To do a descriptive set theoretic study of Banach spaces we need a polish space whose elements are precisely all Banach spaces. Since Banach spaces cannot be coded in just one space, we will restrict our attention to the subclass of those being separable.

1. Coding separable Banach spaces

This coding relies on two results.

Theorem. *Any separable Banach space X is isometrically isomorphic to a closed subspace of $C(2^{\mathbb{N}})$.*

Proof. Consider $(\mathbf{B}_{X^*}, \omega^*)$ the closed unit ball of the dual space with the weak* topology. It is a compact space and since X is separable, it is also metrizable. Then we can fix $f : 2^{\mathbb{N}} \to \mathbf{B}_{X^*}$ a continuous surjection. The mapping we are looking for is

$$\begin{aligned} T : X &\to C(2^{\mathbb{N}}) \\ x &\mapsto g_x, \end{aligned}$$

where $g_x(\sigma) = f(\sigma)(x)$. It remains to check that T is an isometric isomorphism.

- *Linearity* : Follows from the linearity of $f(\sigma)$.
- *Injectivity* : The Hans-Banach theorem provides for each $x \neq y$ a functional x^* of norm 1 such that $x^*(x) \neq x^*(y)$. In particular we have $g_x \neq g_y$ for $x \neq y$.
- *Isometry* : We have
  $$||T(x)|| = \sup_{\sigma \in 2^{\mathbb{N}}} |g_x(\sigma)| = \sup_{\sigma \in 2^{\mathbb{N}}} |f(\sigma)(x)| = \sup_{x^* \in \mathbf{B}_{X^*}} |x^*(x)| \leq ||x||.$$
  To show that the norm of x is actually achieved, recall that the Hans-Banach theorem provides for each $x \in X$, $x^* \in \mathbf{B}_{X^*}$ such that $x^*(x) = ||x||$.

We have proved that any separable Banach space can be seen as a closed subset of $C(2^{\mathbb{N}})$. Nonetheless we still face two difficulties. The first one is that we need a standard Borel structure in the space of closed subsets of $C(2^{\mathbb{N}})$. The second one is that not all closed subsets of $C(2^{\mathbb{N}})$ need be vector spaces so the identification is not $1-1$. The following construction solves the first.

Theorem. (Effros-Borel structure) *Let X be a polish space and denote $F(X)$ the set of closed subsets of X. Consider τ the σ-algebra generated by the sets*

$$\{F \in F(X) : F \cap U \neq \emptyset\},$$

where U ranges over all open sets in X. Then

- *$(F(X), \tau)$ is a standard Borel space.*

- *(Kuratowski-Ryll-Nardzewski) There exist Borel maps*
$$d_n : F(X) \to X$$
such that $(d_n(F))_{n\in\mathbb{N}}$ *is dense in* F.

For the first difficulty note that for closed sets F being a vector subspace can be characterised by a density argument as

$$0 \in F \wedge \forall p, q \in \mathbb{Q}\ \forall n, m \in \mathbb{N}\ pd_n(F) + qd_n(F) \in F.$$

Because the maps d_n are Borel the set

$$\mathsf{SB} = \{X \in F(C(2^{\mathbb{N}})) : X \text{ is a separable Banach space}\}$$

is Borel. Any Borel set is still standard when considered with the restricted σ-algebra.

The coding we were looking for is given by the space SB together with its standard Borel structure and the restriction to SB of the Borel maps $(d_n, n \in \mathbb{N})$. From now on d_n denotes the restriction to SB and not the original map. We are now prepared to do a descriptive set theoretic analysis of properties from separable Banach spaces.

2. Reflexive Banach spaces

The first property of separable Banach spaces we analyse is that of being reflexive. Let us begin by recalling two results that will be needed later on.

Theorem. (Dieudonné) A normed space is reflexive if and only if the closed unit ball is compact with respect to the weak topology.

Theorem. (Mazur) Let X be a normed space and $C \subset X$ a convex subset. Then the norm closure and the weak closure are the same:

$$\overline{C} = \overline{C}^{*}.$$

In order to prove that the set

$$\text{REFL} = \{X \in \mathsf{SB} : X \text{ is reflexive}\}$$

is co-analytic we show that it can be Borel reduced to WF. The structure of the proof is the same for all the properties (P) we study. First, for each Banach space X, we construct a family of trees on X and show that well-foundedness of all the trees is equivalent to the property (P) we analyse. Then we introduce a discrete version of these trees using the maps $(d_n)_{n\in\mathbb{N}}$ described in the previous section and finally we glue all the trees in one so that property (P) holding in X is equivalent to well-foundedness of the final tree. The trees we construct are made up of finite attemps of producing a "nice" sequence in the space X.

