CESARO-SUMMABILITY IN BANACH SPACES

$(x_n)_n$ sequence in X Banach space.

$(x_n)_n$ is Cesàro-summable if $\left(\frac{x_1+\dots+x_n}{n}\right)_n$ converges in norm.

__Rmk__: If $(x_n)_n \subseteq \mathbb{R}$ (or X Banach) is convergent to $x \in \mathbb{R}$ (or X) then $(x_n)_n$ is Cesaro-summable to the same limit.

__THEOREM__ (Rosenthal): let $(x_n)_n$ be a weakly null sequence in some Banach space X. Then either

(1) $(x_n)_n$ has a subsequence $(x_{n_k})_k$ all of whose subsequences are Cesaro-summable

or (2) $(x_n)_n$ has a subsequence $(x_{n_k})_k$ with spreading model $\cong \ell_1$.

__Recall__: $(x_n)_n$ weakly null if $f(x_n) \xrightarrow{n} 0 \quad \forall f \in X^*$.

__THEOREM__ (ERDÖS-MAGIDOR): let $(x_n)_n$ be a bounded sequence in some Banach space X. Then there is a subsequence $(x_{n_k})_k$ such that either:

(a) Every subsequence of $(x_{n_k})_k$ is Cesaro-summable and all of them have the same limit.

or (b) No subsequence of $(x_{n_k})_k$ is Cesaro-summable

__Proof__: $\mathcal{X} := \{ M \in [\mathbb{N}]^\omega : (x_n)_{n\in M}$ is Cesaro-summ$\}$ is $\underset{\sim}{\Pi}^0_4([\mathbb{N}]^\omega)$.

$\mathcal{X}$ Borel in $[\mathbb{N}]^\omega$ $\underset{\text{Galvin-Prikry theorem}}{\Longrightarrow}$ $\exists M\in[\mathbb{N}]^\omega$ st. $[M]^\omega\subseteq\mathcal{X}$ (i)

or

$[M]^\omega\cap\mathcal{X}=\emptyset$ (ii)

If (ii) holds it means that the subsequence $(x_n)_{n\in M}$ has no cesaro-summable subsequences, so we get (b).

Assume (i), then we know that any subsequence of $(x_n)_{n\in M}$ is cesaro-summable. (to an element in $\overline{\text{span}\{x_n : n\in M\}}$)

We consider $Y := \overline{\text{span}\{x_n : n\in M\}}$. Since it's separable then it has a countable basis (wrt norm topology):

fix $(B_k)_{k<\omega}$ be a basis of open balls.

let $\mathcal{X}_k=\{P\in[M]^\omega : (x_n)_{n\in P}$ is cesaro-summ to a point in $B_k\}$ $\forall k<\omega$

$\forall k$ $\mathcal{X}_k$ is Borel in $[M]^\omega$ $\underset{\text{G-P Theorem}}{\Longrightarrow}$

$\exists M_1\in[M]^\omega$ st. $[M_1]^\omega\subseteq\mathcal{X}_1$ or $[M_1]^\omega\cap\mathcal{X}_1=\emptyset$.

[now, since $\mathcal{X}_2$ Borel in $[M]^\omega$ and $[M_1]^\omega\subseteq[M]^\omega\Rightarrow$ $\mathcal{X}_2$ Borel in $[M_1]^\omega$ and we can apply G-P again!]

$\exists M_2\in[M_1]^\omega$ st $[M_2]^\omega\subseteq\mathcal{X}_2$ or $[M_2]^\omega\cap\mathcal{X}_2=\emptyset$

We get $M\supseteq M_1\supseteq M_2\supseteq\cdots M_k\supseteq\cdots$

st. $\forall k$ $\underbrace{[M_k]^\omega\subseteq\mathcal{X}_k}_{\text{any subseq. of } (x_n)_{n\in M_k} \text{ C-S in } B_k}$ or $\underbrace{[M_k]^\omega\cap\mathcal{X}_k=\emptyset}_{\text{any subseq of } (x_n)_{n\in M_k} \text{ C-S outside } B_k}$

