Infinite combinatorics, Banach spaces, and the first Baire class

After the first central part of the course

We started on a kind of slogan: reflexive spaces form a very interesting class, we would like to have conditions for a space X to be in that class.

We started with a theorem of James

Thm (James) A Banach space X with a Schauder basis $\{e_i \mid i \in \mathbb{N}\}$ is reflexive iff that base is both shrinking and boundedly complete.

We mentioned that the natural basis on c_0 is not bddly complete, and the one of ℓ_1 is not shrinking. That is a motivation for studying the spaces containing an isomorphic copy of c_0 and ℓ_1.

It turns out that the question can be made even stronger!

Is it the case that a non-reflexive space contains a copy of c_0 or ℓ_1?

James gave a positive answer, in a specific case.

<u>Thm</u> (James) Let X be a Banach space with an unconditional Schauder basis. Then X is reflexive, or X contains a copy of c_0 or ℓ_1

Recall that a basis is unconditional iff for all x the expansion $\sum a_i e_i$ of x converges unconditionally, iff there is K constant ST for all $(\varepsilon_i)_{i\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$

$$\|\sum \varepsilon_i a_i e_i\| \leq K \|\sum a_i e_i\| .$$

An orthonormal basis is unconditional.
The summing basis (cf Bonino's presentation) is not.
$L_1[0,1]$ does not have an unconditional basis.

So, maybe we can have a complete and simple exampli-
-cation of Banach spaces that have an unconditional
basis? Maybe they all have to contain a copy of
c_0 or a space of the form ℓ_p, for $1 \leq p < \infty$?

No!

<u>Thm</u> (Tsirelson) There is a space with an unconditional Schauder basis that does not contain neither c_0 nor any of the spaces ℓ_p for $1 \leq p < \infty$.

Let me pause for a side note here. Here's a problem when one investigates (closed) subspaces of a Banach space X. Suppose a subspace Y of X has a basis. How do this basis can relate to the basis of X, if it can relate at all?

<u>Thm</u> (Pełczyński) Let X be a Banach space w/ a Sch. basis $\{e_i\}_i$. If Y is an ∞-dim. closed subspace of X, <u>then</u> Y contains an infinite-dim. closed subspace Z w/ a Schauder basis equivalent to a block sequence of $\{e_i\}$.

These techniques are key to prove Tsirelson's theorem, because we know quite well how a basis of c_0 or ℓ_p $(1 \leq p < \infty)$ should behave.

Now, note that by James's result on spaces w/ an unconditional basis, Tsirelson's space is reflexive. So maybe we can recuperate something? Maybe a space is either reflexive or it contains a copy of c_0 or ℓ_1? No.

__Thm__ (James) There is a (sep.) Banach space that is not reflexive and does not contain an isomorphic copy of c_0 or ℓ_1

But James's space contains a reflexive space (ℓ_2), so...

Last try: maybe any Banach space either contains a reflexive space, or a copy of c_0 or ℓ_1? No.

__Thm__ (Gowers) There is a Banach space that does not contain any reflexive space, and does not contain a copy of c_0 or ℓ_1 either.

Let's weaken the question. If a space contains an isomorphic copy of c_0 or ℓ_p $(1 \leq p < \infty)$, does it contain an almost isometric copy of it? or more precisely, given an equivalent norm $\|\cdot\|_0$ of a space $(X, \|\cdot\|)$, does there exist a subspace Y of X st
$$d\big((Y, \|\cdot\|_0), (X, \|\cdot\|)\big) < 1 + \varepsilon?$$
A space that does <u>not</u> satisfy this ppty is called distortable. The problem is the distortion problem.

$\boxed{\text{\underline{Thm} (James, once again) } c_0 \text{ and } \ell_1 \text{ are not distortable}}$

<u>Thm</u> (Odell-Schlumprecht) A sep. ∞-dim. Hilbert space is distortable. Moreover, any Banach space that does not contain c_0 or ℓ_1 contains a distortable subspace.

So, let's weaken even this notion. We say that X is finitely representable in Y if every finite-dim subspace is finitely representable in Y if every finite-dim subspace of X has an ε-almost-isometric copy in Y (for all ε).

The notion of spreading model is a stronger version of this.

<u>Thm</u> (Krivine) One of the spaces c_0 or ℓ_p ($p \geq 1$) is finitely representable in <u>any</u> ∞-dim. Banach space.