Fix X a Banach space, $K \geq 1$ and $\epsilon > 0$. Say that a sequence $(x_i)_{i=1}^l$ is k-Shauder if for any $m \leq l$ and $a_1, \dots, a_l \in \mathbb{R}$,

$$||\sum_{i=1}^m a_i x_i|| \leq k||\sum_{i=1}^l a_i x_i||.$$

Note that a sequence $(x_n)_{n\in\mathbb{N}}$ is basic if and only if for some $k \in \mathbb{N}$ we have that $(x_n)_{n\leq l}$ is k-Shauder for all $l \in \mathbb{N}$. The tree $\mathsf{T}^c(X, \epsilon, K)$ is defined on the sphere S_X and is given by

$$(x_i)_{i=1}^l \in \mathsf{T}(X, \epsilon, K) \textit{ if and only if } (x_i)_{i=1}^l \textit{ is k-Shauder and}$$
$$\textit{for any } a_1, \dots, a_l \in \mathbb{R}^+ \textit{ such that } \sum_{i=1}^l a_i = 1,\ ||\sum_{i=1}^l a_i x_i|| > \epsilon.$$

Lets describe what an infinite branch is coding. From the definition of k-Shauder we get that it provides a basic sequence with basis constant less or equal than k. Since the tree is defined in S_X the sequence is normalized. Lets see that the second condition ensures that it has no weakly null subsequence. Suppose $(x_{\sigma(n)})_{n\in\mathbb{N}}$ is a weakly null sequence, that is

$$x_{\sigma(n)} \to^w 0.$$

By Mazur's theorem the norm closure of $\operatorname{span}\{x_\sigma(n) : n \in \mathbb{N}\}$ is the same as the weak closure. In particular 0 belongs to the norm closure of $\operatorname{span}\{x_{\sigma(n)} : n \in \mathbb{N}\}$ and since the sequence is basic,

$$0 = \sum_{i=1}^\infty a_i x_{\sigma(i)}$$

for some $(a_i) \subset \mathbb{R}$. But then we can find n such that

$$||\sum_{i=1}^l a_i x_i|| < \epsilon,$$

contradicting the definition of the tree. The main result we prove is the following.

Theorem. *A separable Banach space X is reflexive if and only if for any $k \geq 1$ and any $\epsilon > 0$ the tree $\mathsf{T}^c(X, \epsilon, k)$ is well founded.*

Proof. Suppose first that for some ϵ and k there is an infinite branch and denote $(x_n)_{n\in\mathbb{N}}$ the associated basis. We now apply Dieudonné's theorem to show that the space X cannot be reflexive. By contradiction, suppose the unit ball is weakly compact, we want so see that it admits a weakly convergent subsequence $(x_{\sigma(n)})_{n\in\mathbb{N}}$. Note that since X is assumed to be reflexive and it is separable, X^* is also separable. This proves that (B_X, ω) is metrizable and we have found a weakly convergent subsequence $(x_{\sigma(n)})_{n\in\mathbb{N}}$. To get a contradiction our previous discussion ensures that it is enough to show that the subsequence is weakly null. Consider the coordinate functionals $< -, x_m >$ associated to our basic sequence. If y is the weak limit we have that for each $m \in \mathbb{N}$,

$$< x_{\sigma(n)}, x_m > \to < y, x_m > \quad \textit{as } n \to \infty.$$

But since the sequence is basic, $(< x_{\sigma(n)}, x_m >)_{n\in\mathbb{N}} = (0, \dots, 0, 1, 0, \dots, 0, \dots,)$ and $< y, x_m > = 0$ for all $m \in \mathbb{N}$, i.e. $y = 0$.

For the other direction we first recall a corollary of the Eberleyn-Smulian theorem.

Corollary.(Eberleyn-Smulian) Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X. If it does not have a weak limit in X, then it admits a subsequence which is basic.

