LECTURE ON c_0- CHARACTERIZATION THEOREM

Preliminary Results and Notions

Rem: K compact metric space. $\mathcal{B}_1(K) = \{ f : K \to \mathbb{R}, \mathbb{C} : f$ of the first Baire class$\}$.

$(\mathcal{B}_1(K), \|\cdot\|_\infty)$ is a Banach algebra

Def: $f : X \to Y \subseteq \mathbb{R}$ is of the first Baire class or BAIRE-1 if:

$$\{ x : f(x) \in G \} \text{ is } \Sigma_2^0 \quad \forall G \text{ open }, G \subseteq Y$$

i.e. f is a pointwise limit of continuous functions.

Def: $f : K \to \mathbb{C}$ is called a D-function (complex difference of bounded semicontinuous functions) if there exist continuous functions $\varphi_1, \varphi_2, \ldots$ on K s.t.

$$(*) \quad \sup_{x \in K} \sum_{j=1}^{\infty} |\varphi_j(x)| < \infty \quad \text{and} \quad f = \sum_{j=1}^{\infty} \varphi_j \text{ pointwise}$$

Set:

$$\|f\|_D = \inf \left\{ \sup_{x \in K} \sum_{j=1}^{\infty} |\varphi_j(x)| : (\varphi_j)_j \text{ is a sequence in } C(K) \text{ satisfying } (*) \right\}$$

Denote $D(K) = \{ f : K \to \mathbb{C} : f \text{ D-function} \}$

Prop 1: $f \in D(K)$ iff $\exists\, u_1, u_2, u_3, u_4$ bounded lower semi-continuous functions on K s.t. $f = (u_1 - u_2) + i(u_3 - u_4)$

Note: • In the sequel $K = B_{X^*}$ in its weak * topology, where B_{X^*} is the unit ball in X^*

• $X^{**}_{\mathcal{B}_1} = \{x^{**} \in X^{**} : x^{**}|_K \in \mathcal{B}_1(K)\}$

• $X^{**}_D = \{x^{**} \in X^{**} : x^{**}|_K \in D(K)\}$

• $a \in X^{**}_{\mathcal{B}_1}$ (resp. X^{**}_D) is called Baire-1 element (resp. D-element) of X^{**}.

The main result we are interested in is the following Theorem, due to Odell and Rosenthal:

$\boxed{\text{THEOREM (Odell - Rosenthal)}}$: Let X be a seperable Banach space.

(a) $c_0 \hookrightarrow X \iff X^{**}_D \setminus X \neq \phi$

(b) $\ell^1 \hookrightarrow X \iff X^{**} \setminus X^{**}_{\mathcal{B}_1} \neq \phi$

We are interested in proving (a).

$\boxed{\text{Theorem 1}}$: Let X be a seperable Banach space and $K = B_{X^*}$. Let $x^{**} \in X^{**}$.

(a) $x^{**} \in X^{**}_{\mathcal{B}_1} \iff$ there is a weak-Cauchy sequence $(x_j)_j \subset X$, with $x_j \xrightarrow{w^*} x^{**}$. Moreover one can then choose (x_j) with $\|x_j\| \leq \|x^{**}\|$ for all j

(b) $x^{**} \in X^{**}_D \iff$ there exists a sequence $(x_j) \subset X$ s.t. $\sum_{j=1}^{\infty} x_j$ is weakly unconditionally summing (ie. $\sum_{j=1}^{\infty} |x^*(x_j)| < \infty \;\; \forall x^* \in X^*$)

and $\sum_{j=1}^{n} x_j \xrightarrow{w^*} x^{**}$. Moreover, if $x^{**} \notin X$, then given $\varepsilon > 0$ one can choose $(x_j)_j$ so that:

(i) $(x_j)_j$ is equivalent to the c_0-basis;

(ii) $\sup_{x^* \in B_{X^*}} \sum_{j=1}^{\infty} |x^*(x_j)| < \|x^{**}|_K\|_D + \varepsilon$

Rem: (c) of O-R thm immediately follows by (b) of thm. 1.

Def: $(b_j)_j$ and $(e_j)_j$ sequences in a linear space. $(e_j)_j$ is called DIFFERENCE SEQUENCE of $(b_j)_j$ if

- $e_1 = b_1$
- $e_j = b_j - b_{j-1}$ for all j

Def: Denote by SER the Banach space of all convergent series, i.e. $(c_j)_j$ with $\sum_{j=1}^{\infty} c_j$ convergent, under the norm $\|(c_j)_j\|_{SER} = \sup_n |\sum_{j=1}^{n} c_j|$.

