Def Let E be a Banach space with basis $(e_n)_{n\in\omega}$. Given $x\in E$ with $x=\sum_{n=0}^{\infty} x_n e_n$ we define

$$\|x\|_0 = \sup_{N>0} \left\| \sum_{i=0}^{N} x_i e_i \right\|$$

Obs As $\left(\sum_{i=0}^{N} x_i e_i\right)_{N\in\omega}$ is convergent, it is also bounded. Hence $\|\cdot\|_0$ is well-defined and it is a norm on E.

<u>Theorem 1</u> Let E be a Banach space with a basis $(e_n)_{n\in\omega}$. Then $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent norms on E.

<u>Theorem 2</u> Let E be a Banach space and let $(e_n)_{n\in\omega}$ be a sequence in E. Then $(e_n)_{n\in\omega}$ is a basis for E iff

1) each e_n is non-zero
2) $\overline{\operatorname{span}(e_n)} = E$
3) $\exists K>0$ such that $\forall (x_n)_{n\in\omega}$ sequence of scalars and each $N<M$ we have
$$\left\| \sum_{n=0}^{N} x_n e_n \right\| \le K \left\| \sum_{n=0}^{M} x_n e_n \right\|$$

proof of $\Rightarrow$

Suppose $(e_n)_{n\in\omega}$ is a basis. By theorem 1 there exists K such that
$$\|x\| \le \|x\|_0 \le K\|x\| \qquad \forall x\in E$$
We have 1) by the uniqueness of the expansion of $0=\sum_{n=1}^{\infty} 0 e_n$. Condition 2) is clear.
For condition 3) let $(y_n)_{n\in\omega}$ be a sequence of scalars defined by $y_n = x_n$ for $n\le M$ and $y_n=0$ otherwise. Then we have

$$\left\|\sum_{n=0}^{N} x_n e_n\right\| = \left\|\sum_{n=0}^{N} y_n e_n\right\| \le \left\|\sum_{n=0}^{\infty} y_n e_n\right\|_0 \le K\left\|\sum_{n=0}^{\infty} y_n e_n\right\| = K\left\|\sum_{n=0}^{M} x_n e_n\right\|$$

$\boxtimes$

Scansionato con CamScanner

Def The smallest constant which satisfies 3) is the basis constant of the basis $(e_n)_{n\in\omega}$.

Obs by theorem 1 we can always renorm E to give a basis of constant 1.

Lemma 3 Let E be an infinite-dimensional Banach space, let F be a finite-dimensional subspace of E, and let $\varepsilon>0$. Then there exists $x\in E$ s.t. $\|x\|=1$ and

$$\|y\| \leq (1+\varepsilon)\|y+ax\|$$

for all $y\in F$ and all scalars a.

Theorem 4 Every Banach space E contains a basic sequence.

proof We use induction to pick a sequence of norm-one vectors $(x_n)_{n\in\omega}$ such that condition 3) of theorem 2 always holds, with $K=2$ say (the proof works for any $K>1$).

Suppose we have chosen $x_0 \dots x_n$ and $\varepsilon>0$ such that

$$\Big\|\sum_{k=0}^{m} a_k x_k\Big\| \leq (2-\varepsilon)\Big\|\sum_{k=0}^{n} a_k x_k\Big\|$$

for any $m\leq n$ and any scalars $(a_k)_{k=0}^{n}$. Note that we can clearly do this for $n=1$. We now try to find x_{n+1}. We need to ensure that $\|x_{n+1}\|=1$ and that for some $\varepsilon_0>0$ we have that

$$\Big\|\sum_{k=0}^{m} a_k x_k\Big\| \leq (2-\varepsilon_0)\Big\|\sum_{k=0}^{n+1} a_k x_k\Big\|$$

for any $m\leq n$ and any scalars $(a_k)_{k=0}^{n+1}$.

Let F_n be the linear span of $x_0 \dots x_n$, a finite-dimensional subspace of E. Use the Lemma 3 to find x_{n+1} s.t. $\|y\| \leq (1+\delta)\|y+a_{n+1}x_{n+1}\|$ for each $y\in F_n$ and each scalar a_{n+1}, where $\delta>0$ is chosen so that $(2-\varepsilon)(1+\delta)=2-\frac{\varepsilon}{2}$, i.e. $\delta=\frac{\varepsilon}{2(2-\varepsilon)}$.