Suppose that X is non reflexive and take $x^{**} \in X^{**}\backslash X$ with $r = ||x^{**}|| \leq 1$. By the Odell-Rosenthal theorem there are two possibilities. If l_1 embeds via f in X then for some $K \geq 1$ and $\epsilon > 0$, $(f(e_n))_{n\in\mathbb{N}}$ is an infinite branch. If l_1 does not embed, there exists $(x_n)_{n\in\mathbb{N}}$ weak* convergent to x^{**}, meaning that for any $f \in X^*$, $f(x_n) \to x^{**}(f)$. By the Banach-Steinhauss theorem every weak* convergent sequence is bounded, in particular we can admit $(x_n)_{n\in\mathbb{N}}$ in B_X. Since the norm of f is given by

$$r = ||x^{**}|| = \sup_{f\in\mathsf{B}_{X^*}} |x^{**}(f)|,$$

we can find f such that $||f|| \leq 1$ and $x^{**}(f) > \frac{r}{2}$. Because $f(x_n)$ converges to $x^{**}(f)$ there is a subsequence $(x_{\sigma'(n)})$ such that $f(x_{\sigma'(n)}) \geq \frac{r}{2}$. We want to extract a further subsequence being basic. By the Eberlein-Smulian theorem it is enough to see that it has no weak limit. Should it have weak limit x, then for any $g \in X^*$

$$j(x_n)(g) = g(x_n) \to g(x) = x^{**}(g),$$

proving $x = x^{**}$, a contradiction. Then we can extract a subsequence $(x_{\sigma(n)})$ which is basic with constant k. Normalize the sequence by taking $y_n = \frac{x_{\sigma(n)}}{||x_{\sigma(n)}||}$. Ir remains to see that (y_n) is an infinite branch in $T(X, \frac{r}{2}, k)$. For $a_1, \dots, a_l \in \mathbb{R}$ such that

$$\sum_{i=1}^{l} a_i = 1$$

we have

$$\frac{r}{2} = \sum_{i=1}^{l} a_i \frac{r}{2} \leq \sum_{i=1}^{l} a_i f(x_{\sigma(i)}) \leq \sum_{i=1}^{l} a_i \frac{f(x_{\sigma(i)})}{||x_{\sigma(i)}||} = \sum_{i=1}^{l} a_i f(y_i) = |\sum_{i=1}^{l} a_i f(y_i)| =$$
$$|f(\sum_{i=1}^{l} a_i y_i)| \leq ||\sum_{i=1}^{l} a_i y_i|| * ||f|| \leq ||\sum_{i=1}^{l} a_i y_i||.$$

As we said previously, it is time to use the maps d_n in order to define a discrete version of the trees $\mathsf{T}^c(X, \epsilon, k)$. Note that the trees we are working with are defined on the sphere and not in the whole space, it will be then useful to define S_n on SB by

$$S_n(Y) = \frac{d_n(Y)}{||d_n(Y)||}$$

so that for any separable Banach space Y, $(S_n(Y))_{n\in\mathbb{N}}$ is dense in S_Y. Consider $j, k \in \mathbb{N}$ and define $\mathsf{T}^d(X, \frac{1}{j}, k) \subset \mathbb{N}^{<\mathbb{N}}$ as follows:

$$(n_i)_{i=1}^l \in \mathsf{T}^d(X, \frac{1}{j}, k) \textit{ if and only if } (S_{n_1}(X), \dots, S_{n_l}(X)) \in \mathsf{T}^c(X, \frac{1}{j}, k).$$

What is the relation between both trees? An infinite branch $(n_j)_{j\in\mathbb{N}}$ in some $\mathsf{T}^d(X, \frac{1}{j}, k)$ provides an infinite branch $(S_{n_j}(X))_{j\in\mathbb{N}}$ in $\mathsf{T}^c(X, \frac{1}{j}, k)$ by definition. Now, if $(x_j)_{j\in\mathbb{N}}$ is an infinite branch in the continuous tree, a perturbation argument allows to find $\epsilon \leq \frac{1}{j}$, $K \geq k$ and $(n_j)_{j\in\mathbb{N}} \in \mathsf{T}^d(X, \frac{1}{j}, k)$. In particular X is reflexive if and only if for any $j, k \in \mathbb{N}$ the tree $\mathsf{T}^d(X, \frac{1}{j}, k)$ is well founded.

