# Ph.D course

## "Infinite combinatorics, Banach spaces, and the first Baire class"

### Introductory lecture - tentative of a program

Let us first define a little bit what we have in mind w/ this title. The two macro-domains that are infinite combinatorics on one hand, and Banach space theory on the other, seem to have a non-empty intersection which connects with the quite specific domain of Baire class one functions.

Let us explain briefly where the intersection is, through a (very partial) presentation of both domains.

# Infinite combinatorics

We mainly mean by that Ramsey-style results. Let us state the Ramsey theorem.

We denote by $\mathbb{N}$ the set of natural numbers, $[\mathbb{N}]^k$ for $k \in \mathbb{N}$ is the set of $k$-tuple of natural numbers, we identify $N$ w/ $\{i \in \mathbb{N} \mid i < N\}$, and $[\mathbb{N}]^\infty$ is the set of infinite subsets of $\mathbb{N}$.

**Thm** (Ramsey) Let $X \in [\mathbb{N}]^\infty$, $k \in \mathbb{N} \setminus \{0\}$ and $N \in \mathbb{N}$. For any map $c : [X]^k \longrightarrow N$ there exists $Y \in [X]^\infty$ such that $c \restriction [Y]^k$ is constant.

The map $c$ is often called a coloring, and the set $Y$ given by the theorem is homogeneous.

Is there version of Ramsey's theorem for colorings of infinite subsets of $\mathbb{N}$? In general no, but if we only look at "definable" (ie for instance Borel) colorings, then yes (this is the Galvin-Prikry theorem). There is also a transfinite version of Ramsey's theorem (Nash-Williams's theorem).

Further results are Silver's analytic Ramsey theorem, Todorcevic's topological Ramsey space theory and Gowers' strategical Ramseyness.

# Banach spaces

A Banach space is a complete normed linear space (for us, on $\mathbb{R}$). Examples are for instance $\mathbb{R}^n$ w/ the euclidian norm (aka $\ell_2^n$), but as we are looking at the interplay between Banach space theory and infinite combinatorics, we will almost naturally look at more exotic examples.

## Infinite dimensional examples.

For $p \in [1, \infty)$, $\ell_p$ is the space of all $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$ such that $\sum |x_i|^p < \infty$ equipped w/

the norm $\|x\|_p = \left(\sum_{i\in\mathbb{N}} |x_i|^p\right)^{1/p}$

$\ell_\infty$ is the space of all bounded sequences in $\mathbb{R}^{\mathbb{N}}$ w/ the sup norm, $c$ is the subspace of those sequences that have a limit, and $c_0$ is the subspace of $c$ of sequences converging to $0$.

All these <u>except $\ell_\infty$</u> are separable (they have a countable dense subsequence).

We are specially interested in $\ell_1$ and $c_0$. Why? Because they are "more marginal" than the others. Let us explain briefly.

Schauder basis $(e_n)$ is a Schauder basis of $X$

Schauder basis $(e_n)_n$ is a Schauder basis of $X$

if for all $x \in X$ there is a unique sequence $(a_n)_n \in \mathbb{R}^{\mathbb{N}}$

such that $x = \sum_{n \in \mathbb{N}} a_n e_n$,

Schauder bases are very nice objects, unfortunately

Enflo proved that not all separable Banach spaces have one.

(they all embed in one that has such a basis though)

All our classical examples have the same Schauder basis.

Denote by $e_i^*$ the coordinate functional of $e_i$ in $X^*$

- $\{e_i, e_i^* \mid i \in \mathbb{N}\}$ is shrinking if $\overline{\operatorname{span}}\{e_i^* \mid i \in \mathbb{N}\} = X^*$
- it is boundedly complete if $\sum_{i \in \mathbb{N}} a_i e_i$ cvges

  whenever $(a_i)_i$ satisfy $\sup_n \| \sum_0^n a_i e_i \| < \infty$.