Then, for a sequence of scalars $(a_k)_{k=0}^{n+1}$ let $y = \sum_{k=0}^{n} a_k x_k \in F_n$ so that for $m \le n$ we see that

$$\|\sum_{k=0}^{m} a_k x_k\| \le (2-\varepsilon)\|\sum_{k=0}^{n} a_k x_k\| = (2-\varepsilon)\|y\| \le$$

$$\le (2-\varepsilon)(1+\delta)\|y + a_{n+1}x_{n+1}\| = (2-\varepsilon/2)\|\sum_{k=0}^{n+1} a_k x_k\|$$

$\square$

Def A Banach space X is said to be finitely representable in a Banach space Y if for every finite dimensional subspace E of X and every $\varepsilon > 0$ there is a finite dimensional subspace F of Y s.t. $d(E,F) \le 1+\varepsilon$

i.e.

there exists an isomorphism $T: E \to F$ s.t.

$$(1+\varepsilon)^{-1}\|x\| \le \|Tx\| \le (1+\varepsilon)\|x\| \qquad \forall x \in E$$

Prop A Banach space X is finitely representable in Y if and only if there is a subspace Z of some ultraproduct of Y so that for every $\varepsilon > 0$ there is an isomorphism $T: X \to Z$ s.t.

$$(1+\varepsilon)^{-1}\|x\| \le \|Tx\| \le (1+\varepsilon)\|x\|$$

Def Let $(x_n)_{n\in\mathbb{N}}$ be a basic sequence in a Banach space X. A spreading model of (x_n) is a basic sequence $(y_n)_{n\in\mathbb{N}}$ (not necessarily in X) ~~[illegible]~~ such that for every $\varepsilon > 0$ and $K \in \mathbb{N}$ there is an N such that

$$(1+\varepsilon)^{-1}\|\sum_{i=1}^{K} a_i y_i\| \le \|\sum_{i=1}^{K} a_i x_{n_i}\| \le (1+\varepsilon)\|\sum_{i=1}^{K} a_i y_i\|$$

for all $N < n_1 < n_2 \dots < n_K$ and sequences of scalars $(a_i)_{i=1}^{K}$.

Obs Saying that the basic sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ has a spreading model $(y_n)_{n\in\mathbb{N}}$ in Y is stronger than saying that $\overline{\text{span}(y_n)}$ is finitely represented in X.

Ramsey's theorem Let k and r be positive integers. Then for every r-coloring of $[\mathbb{N}]^k$ there exists $X\in[\mathbb{N}]^\infty$ s.t. $[X]^k$ is monochromatic.

Corollary 6 Let K be a totally bounded metric space. For every $\varepsilon>0$, any positive integer k and any function $F:[\mathbb{N}]^k \to K$ there is an $X\in[\mathbb{N}]^\infty$ s.t. $d(F(A),F(B))<\varepsilon$ for every $A,B\in[X]^k$.

proof Since K is totally bounded we may cover it with sets $B_1 \dots B_r$ each of diameter less than ε.
Given $A\in[\mathbb{N}]^k$ let's define $g(A)$ to be the least j s.t. $F(A)\in B_j$.
We then just apply Ramsey's theorem. ☒

(Brunel-Sucheston) Theorem 7
Let $(x_n)_{n\in\mathbb{N}}$ be a normalized basic sequence in a Banach space X, let $\varepsilon>0$ and let k be a positive integer. Then there is an infinite subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that, given any sequence $(a_i)_{i=1}^k$ of scalars, and any pair of sequences $m_1<m_2<\dots<m_k$ and $n_1<n_2<\dots<n_k$ we have

$$\left\|\sum_{i=1}^k a_i y_{m_i}\right\| \le (1+\varepsilon)\left\|\sum_{i=1}^k a_i y_{n_i}\right\|$$

proof Let the basis constant of (x_n) be C. If $n_1<n_2<\dots<n_k$ then we have

$$(2C)^{-1}\max|a_i| \le \left\|\sum_{i=1}^k a_i x_{n_i}\right\| \le k\max|a_i|$$

as $|a_i| = \|P_i(x)-P_{i-1}(x)\| \le 2C\|x\|$

as (x_n) is normalized

Let F_0 be the set of all norms $\|\cdot\|$ on ℓ_∞^K such that

$$(*) \qquad (2C)^{-1}\|x\|_\infty \le \|x\| \le K\|x\|_\infty \quad \text{for every } x\in \ell_\infty^K$$

