(u(0)) = b(u(T)) = \beta(T)$. Hence, by virtue of the monotonicity of β, we deduce that β is constant and

$$\dot\beta(t) = \mu \mid \nabla b(u(t)) \mid^2 \equiv 0.$$

Since $\mu \neq 0$, we have $\nabla b(u(t)) = 0$ hence, from (38) we get that u solves (37). □

Proof of Lyapunov theorem. By means of Lemma 4.3, we can write the thesis as riscriviamo la tesi nel modo seguente:
there exists $r > 0$, two C^1 functions $\omega : (-r,r) to \mathbb{R}$ and $\mu : (-r,r) \to \mathbb{R}$, with $\omega(s) > 0$ for every s, and there exists a family of periodic functions $(u_s)_s$, $s \in (-r,r)$, where $u_s : \mathbb{R} \to \mathbb{R}^n$ such that

- for every s, u_s solves (38) with $\mu = \mu(s)$;
- $\mu(s) \to 0$, $\omega(s) \to \omega_0$ as $s \to 0$;
- for every s, u_s has period $T_s = \frac{2\pi}{\omega(s)}$;
- $\max_{t\in[0,T_s]} \|u_s(t)\| \to 0$ as $s \longrightarrow 0$.

We claim to apply Hopf's theorem with $s_0 = 0$. In order to do that we need to verify assumptions; first of all we remark that from Eq. (41) and assumption $b \in C^2$, we have

$$f'(\xi)y \cdot \nabla b(\xi) + f(\xi) \cdot b''(\xi)y = 0, \ \forall \xi, y \in \mathbb{R}^n. \tag{42}$$

When $\xi = 0$

$$Ay \cdot \nabla b(0) + f(0) \cdot b''(0)y = 0, \ \forall y \in \mathbb{R}^n;$$

Since $f(0) = 0$ and A is not singular we have $\nabla b(0) = 0$, hence

$$\psi(\mu, 0) = f(0) + \mu \nabla b(0) = 0.$$

We are then left to show assumption (c). Consider Eq. (42). Since $f \in C^1$, $f(0) = 0$ and b'' is linear, the map $\xi \to f(\xi) \cdot b''(\xi)y$ is differentiable at $\xi = 0$ and

$$\frac{d}{dt}\left[f(th) \cdot b''(th)y\right]_{t=0} = \left[(f'(th)h) \cdot (b''(th)y) + f(th) \cdot b''(h)y\right]_{t=0} = (Ah) \cdot (By).$$

We can then differentiate (42) in $\xi = 0$, obtaining

$$f''(0)[y, z] \cdot \nabla b(0) + Ay \cdot Bz + Az \cdot By = 0, \ \forall y, z \in \mathbb{R}^n.$$

Since $\nabla b(0) = 0$, we have

$$Ay \cdot Bz + Az \cdot By = 0, \quad \forall y, z \in \mathbb{R}^n,$$

hence, since B is symmetric,

$$A^T B + BA = 0. \tag{43}$$

By assumption (a) it is not restrictive to write

$$A = \begin{pmatrix} S & 0 \\ 0 & R \end{pmatrix}, \qquad \text{where } S = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$$

and R is such that $\pm i\omega_0$ do not belong to its spectrum (since $\pm i\omega_0$ are simple eigenvalues for A).

Since B is symmetric, there exist two symmetric matrices U and C, resp. 2×2 and $(n-2) \times (n-2)$, such that

$$B = \begin{pmatrix} U & M \\ M^T & C \end{pmatrix},$$

From (43) immediately follows that

$$SU = US \qquad SM = MR.$$

Since $\omega_0 \neq 0$ and $SU = US$, we deduce the existence of a real number δ such that $U = \delta I_2$. Let $X, Y \in \mathbb{R}^{n-2}$ be respectively the first and the second line of M From $SM = MR$ follows that X ed Y solve

$$\begin{cases} XR + \omega_0 Y = 0, \\ YR - \omega_0 X = 0. \end{cases}$$

From the second equation $X = (\omega_0)^{-1} Y R$ and replacing in the first we have

$$Y\left(R^2 + (\omega_0)^2 I\right) = 0.$$

Since $\pm i\omega_0$ are not eigenvalues for R, we deduce $X = Y = 0$; hence M is a null matrix and B can be written as.

$$B = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & C \end{pmatrix},$$

with $\delta \neq 0$ being B not singular. Hence

$$A + \mu B = \begin{pmatrix} \mu\delta & -\omega_0 & 0 \\ -\omega_0 & \mu\delta & 0 \\ 0 & 0 & R + \mu C \end{pmatrix}$$

and the eigenvalues $\lambda(\mu)$ corresponding to $\lambda(0) = i\omega_0$ are

$$\lambda(\mu) = \mu\delta + i\omega_0.$$

Hence $\alpha(\mu) = \mu\delta$ and $\alpha'(\mu_0) = \delta \neq 0$, which is assumption (c) $\square$

4.3.2. *Application to the RPC3BP.* We claim to apply Lyapunov Theorem to the RPC3BP in order to find periodic solution (in the rotating frame) near Lagrangian equilibrium points.