We define $m_k:=\min(M_k)$ $\forall k$ and we may assume the sequence is increasing: let $N=\{m_k : k\in\omega\}$

(Note that if $\ell,k\in N$, then $k\in M_\ell$ for all $\ell<k$)

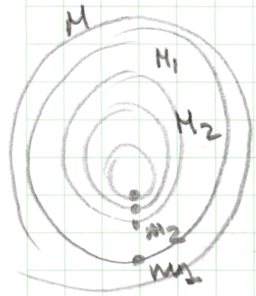
Now, given $\underset{(p_i)}{\overset{\parallel}{P}}\subseteq N$ infinite,

Consider $y_i^P=\dfrac{x_{p_1}+\dots+x_{p_i}}{i}$

- $\forall k$, either $(y_i^P)_i$ is convergent to a point in B_k or outside B_k. (trivial)
- $\forall k$ $\forall P, Q \subseteq \mathbb{N}$ infinite, either both converges to a point in B_k or outside B_k. Indeed:

  $\forall i \in P$, $i \geq k$ then $i \in M_k$

  $\forall j \in Q$, $j \geq k$ then $j \in M_k$



since $[M_k]^\omega \cap \mathcal{X}_k = \phi$ or $[M_k]^\omega \subseteq \mathcal{X}_k$ hold $\forall k$

$\Downarrow$

$(y_i^P)_{i \geq k}$ and $(y_j^Q)_{j \geq k}$ both conv. in B_k or outside

But then (y_i^P) and (y_i^Q) both conv. in B_k or outside B_k because they differ only for a finite (k-many) number of terms and this doesn't change the limit.

So we conclude: $(x_n)_{n \in \mathbb{N}}$ is the subsequence that satisfies (a).

$\blacksquare$

<u>DEF</u>: A Banach space X has the Banach-Sacks property if every bounded sequence $(x_n)_n$ in X has a Cesaro-summable subsequence.

<u>Ex</u>: $L_p[0,1]$ for $1 < p < \infty$

<u>COROLLARY</u>: X Banach space with Banach-Sacks property. Then, every bounded sequence $(x_n)_n$ in X has a subsequence $(x_{n_k})_k$ such that every subsequence of $(x_{n_k})_k$ is Cesaro-sum.

<u>Proof</u>: $(x_n)_n$ is bounded, so any subsequence (x_{n_k}) is still bounded and by B-S property (x_{n_k}) has a cesar-sum subsequence. Applying the theorem, since we cannot be in option (b), we must be in (a). $\blacksquare$

Recall: $B_{X^*} = \{ f \in X^* : \|f\| \leq 1 \}$.

Canonical embedding (isometry)

$$\Pi : X \longrightarrow X^{**}$$
$$x \longmapsto \Pi_x : X^* \longrightarrow \mathbb{R}$$
$$f \longmapsto \Pi_x(f) := f(x)$$

$$\|x\|_X = \|\Pi_x\|_{X^{**}} = \sup_{f \in B_{X^*}} |\Pi_x(f)| = \sup_{f \in B_{X^*}} |f(x)|$$

Given $(x_n)_n \subseteq X$ Banach, we think of x_n as a continuous function on B_{X^*} with the weak* topology.

LEMMA: Suppose $(x_n)_n$ weakly basic subsequence in some X Banach space st. no subsequence of $(x_n)_n$ is Cesaro-summable. Then $(x_n)_n$ has a subsequence $(x_{n_k})_k$ with spreading model isomorphic to ℓ_1.

Dim: By hp, there is no subsequence $(x_{n_k})_k$ s.t. $\frac{x_{n_1} + \dots + x_{n_k}}{k}$ converges uniformly on B_{X^*}.

Given $M \in [\mathbb{N}]^\omega$, $\varepsilon \in \mathbb{R}$, $x^* \in B_{X^*}$, we define:

$M(x^* \geq \varepsilon) = \{ m \in M : x^*(x_m) \geq \varepsilon \}$

$\mathcal{F}_{\geq \varepsilon}(M) = \{ M(x^* \geq \varepsilon) : x^* \in B_{X^*} \}$

$\mathcal{X}_{\geq \varepsilon} = \{ M \in [\mathbb{N}]^\omega : \mathcal{F}_{\geq \varepsilon}(M)$ contains sets of arbitrary large (finite) size $\}$

Similarly we define: $M(x^* \leq \varepsilon)$, $\mathcal{F}_{\leq \varepsilon}(M)$ and $\mathcal{X}_{\leq \varepsilon}(M)$.

