This is due to Kastanas and Tanaka.

The game Given $W \subseteq 2^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}$

$$\begin{array}{llllll} \text{I} : & d_0, A_0 & & d_1, A_1 & & \cdots \\ \text{II} : & & n_0, B_0 & & n_1, B_1 & \end{array} \quad \Bigg| \quad \begin{array}{l} \underline{d} = (d_i)_{i \in \mathbb{N}} \\ X = (n_i)_{i \in \mathbb{N}} \end{array}$$

w/ $d_i \in \{0,1\}$, $n_i \in \mathbb{N}$, A_i and $B_i \in [\mathbb{N}]^{\mathbb{N}}$ for all i.

The Rule

for all $i \in \mathbb{N}$ $n_i < B_i$, $\{n_i\} \cup B_i \subseteq A_i$ and $A_{i+1} \subseteq B_i$

The Winning condition:

I wins iff $(\underline{d}, X) \in W$.

Call P the projection of W on $[\mathbb{N}]^{\mathbb{N}}$.

Thm (Kastanas - Tanaka)

① I has a w.s. $\Rightarrow \exists H \in [\mathbb{N}]^{\mathbb{N}} \; \forall X \in [H]^{\mathbb{N}} \; X \in P$.

② II has a w.s. $\Rightarrow \forall A \in [\mathbb{N}]^{\mathbb{N}} \; \exists H \in [A]^{\mathbb{N}} \; \forall X \in [H]^{\mathbb{N}} \; X \notin P$

Proof: We denote p partial plays, ie finite sequences $((d_0, A_0), (n_0, B_0), (d_1, A_1), \dots)$ finishing by either a move of I or a move of II. For such a partial run we say that p generates $(d_i)_{i \leq k_p}$ and $(n_i)_{i \leq l_p}$ (for appropriate k_p and l_p)

A partial run is consistent with a strategy for one of the players if said player followed the strategy at his turn.

Observe that: if σ is a w.s. for I, p a partial play ending by II's round, and $\sigma(p) = (d, A)$, then I can play (d, A') for any $A' \in [A]^{\mathbb{N}}$ and still win following σ.

(1) Let σ be a w.s. for I. We build, inductively on i, a sequence of finite trees T_i, and $X_i \in [\mathbb{N}]^{\mathbb{N}}$, ST:

- All nodes of T_i are partial plays consistent w/ σ.
- $T_i \subseteq T_{i+1}$
- Setting $n_i = \min X_i$, we have
  All subsets of $\{n_j \mid j < i\}$ are realized in T_i
- All leaves of T_i end with (d, A) for some $A \supseteq X_i$.

Then the set $H = \{n_i \mid i \in \mathbb{N}\}$ is as desired.

Name (d_0, A_0) the first move of I following σ, and set $T_0 = \{(d_0, A_0)\}$ and $X_0 = A_0$.

Suppose T_i and X_i are built, and enumerate the nodes of T_i : $(p_i)_{i<k}$. Define $(d_j)_{j<k}$ and $(Y_j)_{j\leq k}$ by:

$Y_0 = X_i / n_i$, $(d_j, Y_{j+1}) = \sigma(p_j \frown (n_i, Y_j))$

and set $T_{i+1} = T_i \cup \{p_j \frown (n_i, Y_j) \frown (d_j, Y_{j+1}) \mid j < k\}$ and $X_{i+1} = Y_k$

(b) Let τ be a w.s. for II. Build finite trees T_i, $n_i \in \mathbb{N}$, and $C_i \in [\mathbb{N}]^{\mathbb{N}}$ ST

- $T_i \subseteq T_{i+1}$
- The branches of T_i are partial runs consistent w/ τ.
- $n_i < C_{i+1}$ and $n_i \in C_i$
- $\forall s \subseteq \{n_j \mid j < i\}$ $\forall d \in 2^{|s|}$ (d, s) is realized by a branch of T_i.
- Any branch of T_i ends by (n, B) for some $B \supseteq C_i$.

Then $H = \{n_i \mid i \in \mathbb{N}\}$ is as desired.

<u>Lemma</u> (Kastanas) Let $C \in [\mathbb{N}]^{\mathbb{N}}$. For any partial play p ending by (n, B) w/ $B \supseteq C$, there is $A \in [C]^{\mathbb{N}}$ ST $\forall m \in A$ $\forall d < 2$ there are X and Y ST

$$\tau(p \frown (d, X)) = (m, Y) \text{ and } Y \supseteq A \setminus \{m\}$$

Pf: Define (m_i, Y_i) as follows:

$(m_0, Y_0) = z(p^\frown(0, C))$ and $(m_{i+1}, Y_{i+1}) = z(p^\frown(0, Y_i))$

Set then $Y_\infty = \{ m_i \mid i \in \mathbb{N} \}$ and define (m_i', Y_i'):

$(m_0', Y_0') = z(p^\frown(1, Y_\infty))$ and $(m_{i+1}', Y_{i+1}') = z(p^\frown(1, Y_i'))$.

The set $A = \{ m_i' \mid i \in \mathbb{N} \}$ is as desired. ⊠

$T_0 = \{\emptyset\}$ $\quad C_0 = \mathbb{N}$

Enumerate the nodes of T_i: $(p_0 — p_{k-1})$

Apply the lemma to p_0 and C_i to get A_0

p_j — A_{j-1} — A_j

$n_i = \min A_{k-1}$ and $C_{i+1} = A_{k-1} \setminus \{n_i\}$

Lemma ⟹ there are X_j, Y_j, X_j' and Y_j' $(j < k)$ supersets of C_{i+1} st

$z(p_j{}^\frown(0, X_j)) = (n_i, Y_j)$ and

$$z(p_j{}^\frown(1, X_j')) = (n_i, Y_j')$$

$T_{i+1} = T_i \cup \{ p_j{}^\frown(0, X_j)^\frown(n_i, Y_j),\ p_j{}^\frown(1, X_j')^\frown(n_i, Y_j') \mid j < k \}$

[illegible]

□