First of all we need a first integral; with this aim we can consider the Jacobi integral introduced in (24)

$$J(t) = 2\phi(z(t)) - |w(t)|^2$$

where z solves (23) and $w = \dot{z}$. Interpreting J as a function of z and w, its hessian at L_j $j = 1, \ldots, 5$ is

$$J''(L_j) = \begin{pmatrix} 2\phi''(L_j) & 0_2 \\ 0_2 & -I_2 \end{pmatrix}, \tag{44}$$

and it is not singular since

$$\det\left(J''(L_j)\right) = 4\det\left(\phi''(L_j)\right) \tag{45}$$

and by virtue of Eqs. (34) and (36).

In order to verify the assumptions on the the matrices A_j, $j = 1, \ldots, 5$, let us go back to Section 4.2 at page 42. When $j = 1, 2, 3$ the matrix A_j has exactly one pair one purely imaginary eigenvalues, hence Lyapunov theorem can be applied to the collinear Lagrangian points.

Theorem 4.4 (Small oscillations near L_1, L_2 and L_3)**.** *In a neighborhood of every collinear Lagrangia point L_1, L_2 and L_3, the RPC3BR admits a family of solution which are periodic in the rotating frame and their period tends to $\frac{2\pi}{\omega_j}$, where ω_j is the imaginary part of the (unique) pair of purely complex eigenvalue of the matrix A_j.*

Concerning points L_4 and L_5 we go back to the discussion at page 43. We proved that if $\mu \in (0, \mu^*)$ (with $\mu^* \approx 0.0385$ is the smaller soluton of $27\mu(1-\mu) = 1$) then the four eigenvalues are purely imaginary, we term them

$$\pm i\omega', \ \pm i\omega'' \qquad \text{with } 0 < \omega' < \omega''.$$

Indeed we obtained

$$\omega' = \frac{1}{4}\sqrt{1 - \sqrt{1 - 27\mu(1-\mu)}} \quad \text{and} \quad \omega'' = \frac{1}{4}\sqrt{1 + \sqrt{1 - 27\mu(1-\mu)}}$$

and in particular

$$(\omega'')^2 + (\omega')^2 = 1, \qquad (\omega'')^2(\omega')^2 = \frac{27}{4}\mu(1-\mu).$$

Hence if $\mu \in (0, \mu^*)$ we can apply Lyapunov theorem choosing $\omega_0 = \omega''$[7].

If we want to apply Lyapunov theorem to the frequence ω' we need to impose the further non-resonance condition

$$\omega'' \neq k\omega', \quad k \in \mathbb{N}, k \geq 2$$

that is equivalent to

$$\frac{27}{4}\mu(1-\mu) \neq \mu_k := \frac{k^2}{(1+k^2)^2}. \tag{46}$$

Observe that $\mu_k \to 0$ as $k \to \infty$ and

Theorem 4.5 (Small oscillations near L_4 and L_5)**.** *If $\mu \in (0, \mu^*)$, in a neighbourhood of both collinear Lagrangian point L_4 and L_5 the RPC3BR admits a family of solution which are periodic in the rotating frame and their period tends to $\frac{2\pi}{\omega''}$.*

Furthermore if $\mu \neq \mu_k$, then there exists a second family of solution which are periodic in the rotating frame and their period tends to $\frac{2\pi}{\omega'}$.

[7] When we choose as primaries Sun-Jupiter or Earth-Moon we have resp. $\mu \approx 1/1000$ and $\mu \approx 1/82 \approx 0.012$, both smaler then μ^*

Appendix A. Bessel functions

In this section we propose a brief overview on Bessel functions, for a more complete treatment we refer to the book [4, Chapter 3].

Let us consider the so called *Bessel equation*

$$x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) = 0, \quad \alpha \in \mathbb{R}.$$

The set of its solutions is isomorphic to $\mathbb{R}^2$ and we recall here the *Frobenius-Fuchs method* to determine them as a power series. Let us assume that

$$\alpha \in \mathbb{N}$$

and look for solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}, \quad \text{for some real sequence } (a_k)_k.$$