Let K be the unit sphere on ℓ_∞^K and let F be the set of restrictions of norms in F_0 to K.

<u>claim</u>: F is a totally bounded subset of $C(K)$ wrt the uniform metric.

- F is a subset of $C(K)$, as given $(z_n)_{n\in\mathbb{N}} \subseteq K$ s.t. (z_n) is convergent in ℓ_K^∞ to z then (z_n) is also convergent in $\|\cdot\|$ for every $\|\cdot\|$ in F, hence $(\|z_n\|)$ is convergent to $\|z\|$ $\Rightarrow \|\cdot\| : (K, d_{\ell_K}) \to \mathbb{R}_+$ is continuous
- Moreover for any $\|\cdot\|$ in F, and any $x\in K$ we have
$$\|x\| \in [(2C)^{-1}, K] \quad \text{by } (*)$$
- As K is totally bounded (compact) take $(z_i)_{i=1}^r \subset K$ to be a $C^{-1}\varepsilon$-net of K, then note that for any $z\in K$, if we take an z_i s.t. $\|z-z_i\|_\infty < K^{-1}\varepsilon$ we have
$$\big|\|z\|-\|z_i\|\big| \le \|z-z_i\| \le K\|z-z_i\|_\infty < \varepsilon$$
for any $\|\cdot\|$ in F.

As $[(2C)^{-1},K]$ is also totally bounded we have that the claim holds □

Now given a sequence of integers $n_1<n_2<\dots<n_K$, let A be the set $\{n_1 \dots n_K\}$ and let $F(A)$ to be the norm $\|\cdot\|_A$ on ℓ_∞^K in F defined by

$$\|(a_1 \dots a_K)\|_A = \Big\|\sum_{i=1}^K a_i x_{n_i}\Big\|$$

By corollary 6 there exists $Z\in[\mathbb{N}]^\infty$ with $Z=\{n_1,n_2,\dots\}$ such that for any $A,B\in[Z]^K$, the distance $d(\|\cdot\|_A,\|\cdot\|_B)$ in $C(K)$ is at most ε/K

Since there is no loss of generality in assuming $(a_1\dots a_K)$ in the statement of the theorem to belong to the unit sphere of ℓ_∞^K and since for any such sequence and any norm $\|\cdot\|$ in F we have $\|(a_1 \ldots a_K)\| \le C^K$, ~~then we have that.~~

~~for any $n_1 < \ldots n_K$, $m_1 < \ldots m_K$ and any $(a_i)_{i=1}^K$.~~

$\cancel{\|\sum^K}$

then if we enumerate $(x_{n_1}, x_{n_2} \ldots)$ as $(y_1, y_2 \ldots)$ and consider $n_1 < \cdots n_K$, $m_1 < \cdots < m_K$ and $(a_i)_{i=1}^K$

$$\left| \left\|\sum_{i=1}^K a_i y_{n_i}\right\| - \left\|\sum_{i=1}^K a_i y_{m_i}\right\| \right| < \varepsilon (2C)^{-1}$$

$$\Rightarrow \left\|\sum_{i=1}^K a_i y_{n_i}\right\| \le \left\|\sum_{i=1}^K a_i y_{m_i}\right\| + \varepsilon (2C)^{-1} \qquad \text{as } \|\cdot\| \geq (2C)$$