The SUMMING BASIS refers to the unit vector basis for SER, i.e. the sequence $(b_j)_j$ s.t. $\boxed{b_j(i) = \delta_{ij}}$, $\forall i, j$.

Rem: SER $\cong c_0$ now. Indeed, if $(e_j)_j$ = standard unit vector basis for c_0, then setting $b_j = \sum_{i=1}^{j} e_i \Rightarrow (b_j)_j$ is equivalent to the summing basis, indeed:

$$\|\sum_{j=1}^{n} c_j b_j\| = \sup_k |\sum_{j=k}^{n} c_j| \quad \text{for all } n$$

so $(b_j)_j$ is 2-equivalent to the summing basis and $(b_j)_j$ is a basis for c_0.

Def: $(b_j)_j$ a given sequence in a Banach space.

(a): $(b_j)_j$ is called NON-TRIVIAL WEAK-CAUCHY if $(b_j)_j$ is weak-Cauchy and non-weakly convergent.

(b): $(b_j)_j$ is called an (s)-sequence if $(b_j)_j$ is a weak Cauchy basic sequence so that $\sum_{j=1}^{\infty} c_j b_j < \infty \Rightarrow \sum_{j=1}^{\infty} c_j < \infty$

(*) "s" stands for "summing"

Rem: An (s)-sequence is NTWC (Non-Trivial Weak-Cauchy), hence its closed linear span cannot be weakly sequentially complete.

Prop 2: Let $(x_j)_j$ be a NTWC sequence in a Banach space. Then $(x_j)_j$ has an (s)-subsequence.

$\boxed{\text{Proof}}$ (Sketch): First recall: X Banach and Y linear subspace of X^*. Y is said to ISOMORPHICALLY NORM X if $\exists$ constant $\eta > 0$ s.t.

$$(*) \quad \boxed{\eta \|x\| \leq \sup_{y \in B_Y} |y(x)| \qquad \text{for all } x \in X}$$

We say (also) that Y η-norms X if (*) holds. (Of course $\eta \leq 1$).

- Rem: $(b_j)_j$ is called λ-basic if it is basic with basic constant at most λ.

Lemma 1: X be a Banach space. $(x_j)_j$ a seminormalized sequence in X and Y an isomorphically norming subspace of X^* so that

$$\boxed{y(x_j) \xrightarrow[j \to \infty]{} 0 \quad \forall y \in Y}$$

Then $(x_j)_j$ has a basic subsequence. In fact, if Y η-norms X, then given $0<\varepsilon<\eta$, $(x_j)_j$ has a $\frac{1}{\eta-\varepsilon}$ basic subsequence.

<u>Lemma 2</u>: X be a Banach space and $G \in X^{**} \setminus X$. Then $G^\perp$ isomorphically norms X, where $G^\perp = \{x^* \in X^* : G(x^*)=0\}$

Now pick a NTWC sequence $(x_j)_j \subset X$, and define $G \in X^{**}$ by:

$$\boxed{G(f) = \lim_{j\to\infty} f(x_j)} \quad , \forall f \in X^*$$

then $G \in X^{**} \setminus X$ since $(x_j)_j$ is non trivial, hence by <u>Lemma 2</u> isomorphically norms X and by <u>Lemma 1</u> $(x_j)_j$ has a basic subsequence $(y_j)_j$

Now choose $f \in X^*$ with $G(f)=1$. Hence $f(y_j) \xrightarrow[j\to\infty]{} 1$. Then given $\tau>0$ choose $(b_j)_j$ a subseq. of $(y_j)_j$ with

$$\left|1 - f(b_j)\right| < \frac{\tau}{2^j} \quad \forall j$$

We are done if we show that $(b_j)_j$ is an (s)-subsequence. But this follows if we show ~~that~~ there exists a $\beta<\infty$ so that:

$$\left|\sum_{j=1}^{n} c_j\right| \leq \beta \left\|\sum_{j=1}^{n} c_j b_j\right\| \quad \forall n, \forall c_1, \dots, c_n \text{ scalars}$$