Since, at least formally,

$$y'(x) = \sum_{k=0}^{\infty}(k+\alpha)a_k x^{k+\alpha-1}, \qquad y''(x) = \sum_{k=0}^{\infty}(k+\alpha)(k+\alpha-1)a_k x^{k+\alpha-2},$$

replacing in the equation we obtain

$$\begin{aligned}
0 &= x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) \\
&= \sum_{k=0}^{\infty}(k+\alpha)(k+\alpha-1)a_k x^{k+\alpha} + \sum_{k=0}^{\infty}(k+\alpha)a_k x^{k+\alpha} \\
&\quad + \sum_{k=0}^{\infty} a_k x^{k+\alpha+2} - \alpha^2 \sum_{k=0}^{\infty} a_k x^{k+\alpha} \\
&= \sum_{k=0}^{\infty}[(k+\alpha)^2 - \alpha^2]a_k x^{k+\alpha} + \sum_{k=0}^{\infty} a_k x^{k+\alpha+2} \\
&= \sum_{k=0}^{\infty} k(k+2\alpha)a_k x^{k+\alpha} + \sum_{k=0}^{\infty} a_k x^{k+\alpha+2} \\
&= 0 \cdot a_0 + (1+2\alpha)a_1 + \sum_{k\geq 2}[(k^2+2\alpha k)a_k + a_{k-2}]x^{k+\alpha}
\end{aligned}$$

Imposing now that every coefficient vanish, we obtain $a_1 = 0$ (since α is a natural number) and, for every $k \geq 2$,

$$a_k = -\frac{a_{k-2}}{k(2\alpha+k)}, \qquad \forall k = 0, 1, 2, \dots.$$

We do not have any restriction on the choice of a_0. We then deduce that $a_k = 0$ when k is odd, and, for every $j \geq 1$

$$a_{2j} = -\frac{a_{2j-2}}{2^2 j(j+\alpha)} = (-1)^j \frac{a_0\, \alpha!}{2^{2j} j!(j+\alpha)!}.$$

Choosing now

$$a_0 = \frac{1}{2^\alpha\, \alpha!}$$

we have

$$a_{2j} = \frac{(-1)^j}{j!(j+\alpha)!} \frac{1}{2^{2j+\alpha}},$$

and we obtain the *Bessel function of order* $\alpha \in \mathbb{N}$

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+\alpha)!} \left(\frac{x}{2}\right)^{2j+\alpha}.$$

For the proof of the next result we refer to the book [4, Chapter 3.3, p.64].

Proposition A.1. *For every $n \in \mathbb{N}$, the following equality holds*

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(nu - x \sin u)\, du.$$

Appendix B. A very brief recap on linearization method

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be of class $C^1(\Omega)$ and consider the autonomous dynamical system

$$\dot{x} = f(x). \tag{47}$$

We will indicate with $x(t; x_0)$ the solution of the Cauchy problem associated to (47) with initial condition $x(0) = x_0$. The point $x^* \in \Omega$ is an *equilibrium point* is $f(x^*) = 0$. We can characterize an equilibrium point with respect to the behaviour of solutions of the dynamical system near it. We say that an equilibrium x^* is *stable* if a solution that starts close to it remain close to it when the time evolves, i.e.

$$\forall \epsilon > 0 \; \exists \delta > 0 \;\text{ such that}$$
$$x_0 \in \Omega \text{ and } |x^* - x_0| < \delta \implies |x(t; x_0) - x^*| < \epsilon, \forall t > 0.$$

An equilibrium x^* is *asymptotically stable* if it is stable and

$$\exists r > 0 \;\text{ such that }\; x_0 \in \Omega \text{ and } |x^* - x_0| < r \implies \lim_{t \to 0} x(t; x_0) = x^*.$$

An equilibrium x^* is *unstable* if it is not stable.

Let now x^* be an equilibrium point; since f is of class C^1 we can write

$$f(x^* + \eta) = J_f(x^*)\eta + o(\|\eta\|), \text{ as } \;\|\eta\| \to 0,$$

whre $J_f(x^*)$ the jacobian matrix of f at x^*. In order to study the behaviour the solutions near x^* we replacing $x = x^+ + \eta$ in (47) and obtain

$$\dot{\eta} = J_f(x^*)\eta + o(\|\eta\|) \text{ as } \;\|\eta\| \to 0.$$

Neglecting the term $o(\|\eta\|)$ we obtain the *linearized system*

$$\dot{\eta} = A\eta \quad \text{ with } A = J_f(x^*). \tag{48}$$

We can study stability of the origin for this linear system considering the eigenvalues of the matrix A:

- if there exists at least one eigenvalue with strictly positive real part, then the origin is unstable;
- if every eigenvalue of A has strictly negative real part, then the origin is asymptotically stable;
- if every eigenvalue has negative real part then the origin is stable.

The origin (as an equilibrium point of the linearized system (48)) is termed *hyperbolic* if every eigenvalue of the matrix A has non-vanishing real part. The next result state when stability property of the origin are inherited from x^* (equilibrium point for (47)).

Theorem B.1 (Hartman-Grobman Theorem)**.** *Assume that the origin is an hyprbolic equilibrium point for the linearized system* (48)*. If the origin is asymptotically stable/unstable then x^* is asymptotically stable/unstable as an equilibrium point for* (47)*.*

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