$$\le (1+\varepsilon) \left\|\sum_{i=1}^K a_i y_{m_i}\right\| \qquad \boxtimes$$

Using a further diagonalization, Brunel and Sucheston obtained a stronger result

Theorem Let $(x_n)_{n \in \mathbb{N}}$ be a normalized basic sequence. Then there is a subsequence $(y_1, y_2 \ldots)$ of $(x_1 \ldots)$ with the following property. Given any positive integer K, any sequence $(a_1 \ldots a_K)$ of scalars an any sequence $A_1, A_2 \ldots$ of sets in $[\mathbb{N}]^K$ such that $\min A_i$ tends to infinity as $i \to \infty$, the sequence $\left(c_i = \left\|\sum_{j=1}^K a_j y_{n_{ij}}\right\|\right)$ converges, where $A_i = \{n_{i1}, \ldots, n_{iK}\}$ with $n_{i1} < \cdots < n_{iK}$

proof

By the previous theorem we can choose subsequences $S_i = (x_{i1}, x_{i2}, \ldots)$ of the sequence $S_0 = (x_1, x_2 \ldots)$ with the following properties. First, S_i is a subsequence of S_{i-1} for every $i \geq 1$ and satisfies the conclusion of the previous theorem with $K = i$ and $\varepsilon = i^{-1}$.

$$
\begin{array}{cccccccccc}
S_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\
\cup & & & & & & & & & \\
S_1 & & x_{11} & & x_{12} & x_{13} & & x_{14} & x_{15} & x_{16} \\
\cup & & & & & & & & & \\
S_2 & & x_{21} & & & x_{22} & & x_{23} & & x_{24} \\
\cup & & & & & & & & & \\
S_3 & & & & & x_{31} & & x_{32} & & x_{33} \\
\vdots & & & & \vdots & & & & &
\end{array}
$$

Let S be the diagonal subsequence $(x_{11}, x_{22}, x_{33}, \dots)$.
Now consider $A_1, A_2, \dots$ sequence of elements of $[N]^k$ such that $\min A_i \xrightarrow[i\to\infty]{} \infty$.

<u>claim:</u> $(c_i)_{i\geq 1}$ is a Cauchy sequence.

<u>proof</u>

Fix an $\varepsilon>0$. Take a positive integer i s.t. $\forall p\geq i$ we have $\min A_p > \max\{k, \frac{\varepsilon}{k\max|a_i|}\}$,
such integer always exists
as $\min A_i$ ~~[illegible]~~ goes to infinity
by hypothesis.

Now consider ~~[illegible]~~ $l>p\geq i$

$$|c_l - c_p| = \left| \left\|\sum_{j=1}^{k} a_j x_{n_{l_j} n_{l_j}}\right\| - \left\|\sum_{j=1}^{k} a_j x_{n_{p_j} n_{p_j}}\right\| \right|$$

$$\leq \frac{\varepsilon}{k\max|a_i|} \left\|\sum_{j=1}^{k} a_j x_{n_{p_j}}\right\| \leq \varepsilon$$

$\square$

a) $(x_{n_{p_j} n_{p_j}})_{j=1}^{k}$ and $(x_{n_{l_j} n_{l_j}})_{j=1}^{k}$ are subsequences of $S_{\max\{k, \frac{\varepsilon}{k\max|a_i|}\}}$

Consider the set c_{00} of sequences of scalars being eventually constantly 0, and let's define a norm on it as follows

$$\|(a_1 \dots a_k)\| = \lim_{i\to\infty} c_i$$

where the sequence (c_i) is defined as in the previous theorem.

Obs Arguing as in the previous proof we can see that the limit of the (c_i) do <u>not</u> depend on the choice of the A_i as long as $\lim\limits_{i\to\infty} \min A_i = +\infty$.

Hence the norm is well-defined on c_{00}.

Let Z be the completion of c_{00} under this limiting norm.

claim: the canonical basis $(e_n)_n$ of c_0 is a basis of Z

proof: we just need to check condition 3) of theorem 2.

Take $(a_i)_{i\in\mathbb{N}}$ be a sequence of scalars and take C to be the basis constant of S (of the prev. theorem), and $N<M$.