Given $c_1, \dots, c_n$ scalars and setting $x_n = \sum_{j=1}^{n} c_j b_j$ we have ~~that~~:

$$\left|\sum_{j=1}^{n} c_j\right| = \left|\sum_{j=1}^{n} c_j f(b_j) + \sum_{j=1}^{n} c_j (1 - f(b_j))\right| \leq$$

$$\leq \|f\| \|x_n\| + \left|\sum_{j=1}^{n} \sum_{k=1}^{n} b_j^*(b_k c_k) \frac{\tau}{2^j}\right| \leq$$

$\leq \|f\| \, \|x_n^*\| + \|x_n\| \sup_j \|b_j^*\| \left| \sum_{j=1}^{n} \frac{\tau}{2j} \right| < \infty$ $\tilde{C}$

Thus we can choose $\beta = \|f\| + \sup \|b^*_j\| \, \tilde{C}$

□

Def: A sequence $(e_j)_j$ in a Banach space is called a (c)-sequence provided that it is a seminormalised basic sequence so that $\left(\sum_{j=1}^{n} e_j\right)_n$ is weak-Cauchy.

Prop. 3: Let $(b_j)_j$ and $(e_j)_j$ sequences in a Banach space with $(e_j)_j$ the difference sequence of $(b_j)_j$. Then:

$$\boxed{(b_j)_j \text{ is (s)} \iff (e_j)_j \text{ is (c)}}$$

Rem: · $[x_j]$ denotes the closed linear span of $(x_j)_j$.

· If $(x_j)_j$ is basic, $(x^*_j)_j$ denotes its sequence of biorthogonal functionals in $[x_j]^*$: $x^*_j(x_i) = \delta_{ij}$, $\forall i, j$.

- $(b_j)_j$ and $(e_j)_j$ are both bases of $[b_j]$

Proof (sketch):

($\Rightarrow$) Suppose $(b_j)_j$ an (s)-sequence and let $(P_k)_k$ be its basis projections.

$$P_k : [b_j] \longrightarrow [b_j] \quad : P_k(x) = \sum_{j=1}^{k} c_j b_j \quad \text{if} \quad x = \sum_{j=1}^{\infty} c_j b_j$$

· let λ be a basis constant of $(b_j)_j$ (i.e. $\lambda = \sup_k \|P_k\|$)

· let s the summing functional on $[b_j]$, i.e.:

$$s\left(\sum_{j=1}^{\infty} c_j b_j\right) = \sum_{j=1}^{\infty} c_j \quad \text{for all } x \in [b_j] \text{ with } x = \sum_{j=1}^{\infty} c_j b_j$$

Define e^*_n for all n by:

$$e_n^* = S - \sum_{i=1}^{n-1} b_i^* \quad \text{for } n>1 \text{ and } e_1^* = S$$

Then $(e_n^*)_n$ is biorthogonal to $(e_j)_j$. Moreover since for all n,

$$\sum_{i=1}^{n} b_i^* = S P_n$$

note $e_n^* = \sum_{j=n}^{\infty} b_j^*$ and $S = \sum_{j=1}^{\infty} b_j^*$

it follows that

$$\sup_n \|e_n^*\| \le \|S\| (1+\lambda)$$

Finally to see that $(e_j)_j$ is basic, since $(e_j)_j$ is trivially linearly independent, it is sufficient to estimate the norms of its basis projections on its linear span, that is:

$[e_j] = X_0$ and for each k set

$$Q_k\left(\sum_{j=1}^{\infty} c_j e_j\right) = \sum_{j=1}^{k} c_j e_j \quad , \text{for all } \sum_{j=1}^{\infty} c_j e_j \text{ in } X_0$$

we need to estimate $\sup_k \|Q_k\|_{X_0}$.

Let $k<n$ and $x = \sum_{j=1}^{n} c_j e_j$. Then:

$$\sum_{j=1}^{n} c_j e_j = c_1 b_1 + c_2(b_2 - b_1) + \dots + c_n(b_n - b_{n-1}) =$$

$$\boxed{= (c_1 - c_2) b_1 + \dots + (c_{n-1} - c_n) b_{n-1} + c_n b_n}$$

Hence:

$$\sum_{j=1}^{k} c_j e_j = P_{k-1} x + e_k^*(x) b_k \quad , \text{having set } P_0 = 0$$

That is, we have proved:

$$\boxed{Q_k = P_{k-1} + e_k^* \otimes b_k} \quad (*)$$

(where $(x^* \otimes x)(y) = x^*(y) x$, $\forall y \in X$ Banach space)

Hence:

$$\sup_k \|Q_k\| \leq \lambda + (1+d)\|s\| \sup_k \|b_k\|$$

that is, $\sup_k \|Q_k\| < \infty$.