Shrinking $\Rightarrow$ $\{e_i^*\}$ is a Schauder basis of $X^*$.

The standard unit basis of $c_0$ and $\ell_p$ $(p>1)$ are shrinking, the one of $\ell_1$ is <u>not</u>.

The standard basis of $\ell_p$ $(p \geq 1)$ is boundedly complete the one of $c_0$ is <u>not</u>

<u>Thm</u> (James) $X$ Banach, Schauder basis $\{e_i\}$.

$X$ reflexive $\Longleftrightarrow$ $\{e_i\}$ shrinking and bddly complete.

We will therefore be interested in characterizations of spaces containing $\ell_1$ and $c_0$, by Rosenthal.

And specially because these enjoy proofs using infinite combinatorics!

Thm (Rosenthal) $(x_n)_n$ infinite normalized sequence in $X$ then $(x_n)$ has an $\infty$ subseq. that is weakly Cauchy or equiv. to the natural basis of $\ell_1$

(this uses G.P)

So what can we say without Schauder basis?
Say that $\{e_n\}$ in $X$ is a basic sequence if it is a Schauder basis of $\overline{\text{span}}\{e_n\}$.
Now every Banach space contains a basic sequence (this was already known to Banach).
An elaboration on this is the notion of spreading model of a basic sequence.
The Brunel-Sucheston theorem (an application of the original Ramsey theorem) guarantees the existence of a spreading model. Rosenthal's theorem then becomes

Thm (Rosenthal) $(x_n)_n$ weakly null seq. in $X$ then either $(x_n)_n$ has a subseq. all of whose subseq are Cesàro-summable,

or $(x_n)_n$ has subseq w/ spreading model $\cong \ell_1$

Then

Thm (Odell-Rosenthal) X separable Banach space

1. $c_0 \hookrightarrow X$ iff $X^{**}_{\mathcal{D}} \setminus X \neq \emptyset$
2. $\ell_1 \hookrightarrow X$ iff $X^{**} \setminus X^{**}_{\mathcal{B}_1} \neq \emptyset$

## To summarize what is to come

1. 2nd lecture: game-theoretical proof of the Galvin-Prikry thm.
2. 3rd/4th Lectures: first $\ell_1$-thm

   1 will be done by R. Camoy, 2 by Salvatore Scamperti.

Then there should be: the Brunel-Sucheston thm from the Ramsey thm (1 lecture), maybe Rosenthal's second $\ell_1$ thm (1/2 lectures), and the Odell/Rosenthal result (at least 2 lectures, maybe can be divided between 2 people). Then:

## Further possible directions

- Stronger characterizations of spaces containing $c_0$
- Further descriptive set theory, ex. The class REFL is $\Pi^1_1$
- Gowers' dichotomy
- Further infinite combinatorics (ultrafilter, P-ideals) and how it connects to Banach space theory.
- Topological Ramsey spaces, towards Todorcevic's works on Rosenthal compacta and its link to Banach spaces.
- ---- suggestions ?

Observe that any of these further directions is already, more than enough material for the entire course!

## References

Reference books with a accent on infinite-dimensional Banach spaces

① "Functional analysis and infinite-dimensional geometry" by Fabian, Habala, Hájek, Santalucía, Pelant, and Zizler.

② "Classical Banach spaces I: Sequence spaces" by Lindenstrauss and Tzafriri.

More specialized books for the course.

① "Ramsey methods in analysis" by Argyros and Todorcevic

② "Banach spaces and Descriptive Set Theory: Selected topics" by Pandelis Dodos.

And a few articles.

In the "Handbook of the geometry of Banach spaces, vol. 2":

- Chapter 23 "Descriptive set theory and Banach spaces" by Argyros, Godefroy, and Rosenthal.
- Chapter 24 "Ramsey methods in Banach spaces" by Gowers.
- "Analytic P-ideals and Banach spaces" by Borodulin-Nadzieja and Farkas, in the journal of functional analysis.