Then we have

$$\Big\|\sum_{n=1}^{N} a_n e_n\Big\|_Z = \|(a_1 \dots a_N)\|_Z =$$

for some seq. A_i s.t. $\min A_i \to +\infty$

$$= \lim_{i\to\infty}\Big\|\sum_{j=1}^{N} a_j y_{n_{i_j}}\Big\|$$

now can extend each $A_i \subseteq A_i' \in [\mathbb{N}]^{M}$ s.t. $\min A_i = \min A_i'$ and we have

$$\Big\|\sum_{j=1}^{N} a_j y_{n_{i_j}}\Big\| \le C\Big\|\sum_{j=1}^{M} a_j y_{n'_{i_j}}\Big\|$$

$$\Rightarrow \Big\|\sum_{n=1}^{N} a_n e_n\Big\|_Z = \lim_{i\to\infty}\Big\|\sum_{j=1}^{N} a_j y_{n_{i_j}}\Big\| \quad \text{which exists!}$$

limit does not depend on A_i

$$\le \lim_{i\to\infty} C\Big\|\sum_{j=1}^{M} a_j y_{n'_{i_j}}\Big\| = \Big\|\sum_{n=1}^{M} a_n e_n\Big\|_Z \qquad \boxtimes$$

$\boxed{\text{2 obs}}$ → note does not dep. e.

Obs Note that by definition of the limiting norm we have

$$\Big\|\sum_{i=1}^{K} a_i e_{m_i}\Big\|_Z \le \Big\|\sum_{i=1}^{K} a_i e_{n_i}\Big\|_Z$$

whenever $K\in\mathbb{N}$ and $n_1<\dots<n_K$ and $m_1<\dots<m_K$.

→ the canonical basis of Z is said to be 1-spreading.

Moreover note that any space which is finitely represented in Z (wrt its basis) it is finitely represented in $\overline{\operatorname{span}(x_n)}$.

⇒ when considering questions about finite representability we can often confine ourselves to spaces with a 1-spreading basis.

<u>claim</u>: Let $(y_i)_{i\in\mathbb{N}}$ be the (sub) sequence of the previous theorem. Then the canonical basis $(e_n)_n \subset Z$ is a spreading model of (y_i).
Therefore every basic sequence has a subsequence with a spreading model.

<u>proof</u>

without loss of generality we can restrict ourselves with sequences of scalars having modulus at most 1 (unit sphere of c_{00} $\|(a_1 \dots a_K)\|_{c_{00}} = 1$)

Note that the rate of convergence of the c_i's in the previous theorem do <u>not</u> depend on the sequence of scalars, only on K and $\min A_i$.

I.E. If we fix $K\in\mathbb{N}$ and $\varepsilon>0$, then we have that $\exists N$ s.t.

$$\left|\, \Big\|\sum_{i=1}^{K} a_i y_{n_i}\Big\|_X - \Big\|\sum_{i=1}^{K} a_i e_i\Big\|_Z \right| < \varepsilon \qquad \Big\|\sum_{i=1}^{K} a_i e_i\Big\|_Z = \lim_{i\to\infty} \Big\|\sum_{j=1}^{K} a_j y_{n_{i_j}}\Big\|$$

for all $N<n_1<n_2<\dots<n_K$ and sequences of scalars $(a_i)_{i=1}^{K}$.

$\square$

Take C, as before, to be the basis constant of S, then

~~⇒ If we fix ε>0 and K∈N then we know that ∃N s.t.~~
~~$\left|\|\sum a_i y_{n_i}\|_X - \|\sum_{i=1}^{K} a_i e_i\|_Z\right| <$~~

$$\Big\|\sum_{i=1}^{K} a_i e_i\Big\|_Z \geq (2C)^{-1}\max|a_i| = (2C)^{-1}$$

⇒ if we take $\varepsilon = \varepsilon'(2C)^{-1}$ we have

$$\left\|\sum_{i=1}^{k} a_i y_{n_i}\right\|_X \leq \left\|\sum_{i=1}^{k} a_i e_i\right\|_Z + \varepsilon'(RC)^{-1}$$

$$\leq (1+\varepsilon')\left\|\sum_{i=1}^{k} a_i e_i\right\|_Z$$

the other inequality holds similarly $\square$