Now, $(b_k)_k$ NTWC $\Rightarrow$ semi-normalized and basic $\Rightarrow$ its difference sequence $(e_j)_j$ is also seminormalized.

Moreover, since $\sum_{j=1}^{n} e_j = b_n$ for all n, $\left(\sum_{j=1}^{n} e_j\right)_n$ is weak-Cauchy and hence $(e_j)_j$ is (c)-sequence.

($\Leftarrow$) Suppose $(e_j)_j$ (c)-seq. Then trivially $(b_j)_j$ is weak-Cauchy.

Since $(e_j)_j$ is seminormalised, $(e^*_j)_j$ is bounded.

Now use $(*)$ to obtain that if $(P_k)_k$ is the sequence of basis projections of $(b_j)_j$ on X_0, then, for all k:

$$\boxed{P_k = Q_{k+1} - e^*_{k+1} \otimes b_{k+1}}$$

hence $\sup_k \|P_k\| \leq \sup_k \|Q_k\| + \sup_k \|e^*_k\| \|b_k\| < \infty$.

Thus $(b_k)_k$ is a basic sequence, since $e^*_1(b_j) = 1$ for all j, e^*_1 is indeed the summing functional s on $[b_j]$, whence $(b_j)_j$ is (s).

□

<u>Def</u>: Let $(x_j)_j$ and $(f_j)_j$ be sequences in a Banach space. Then $(x_j)_j$ is called WUC (Weakly-Unconditionally-Cauchy) if

$$\|(x_j)_j\|_{WUC} = \sup\left\{ \sum_{j=1}^{\infty} |x^*(x_j)| : x^* \in B_{X^*} \right\} < \infty$$

$(f_j)_j$ is called DUC (Difference-weakly-Unconditionally Cauchy) if $(f_j - f_{j-1})_{j=1}^{\infty}$ is WUC (where $f_0 = 0$)

We set:

$$\|(f_j)_j\|_{DUC} = \|(f_j - f_{j-1})_j\|_{WUC}$$

Banach-Steinhaus

Rem: $\sum_{j=1}^{\infty} |x^*(x_j)| < \infty \quad \forall x^* \in X^* \Rightarrow (x_j)_j$ WUC. Since the seq of partial sums of WUC seq. is weak-Cauchy, then DUC seq are weak-Cauchy.

Def: $(x_j)_j$, $(y_j)_j \subset X$ seq. in Banach space. $(y_j)_j$ is called a <u>CONVEX BLOCK BASIS</u> of $(x_j)_j$ if there exist:

$0 = n_0 < n_1 < n_2 < \dots$ and $(\lambda_j)_j$ non negative scalars so that for all j:

$$\sum_{n_{j-1} < i \leq n_j} \lambda_i = 1 \quad \text{and} \quad y_j = \sum_{n_{j-1} < i \leq n_j} \lambda_i x_i$$

Prop. 4: Let $(y_j)_j$ be a convex block basis of a DUC sequence $(x_j)_j$. Then $(y_j)_j$ is also DUC and $\|(y_j)_j\|_{DUC} \leq \|(x_j)_j\|_{DUC}$

Rem: $(x_j)_j$, $(y_j)_j$ DUC $\Rightarrow (x_j + y_j)_j$ DUC and $(\lambda x_j)_j$ DUC, $\forall \lambda$ scalar.

$\Rightarrow$ DUC seq. with $\|\cdot\|_{DUC}$ are Banach space

Note that if $(x_j)_j$ is DUC and $\sum \|x_j - y_j\| < \infty$, then since $(x_j - y_j)_j$ is DUC, then $(y_j)_j$ is DUC too.