Finally, we glue all the discrete trees in just one as follows:

$$(n_i)_{i=1}^l \in \mathsf{T}_R(X) \textit{ if and only if } n_1 = < j, k > \textit{ and } (n_i)_{i=2}^l \in \mathsf{T}^d(X, \frac{1}{j}, k).$$

Since an infinite branch $< j, k >^\frown (n_i)_{i\in\mathbb{N}}$ in $\mathsf{T}_R(X)$ is giving an infinite branch in $\mathsf{T}^d(X, \frac{1}{j}, k)$, and vice-versa, X is reflexive if and only if the tree $\mathsf{T}_R(X)$ is well founded. To conclude that REFL is co-analytic it is enough to note that the map

$$X \mapsto \mathsf{T}_R(X)$$

is Borel.

3. Natural properties

In this section we study the complexity of several natural properties. Recall that we denote SB the standard Borel space of all separable Banach spaces. This construction was done for the particular case in which $X = C(2^{\mathbb{N}})$, but can be done for any separable Banach space X in which case we obtain $\mathsf{SUBS}(X)$ the space of all Banach subspaces of X. By the same argument as before it comes equipped with the Effros-Borel structure making it a standard Borel space and has a family of Borel selectors $(d_n)_{n\in\mathbb{N}}$.

Inclusion. *For any separable Banach space X the set $\{(Y, Z) : Y \subset Z\}$ is Borel in $\mathsf{SUBS}(X) \times \mathsf{SUBS}(X)$.*

We have $Y \subset Z$ if and only if for any $n \in \mathbb{N}$, $d_n(Y) \in Z$.

Membership. *For any separable Banach space X the set $\{(y, Y) : y \in Y\}$ is Borel in $X \times \mathsf{SUBS}(X)$.*

We have $y \in Y$ if and only if for any $n \in \mathbb{N}$ there exists some $m \in \mathbb{N}$ such that $|d_m(Y) - y| < \frac{1}{n}$.

Dense span. *For any separable Banach space X the set $\{((y_n)_{n\in\mathbb{N}}, Y) : \overline{span}\{y_n : n \in \mathbb{N}\} = Y\}$ is Borel in $X^{\mathbb{N}} \times \mathsf{SUBS}(X)$.*

We have $\overline{span}\{y_n : n \in \mathbb{N}\} = Y$ if and only if

- for any $n \in \mathbb{N}$, $y_n \in Y$ and

- for any $n, m \in \mathbb{N}$ there exists some $l \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_l \in \mathbb{Q}$ such that $|d_m(Y) - \sum_i^l \lambda_i y_i| < \frac{1}{n}$.

k-equivalence. *For any separable Banach spaces* X, Y *the set* $\{((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) : x_n \text{ and } y_n \text{ are equivalent}\}$ *is Borel in* $X^{\mathbb{N}} \times Y^{\mathbb{N}}$.

We have that x_n and y_n are equivalent if and only if for some $k \in \mathbb{N}$ and for all $l \in \mathbb{N}$, $\lambda_1, \dots, \lambda_l \in \mathbb{Q}$,

$$\frac{1}{k}||\sum_i^l \lambda_i x_i|| \leq ||\sum_i^l \lambda_i y_i|| \leq k||\sum_i^l \lambda_i x_i||.$$

In particular the relation of k-equivalence is closed.

Isomorphism. *The set* $\{(X, Y) : X \text{ and } Y \text{ are isomorphic}\}$ *is analytic in* $\mathsf{SB} \times \mathsf{SB}$.

We have that X and Y are isomorphic if and only if there exist $(x_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ in $C(2^{\mathbb{N}})^{\mathbb{N}}$ such that

- $\overline{span}\{x_n : n \in \mathbb{N}\} = X$, $\overline{span}\{y_n : n \in \mathbb{N}\} = Y$ and
- for some $k \in \mathbb{N}$ the sequences are k-equivalent.

Basic sequence. *The set* $\{(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is a basic sequence}\}$ *is Borel in* $\mathsf{SB}^{\mathbb{N}}$.

We have that $(x_n)_{n\in\mathbb{N}}$ is a basic sequence if and only if for some $k \in \mathbb{N}$ for all $m, l \in \mathbb{N}$ and all $\lambda_1, \dots, \lambda_l \in \mathbb{Q}$, $||\sum_i^m \lambda_i x_i|| \leq k||\sum_i^l \lambda_i x_i||$.

Spaces with a Shauder basis. *The set* $\{X : X \text{ has a Shauder basis}\}$ *is analytic in* SB.

We have that X has a Shauder basis if and only if there exist $(x_n)_{n\in\mathbb{N}}$ such that

- $\overline{span}\{x_n : n \in \mathbb{N}\} = X$
- $(x_n)_{n\in\mathbb{N}}$ is basic.