Rmk: Assuming X is separable, by Keller's theorem B_{X^*} is homeomorphic to the Hilbert cube $[0,1]^{\mathbb{N}}$, so B_{X^*} is a polish space.

Now, $\mathcal{X}_{\geq\varepsilon}$ is analytic in $[\mathbb{N}]^\omega$: given $M\in[\mathbb{N}]^\omega$,

$$M\in\mathcal{X}_{\geq\varepsilon} \leftrightarrow \forall k \ \exists x^*\in B_{X^*} \ |M(x^*\geq\varepsilon)|\geq k$$
$$\leftrightarrow \forall k \ \exists x^*\in B_{X^*} \ (M,x^*)\in A_{k,\varepsilon}$$

where $A_{k,\varepsilon} := \{(M,x^*)\in[\mathbb{N}]^\omega\times B_{X^*} :$
$$\exists m_1 \dots \exists m_k \ (x^*(m_1)\geq\varepsilon \wedge \dots \wedge x^*(m_k)\geq\varepsilon)\}.$$

So, $\mathcal{X}_{\geq\varepsilon} = \bigcap_{k\in\omega} p_1(A_{k,\varepsilon})$ where p_1 = projection on the 1^{st} component

Since $A_{k,\varepsilon}$ is Borel and analytic sets are closed under countable intersections we get that $\mathcal{X}_{\geq\varepsilon}$ is analytic. Clearly $\mathcal{X}_{\leq\varepsilon}$ is analytic too.

CLAIM: There exists $\varepsilon>0$ s.t. either $\mathcal{X}_{\geq\varepsilon}$ or $\mathcal{X}_{\leq-\varepsilon}$ contains a set of the form $[M]^\omega$ for some $M\subseteq\mathbb{N}$.

Proof: Assume towards a contradiction that $\forall\varepsilon>0$ no set of such form is contained in $\mathcal{X}_{\geq\varepsilon}\cup\mathcal{X}_{\leq-\varepsilon}$.

Since $\mathcal{X}_{\geq\varepsilon}\cup\mathcal{X}_{\leq-\varepsilon}$ analytic $\forall\varepsilon$, we apply Silver's theorem to $\mathcal{X}_{\geq 1}\cup\mathcal{X}_{\leq -1}$ and we find $M_1\in[\mathbb{N}]^\omega$ s.t. $[M_1]^\omega\subseteq \cancel{\mathcal{X}_{\geq 1}\cup\mathcal{X}_{\leq -1}}$ or } by ass.!
$$[M_1]^\omega\cap(\mathcal{X}_{\geq 1}\cap\mathcal{X}_{\leq -1})=\emptyset$$
iterating Silver's theorem $\forall k$, we get a decreasing sequence of infinite subsets of $\mathbb{N}$
$$M_1\supseteq M_2\supseteq \dots \supseteq M_k\supseteq \dots \quad \text{s.t.}$$
$$[M_k]^\omega\cap\left(\mathcal{X}_{\geq\frac{1}{k}}\cup\mathcal{X}_{\leq-\frac{1}{k}}\right)=\emptyset.$$
Let M be infinite s.t. $\ell\in M_k$ $\forall k>\ell$, $k,\ell\in M$.
By assumption, $[M]^\omega\cap(\mathcal{X}_{\geq\varepsilon}\cup\mathcal{X}_{\leq-\varepsilon})=\emptyset$ $\forall\varepsilon$
neither $\mathcal{F}_{\geq\varepsilon}(M)$ nor $\mathcal{F}_{\leq-\varepsilon}(M)$ contains arbitrarily large sets, hence there's a bound in the size of

the sets of the form $M(x^* \geq \varepsilon)$ or $M(x^* \leq \varepsilon)$ for all $x^* \in B_{X^*}$. More precisely: $\forall \varepsilon > 0 \; \exists\, n(\varepsilon)$ s.t.

$$|M(x^*, \varepsilon)| \leq n(\varepsilon)$$

where $M(x^*, \varepsilon) := \{ m \in M : |x^*(x_m)| \geq \varepsilon \}$

$$= M(x^* \geq \varepsilon) \sqcup M(x^* \leq \varepsilon)$$

Let $(m_k)_k$ be increasing enumeration of M.