Proof: Let $e_1 = x_1$, $e_j = x_j - x_{j-1}$ for $j > 1$. Then it follows that given $k < l$ and scalars $\lambda_{k+1}, \dots, \lambda_l$ with $\sum_{j=k+1}^{l} \lambda_j = 1$, and $y = \sum_{i=k+1}^{l} \lambda_i x_i$, then setting $p_j = \sum_{i=j}^{l} \lambda_i$ for all $k < j \leq l$, we have that: $\qquad \rho_{k+1} = 1 \qquad$ and

$$y = \sum_{i=1}^{k} e_i + \sum_{j=k+1}^{l} \rho_j e_j$$

Hence, also given $m > l$ and scalars $\lambda_{l+1}, \dots, \lambda_m$ with $\sum_{j=l+1}^{m} \lambda_j = 1$ and $\tilde{y} = \sum_{i=l+1}^{m} \lambda_j x_j$, then setting $\tilde{\rho}_j = \sum_{i=j}^{m} \lambda_j$, for all $l < j \leq m$ we have that:

$$\boxed{\tilde{y} - y = \sum_{j=k+1}^{l} (1-\rho_j) e_j + \sum_{j=l+1}^{m} \tilde{\rho}_j e_j} \qquad (*)$$

Now let $(y_j)_j$ be convex block basis of $(x_i)_i$. We may choose $0 = n_0 < n_1 < n_2 < \dots$ and non-negative scalars λ_j so that for all i:

$$y_i = \sum_{j=n_{i-1}+1}^{n_i} \lambda_j x_j \qquad \text{and} \qquad \sum_{j=n_{i-1}+1}^{n_i} \lambda_j = 1$$

Set: $\rho_j^i = \sum_{k=j}^{n_i} \lambda_k$ for all $n_{i-1} < j \leq n_i$

it follows by $(*)$ that:

$$(**) \qquad \boxed{\begin{array}{l} y_i = \sum_{j=1}^{n_i} \rho_j^i e_j \qquad \text{and} \\ y_{i+1} - y_i = \sum_{j=n_{i-1}+1}^{n_i} (1-\rho_j^i) e_j + \sum_{j=n_i+1}^{n_{i+1}} \rho_j^i e_j \qquad \forall i \end{array}}$$

Now set $y_0 = 0$ and let $x^* \in B_{X^*}$, we have by $(**)$ and the fact that $0 \leq \rho_j^i \leq 1$ $\forall i, j$:

$$\sum_{i=0}^{\infty} |x^*(y_{i+1} - y_i)| \leq$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=n_{i-1}+1}^{n_i} \underbrace{\rho_j^i}_{\leq 1} |x^*(e_j)| + \sum_{i=1}^{\infty} \sum_{j=n_i+1}^{n_{i+1}} (1-\rho_j^i) |x^*(e_j)| \leq$$

$$\leq \sum_{j=1}^{\infty} |x^*(e_j)| \leq \|(e_j)_j\|_{WUC} = \|(x_j)_j\|_{DUC}.$$

□

Prop. 5: (a) $(x_j)_j \subset X$ is equivalent to the summing basis $\Longleftrightarrow$ $(x_j)_j$ is (s) and DUC.

(b) Let $(y_j)_j$ and $(x_j)_j$ sequences in a Banach space so that:

(i) $y_j - x_j \xrightarrow[j\to\infty]{w} 0$

(ii) $(y_j)_j$ is DUC and non weakly-convergent

Then some convex block basis of $(x_j)_j$ is equivalent to the summing basis.

Proof: (a): ($\Rightarrow$) the summing basis is DUC and (s), so any equiv seq. is the same.

($\Leftarrow$) Let $(b_j)_j$ be (s) and DUC and $(e_j)_j$ its difference sequence.

By Prop. 3 $(e_j)_j$ is seminormalised, then $(e^*_j)_j$ is uniformly bounded, hence let $\lambda = \sup_j \|e^*_j\|$ and given scalars $c_1, \dots, c_n$ we have that

$$\max_{1\leq j\leq n} |c_j| \leq \lambda \left\| \sum_{j=1}^{n} c_j e_j \right\|$$

On the other hand $(e_j)_j$ is WUC. So choose $x^* \in B_{X^*}$ with:

$$\boxed{\left\| \sum_{j=1}^{n} c_j e_j \right\| = \left| x^* \left(\sum_{j=1}^{\infty} c_j e_j \right) \right|} \quad (1)$$

we have

$$(2)\quad \boxed{\left\|\sum_{j=1}^{n} c_j e_j\right\| \leq \sum_{j=1}^{n} |c_j|\,|x^*(e_j)| \leq \max|c_j|\,\|(e_j)_j\|_{WUC}}$$

(1) + (2) $\Rightarrow$ $(e_j)_j$ equivalent to the c_0-basis $\Rightarrow$ $(b_j)_j$ equivalent to the summing basis.

(b) Since DUC sequences are weak-Cauchy we have that $(y_j)_j$ is NTWC. Hence by Prop 2 it has a (s)-subseq. $(y'_j)_j$.
Since a subseq. is trivially a convex block basis for the sequence, we have that by Prop. 4 $(y'_j)_j$ is DUC.
But by (a) of this theorem $(y'_j)_j$ is equivalent to the summing basis.
Since $y_j - y'_j \xrightarrow{w} 0$, also $y'_j - x_j \xrightarrow{w} 0$. So applying Mazur's Lemma $\Rightarrow \exists$ convex block bases $(u_j)_j$ of $(x_j)_j$ and $(v_j)_j$ of $(y'_j)_j$ respectively so that.

$$\left(\|u_j - v_j\| \xrightarrow[j\to\infty]{} 0\right) \quad \sum_{j=1}^{\infty} \|u_j - v_j\| < \infty$$

Now: $(v_j)_j$ is obviously equivalent to the summing basis (we apply Prop 4 knowing that $(v_j)_j$ is (s) and DUC).
So also $(u_j)_j$ is equivalent to the summing basis.. Note that if $\varepsilon > 0$ is given in advance, we may ensure that $\sum_{j=1}^{\infty} \|u_j - v_j\| < \varepsilon$
and so by Prop 4:

$$\|(u_j)_j\|_{DUC} < \|(y_j)_j\|_{DUC} + \varepsilon$$

□

• Now let K be a compact metric space and identify $C(K)^*$ with:

$$M(K) \sim C(K)^*$$

the space of all scalar-valued Borel measures on K.

<u>Lemma A</u>: Let K be a compact metric space and $Y = C(K)$. Let $E = B_{Y^*}$ in the w^*-topology and regard K as canonically embedded in E. Let $F \in X^{**}$ with $F|_E \in \mathcal{B}_1(E)$. Then:

$$F(\mu) = \int_K F(k)\, d\mu(k)$$

for all $\mu \in Y^* = M(K)$.

<u>Proof</u>: Let S be closed subset of K. $\mathcal{P}(S) = \{\mu$ prob. measure on K (i.e., $\mu \in M(K)$ $|$ $\mu(K \setminus S) = 0\}$.

$\mathcal{P}_a(S) = \{$ <u>purely atomic members</u> of $\mathcal{P}(S)\}$

↓

i.e. if every measurable set of positive measures contains an atom $\Rightarrow$ atom A if $\mu(A) > 0$ and $\forall$ measurable subset $B \subset A$ with $\mu(B) < \mu(A) \Rightarrow \mu(B) = 0$

for μ in $\mathcal{P}(K)$: $\mathcal{P}_\mu = \{\lambda$ measure in $\mathcal{P}(K)$ $|$ $\lambda \ll \mu\}$
absolutely cont.

Let supp $\mu = \{k \in K \mid \forall U_k$ open neighborhood of k, $\mu(U_k) > 0\}$

→ supp μ is closed in K.

It holds that:

(*) $\boxed{\mathcal{P}_\mu \text{ and } \mathcal{P}_a(S) \text{ are both } w^*\text{-dense in } \mathcal{P}(S);\ S = \operatorname{supp}\mu}$

having noticed that $W = \mathcal{P}_\mu$ or $W = \mathcal{P}_a(S)$ is convex and $\|f\|_\infty = \sup_{\nu \in W}$

$|\int f \, d\nu|$ for all $f \in C(S)$.

then Hahn - Banach (f) Thm. applies and (*) holds.

Define G on $M(K)$ by:

$$G(\mu) = \int_K F(k) \, d\mu(k) \qquad \text{for all } \mu \in M(K).$$

. <u>Bounded convergence theorem</u>: Let $(f_n)_n$ a sequence of measurable functions on a set of finite measure E. Suppose $(f_n)_n$ is uniformly pointwise bounded on E, that is there is a $M \geq 0$ for which $\|f_n\| \leq M$ $\forall n$.

If $f_n \longrightarrow f$ pointwise on E $\Rightarrow$ $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$

So $G|_E = G|_{B_{Y^*}} = G|_{B_{M(K)}} \in \mathcal{B}_1(E)$, hence also $H|_E \in \mathcal{B}_1(E)$,

where $H = F - G$. We want to prove that $H = 0$.