Then

$$(y_k^M)_k = \left(\frac{x_{m_1} + \dots + x_{m_k}}{k} \right)_k$$

converges uniformly on B^*, because B^* is compact w.r.t. the weak* topology (Banach-Alaoglu theorem) and $(y_k)^M$ converges pointwise on B_{X^*}:

$\forall x^* \in B_{X^*} \; \forall \varepsilon > 0 \; \exists m_{\varepsilon, x^*} \; \forall k \geq m_{\varepsilon, x^*} \; (k \in M \rightarrow |x^*(x_{m_k})| < \varepsilon)$

and since $|x^*(y_k)| = \frac{1}{k} |x^*(x_{m_1}) + \dots + x^*(x_{m_k})|$

$$\leq \frac{1}{k} |x^*(x_{m_1})| + \dots + |x^*(x_{m_k})|$$

we are done. But this contradicts our hp. $\boxed{\equiv}$

Fix $\varepsilon > 0$ s.t. $\mathcal{X}_{\geq \varepsilon}$ or $\mathcal{X}_{\leq -\varepsilon}$ contains a set as in the claim. W.l.o.g., assume it is contained in $\mathcal{X}_{\geq \varepsilon}$ and that for every infinite $M \in [\mathbb{N}]^\omega$, $\mathcal{Y}_{\geq \varepsilon}(M)$ contains sets of arbitrary large cardinality.

FACT: let $\mathcal{F}$ be a hereditary family of (finite) subsets of $\mathbb{N}$. Assume that every infinite subset of $\mathbb{N}$ contains members of $\mathcal{F}$ of arbitrary large finite cardinality. Then for any $M \in [\mathbb{N}]^\omega$ and for any $k>0$ there exists $N \in [M]^\omega$ s.t. $[N]^k \subseteq \mathcal{F}$.

Proof: Define a partition $[M]^k = P_0 \cup P_1$ where

$$P_0 = \{A \subseteq M : |A| = k, A \in \mathcal{F}\} \text{ and } P_1 = [M]^k \setminus P_0 .$$

By Ramsey theorem $\exists N \subseteq M$ infinite P_0-homogeneous subset. By hp, N contains a member of $\mathcal{F}$ of cardinality k

$\Longrightarrow [N]^k \subseteq P_0 \Longrightarrow [N]^k \subseteq \mathcal{F}$. ∎

So, $\forall k>0 \ \forall M \in [\mathbb{N}]^\omega \ \exists N \in [M]^\omega$ st. $[N]^k \subseteq \mathcal{F}_{\geq \varepsilon}(N)$.

Iterating this argument and diagonalizing, we take an infinite set $M \subseteq \mathbb{N}$ st.

$$\mathcal{F}_{\geq\varepsilon}(M) = \{E \subseteq M : |E| < |\{m \in M : m < \min(E)\}|\}$$

$$\shortparallel$$

$$\mathcal{S}(M) \quad \text{(Schreier space on } M\text{)}$$

For $E \in \mathcal{S}(M)$, then we can pick $x^* \in B_{X^*}$ st. $E \subseteq B(x^* \geq \varepsilon)$.