By contradiction $H \neq 0$. So, since the linear span of $\mathcal{P}(K)$ equals $M(K)$ $\Rightarrow$ there exists a $\nu \in \mathcal{P}(K)$ with $H(\nu) \neq 0$.

Assume $H(\nu) > 0$ (eventually multiplying H by -1, if necessary)

Now denote $Z = \{\lambda \in M(K) \mid \lambda \text{ absolutely continuous with respect to } \nu\}$

λ abs. cont. with resp. to ν $\Leftrightarrow$ $\lambda(A) = 0$ for any measurable A s.t. $\mu(A) = 0$

$\lambda \ll \nu$

Radon - Nikodym theorem : $\nu \ll \mu \Rightarrow$ there exist a bounded measurable function f on K (ν, μ measures on K) s.t. $\forall A$ measurable:

$$\nu(A) = \int_A f \, d\mu$$

$\to$ so by Radon - Nikodym thm we identify Z by $L^1(\nu)$.

Now $H|_Z$ is a bounded linear functional on Z so by Riesz - representation thm there is a bounded Borel-measurable function φ so that:

$$\boxed{H(\lambda) = \int_K \varphi \, d\lambda \quad \text{for all } \lambda \in Z} \quad (1)$$

Since $H(\nu) = \int_K \varphi \, d\lambda > 0 \implies \int_K (\varphi)^+ \, d\lambda > 0$

where $(\varphi)^+$ denotes the positive part of φ.

Thus we may choose $c > 0$ so that $\nu(L) > 0$, where:

$$L = \{ k \in K \,;\, \varphi(k) > c \}$$

Thus if $\lambda \in \mathcal{P}(K)$ is such that $\lambda(K \setminus L) = 0$, then

$$\boxed{\int_K \varphi \, d\lambda = \int_{K \setminus L} \varphi \, d\lambda + \int_L \varphi \, d\lambda \geq c} \quad (2)$$

- $\int_{K \setminus L} \varphi \, d\lambda = 0$ since φ bounded and $\lambda(K \setminus L) = 0$
- $\int_L \varphi \, d\lambda \geq c$ for how we have chosen L

Finally let $\mu \in \mathcal{P}(K)$ defined by:

$$\mu(B) = \frac{\nu(B \cap L)}{\nu(L)} \qquad \forall \text{ Borel } B$$

and let $\boxed{S = \operatorname{supp} \mu}$

From (1) and (2) we have that:

$$H(\lambda) \geq c \qquad \text{for all } \lambda \in \mathcal{P}_\mu$$

On the other hand if $k \in K$ and δ_k = point-mass probability at k, then $G(\delta_k) = F(\delta_k)$, and thus

$$H(\lambda) = 0 \qquad \text{for all } \lambda \in \mathcal{P}_a$$

But we are done because $\quad H = 0 \quad$ in $\mathcal{P}_a$ $\quad$ and

$$H \geq c \qquad \text{in } \mathcal{P}_\mu$$

$\mathcal{P}_a$ and $\mathcal{P}_\mu$ are dense subsets of $\mathcal{P}(S)$. So H has no point of conformity, but this is in contradiction of being 1-Baire on K. $\square$

We can now give the proof of <u>THM 1</u>.

$\boxed{\text{PROOF of THM 1}}$: Let X be a separable Banach space, $K = B_{X^{**}}$

Let $Y = C(K)$ and regard $X \subset Y$ by the Hahn-Banach thm.

(a) ($\Leftarrow$) Let $f = x^{**}|_K$. If there is a Cauchy sequence $(x_j)_j \xrightarrow{w^*} x^{**}$ then trivially $f \in \mathcal{B}_1(K)$, since the x_j may be regarded as continuous functions on K, and of course $x_j|_K \to f$ pointwise.

($\Rightarrow$) Suppose $f \in B_1(K)$. Choose $(f_j)_j$ seq. of continuous functions on K with $f_j \to f$ pointwise

where $f = x^{**}|_K$.

WLOG (without loss of generality) assume $\|x^{**}\| = 1$

Let $\tau : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\tau(z) = \begin{cases} z & \text{for } |z| \leq 1 \\ \frac{z}{|z|} & \text{otherwise} \end{cases}$

then replacing f_j by $\tau \circ f_j$ we may assume that $\|f_j\| \leq 1$ for all j.

<u>CLAIM:</u> fixing n we have that $dist(X, conv(\underbrace{\{f_j : j \geq n\}}_{(v_j)})) = 0$

If CLAIM is false, by Hahn–Banach separation theorem there are $y^* \in Y^*$ and $r > 0$ with

$$\boxed{y^*(x) = 0 \ \forall x \in X \quad \text{and} \quad Re\, y^*(f_j) \geq r \quad \text{for all } j.}$$

By Riesz representation theorem there is a real valued Borel measure μ on K representing y^*. We thus have that

$$\boxed{\begin{array}{l} \int_K x(k)\, d\mu(k) = 0 \quad (\text{it represents } y^*(x) \text{ for all } x \in X) \\ \\ \text{and} \quad \int_K f_j(k)\, d\mu(k) \geq r \quad \text{for all } j \end{array}} \quad (*)$$

$\Rightarrow \mu \in X^{\perp}$ and identifying X^{**} with $X^{\perp\perp}$ in Y^{**} we have

that $\underset{X^{\perp\perp}}{\underset{\cap}{x^{**}}}(\underset{X^{\perp}}{\underset{\cap}{\mu}}) = 0$

Now denoting $E = B_{Y^*}$, then $x^{**}|_E \in \mathcal{B}_1(E)$. Indeed, letting $T: X \to Y$ denote the canonical isometric injection, we have

that $\boxed{T^*(E) = K}$ and thus $(f_n \circ (T^*|_E))$ is a sequence

of continuous functions on E converging pointwise to $x^{**}|_E$.

Thus by Lemma A.

$$x^{**}(\mu) = \int_K x^{**}(k)\, d\mu(k) = 0$$

but by (*)

$$\int_K x^{**}(k)\, d\mu(k) \geq r > 0$$

having employed bounded conv. thm (recall that $f_j \circ (T^*|_E) \xrightarrow{w^*} x^{**}|_E$)

Which is a contradiction $\Rightarrow$ CLAIM is true.

$\Rightarrow$ there exists a convex block basis $(u_j)_j$ of $(v_j)_j$ and a sequence $(x_j)_j$ in X so that

$$\|u_j - x_j\| \longrightarrow 0 \quad \text{and } \|x_j\| \leq 4 \quad \forall j.$$

but
$u_j \longrightarrow f$ pointwise, hence also $x_j \longrightarrow f$ pointwise on K.

Employing once again Riesz repr. theorem we obtain $(x_j)_j$ weak Cauchy and $x_j \longrightarrow x^{**}$ w^*.

(b) ($\Leftarrow$) if (x_j) is as in (a) $\Rightarrow$ $(x_j)_j$ is WUC $\Rightarrow$ $f = x^{**}|_K \in$ (10)

$\in D(K)$.

($\Rightarrow$) Suppose $x^{**} \in X^{**}_D$. Given $\varepsilon > 0$ we may choose a sequence in $C(K)$ (with $f_0 = 0$) so that:

$f_j \to f$ pointwise and

$$\sum_{j=1}^{\infty} |(f_j - f_{j-1})(k)| < \|f\|_D + \varepsilon \quad \forall k \in K$$

By Riesz repr. thm we have:

$$\|(f_j)\|_{DUC} < \|f\|_D + \varepsilon$$

and moreover $(f_j)_j$ is weak Cauchy with $f_j \to x^{**}$ w*.

By part (a) $\Rightarrow$ $\exists$ weak Cauchy sequence $(x_j)_j$ in X with

$x_j \to x^{**}$ w*.

But then $f_j - x_j \to 0$ weakly and if $x^{**} \notin X$ we have $(f_j)_j$ non weakly conv.

By (b) of Prop 5: $\Rightarrow$ some convex block basis $(x'_j)_j$ of $(x_j)_j$ is equivalent to the summing basis.

Also by Prop 5. for a given $\varepsilon > 0$ $(x'_j)_j$ may be chosen so that the thesis holds.

$\square$

Ref:

- A characterisation of Banach spaces containing c_0, Rosenthal
- A double dual characterisation of separable Banach spaces containing ℓ^1, Odell and Rosenthal
- Handbook of the Geometry of Banach spaces, vol 2
- Lectures on Choquet's Theorem, R. Phelps.