Given any sequence $(a_n)_{n\in E}$ of non negative scalars then :

$$\Big\|\sum_{n\in E} a_n x_n\Big\| \geq \sup_{x^*\in B_{X^*}} \Big|x^*\Big(\sum_{n\in E} a_n x_n\Big)\Big| \geq \Big|x^*\Big(\sum_{n\in E} a_n x_n\Big)\Big| \overset{\text{linearity}}{=}$$

$$= \Big|\sum_{n\in E} a_n x^*(x_n)\Big| \geq \varepsilon \Big|\sum_{n\in E} a_n\Big| = \varepsilon \sum_{n\in E} a_n . \qquad (*)$$

$\forall n\in E$ $x^*(x_n)\geq\varepsilon$ because $E\subseteq M(x^*\geq\varepsilon)$

Let (M_k) increasing enumeration of M. We need the following theorem:

<u>THEOREM</u> (Brunel-Sucheston): Every normalized weakly null basic sequence in some Banach space X has a subsequence with a spreading unconditional model. (Ramsey methods in analysis p. 133)

<u>DEF</u>: A sequence $(e_i)\subseteq X$ is an unconditional basic sequence if there is $C<\infty$ st. for all finite sequences $(a_i)_{i=1}^k$ of scalars and $(\varepsilon_i)_{i=1}^k \in \{\pm 1\}^k$

$$\Big\|\sum_{i=1}^k \varepsilon_i a_i e_i\Big\| \leq C\Big\|\sum_{i=1}^k a_i e_i\Big\| .$$

Applying Brunel-Sucheston theorem to $(x_{m_k})_k$, we find a subsequence $(x_{m_{k_i}})_i$ with an unconditional spreading model $(e_i)_i$.

Since $(e_i)_i$ is a spreading model of $(x_{m_{k_i}})_i$, for every $k>0$ $\exists N$ for every choice of scalars $(a_i)_{i=1}^k$

$$(1+\varepsilon)^{-1}\Big\|\sum_{i=1}^k a_i e_i\Big\| \leq \Big\|\sum_{i=1}^k a_i x_{m_{k_{j_i}}}\Big\| \leq (1+\varepsilon)\Big\|\sum_{i=1}^k a_i e_i\Big\|$$

$\forall j_1,\dots,j_k$ s.t. $N<j_1<\dots<j_k$.

So, for every choice of <u>positive</u> scalars $(a_i)_{i=1}^k$, by (*) we get

$$\varepsilon\sum_{i=1}^k |a_i| \leq \Big\|\sum_{i=1}^k a_i x_{m_{k_{j_i}}}\Big\| \leq \Big\|\sum_{i=1}^k a_i e_i\Big\|(1+\varepsilon) .$$

Hence $\dfrac{\varepsilon}{\varepsilon+1}\displaystyle\sum_{i=1}^k |a_i| \leq \Big\|\sum_{i=1}^k a_i e_i\Big\|$ $\forall k$ $\forall (a_i)_{i=1}^k$ positive scalars.

Since $(e_i)_i$ is unconditional, there's $C < \infty$ s.t. for every sequence of scalars $(a_i)_{i=1}^k$

$$\left\| \sum_{i=1}^k a_i e_i \right\| \leq C \left\| \sum_{i=1}^k a_i e_i \right\| ,$$

therefore we conclude that $\forall k \ \forall (a_i)_{i=1}^k$

$$\frac{\varepsilon}{C(\varepsilon+1)} \sum_{i=1}^k |a_i| \leq \left\| \sum_{i=1}^k a_i e_i \right\| \leq \sum_{i=1}^k |a_i| \, \|e_i\| \leq \sum_{i=1}^k |a_i|$$

As we proved that (e_i) is equivalent to the unit vector basis of ℓ_1, we are done. ∎

PROOF (ROSENTHAL'S THEOREM):

Let $(x_n)_n$ be a weakly null sequence. WLOG we can assume that $(x_n)_n$ is a normalized basic sequence in X, indeed the following lemma holds:

Lemma Every normalized weakly null sequence in some Banach space X has a subsequence $(x_{n_k})_k$ that is a Schauder basis of its closed linear span.

Proof ("Ramsey methods in analysis", Argyros - Todorcevic, p. 169).

$(x_n)_n$ is weakly null, so in particular it's bounded.

If the space X has the Banach-Saks property, (1) follows from the Corollary of Erdös-Magidor theorem.

Otherwise, if (x_n) has no Cesaro-summable subsequences, (2) is given by the lemma.