Our goal is to show that the following families are Π^1_1-complete:

- Reflexive separable Banach spaces,
- Banach spaces with separable dual,
- Non universal separable
- Separable not containing l_1.

Actually we can prove something stronger.

• Theorem (Bossard): Any $A \subset SB$ analytic and containing REFL up to isomorphism contains an universal space for SB.

The theorem will follow from the following construction.

• Theorem (Bossard?) There exists a Borel map

$$\Phi : Tr \longrightarrow SB$$

$$T \in WF \to \Phi(T) \text{ is reflexive}$$

$$T \in IF \to \Phi(T) \text{ is universal.}$$

Let's prove why this theorem implies

the first one. Since Φ is Borel, $\Phi^{-1}(A_{\cong})$ is analytic. We have $WF \subset \Phi^{-1}(A_{\cong})$, but WF is not analytic, proving that there exists $T \in IF \cap \Phi^{-1}(A_{\cong})$. Then $\Phi(T) \in A_{\cong}$ and is universal.

We now proceed to construct Φ.

The construction of $\Phi(T)$ goes as follows:

"The idea is to generalize the space l_2 from infinite sequences to trees, changing the role of $(e_n)_{n\in\mathbb{N}}$ with a basis $(x_n)_{n\in\mathbb{N}}$ for $C(2^{\mathbb{N}})$."

Consider $c_{00}(\mathbb{N}^{<\mathbb{N}}) = \{ y : \mathbb{N}^{<\mathbb{N}} \to \mathbb{R} : \operatorname{supp}(y) \text{ is finite} \}$. If we denote, for $s \in \mathbb{N}^{<\mathbb{N}}$,

$$\begin{aligned} \chi_s : \mathbb{N}^{<\mathbb{N}} &\to \{0,1\} \\ s &\to 1 \\ r \neq s &\to 0 \,, \end{aligned}$$

then $c_{00}(\mathbb{N}^{<\mathbb{N}}) = \text{span}(\{x_s : s \in \mathbb{N}^{<\mathbb{N}}\})$.

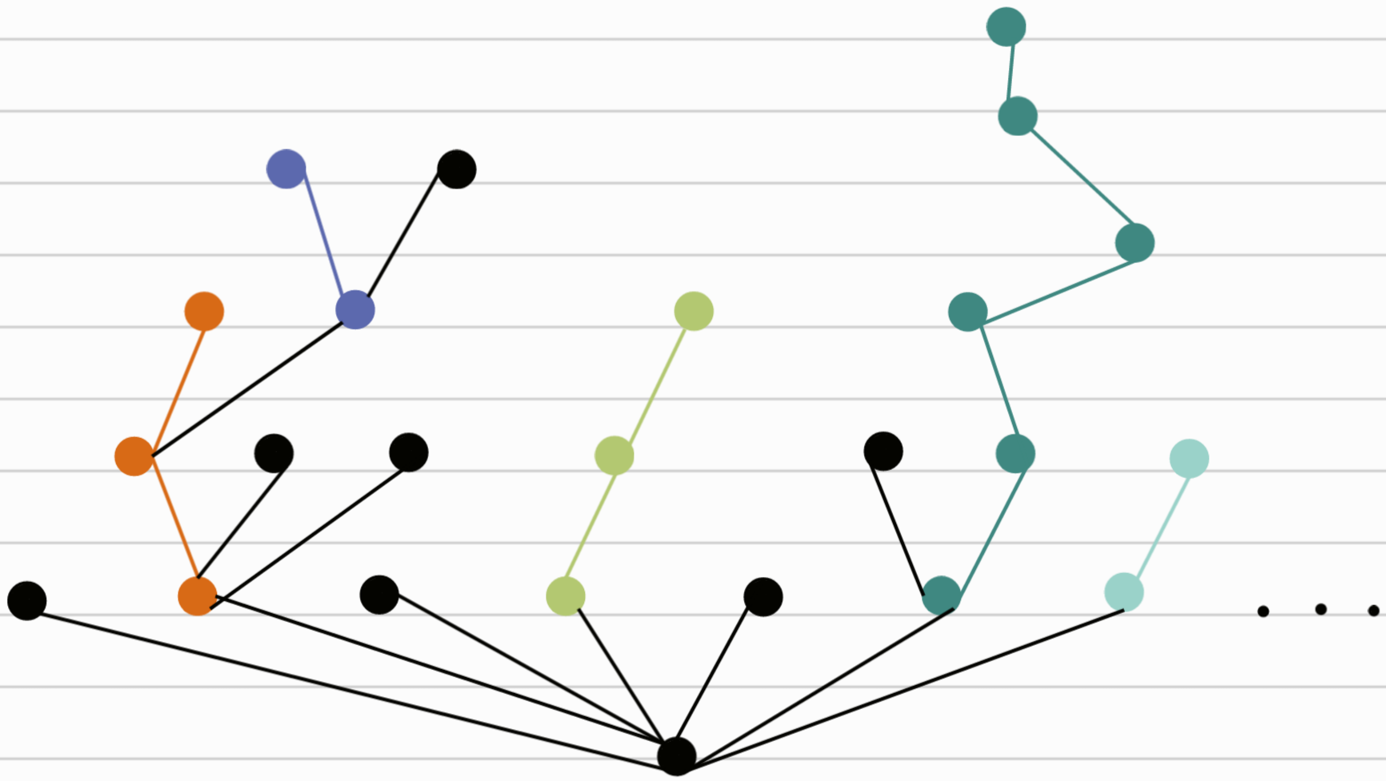
Recall that l_2 is the completion of c_{00} under the norm $\|\cdot\|_2$.

We want to d define something equivalent for $c_{00}(\mathbb{N}^{<\mathbb{N}})$, for which we need the following definitions.

An interval on $\mathbb{N}^{<\mathbb{N}}$ is a set $I = [s,t] = \{r \in \mathbb{N}^{<\mathbb{N}} : s \preceq r \preceq t\}$, where $s \preceq t$, $s, t \in \mathbb{N}^{<\mathbb{N}}$.

The key definition is the following: $(I_j : j \le k)$ is an <u>admissible choice of intervals</u> if every branch intersects at most one I_j.

a.c.i.



is an interval,

{ } is not admissible,

{ } is admissible.

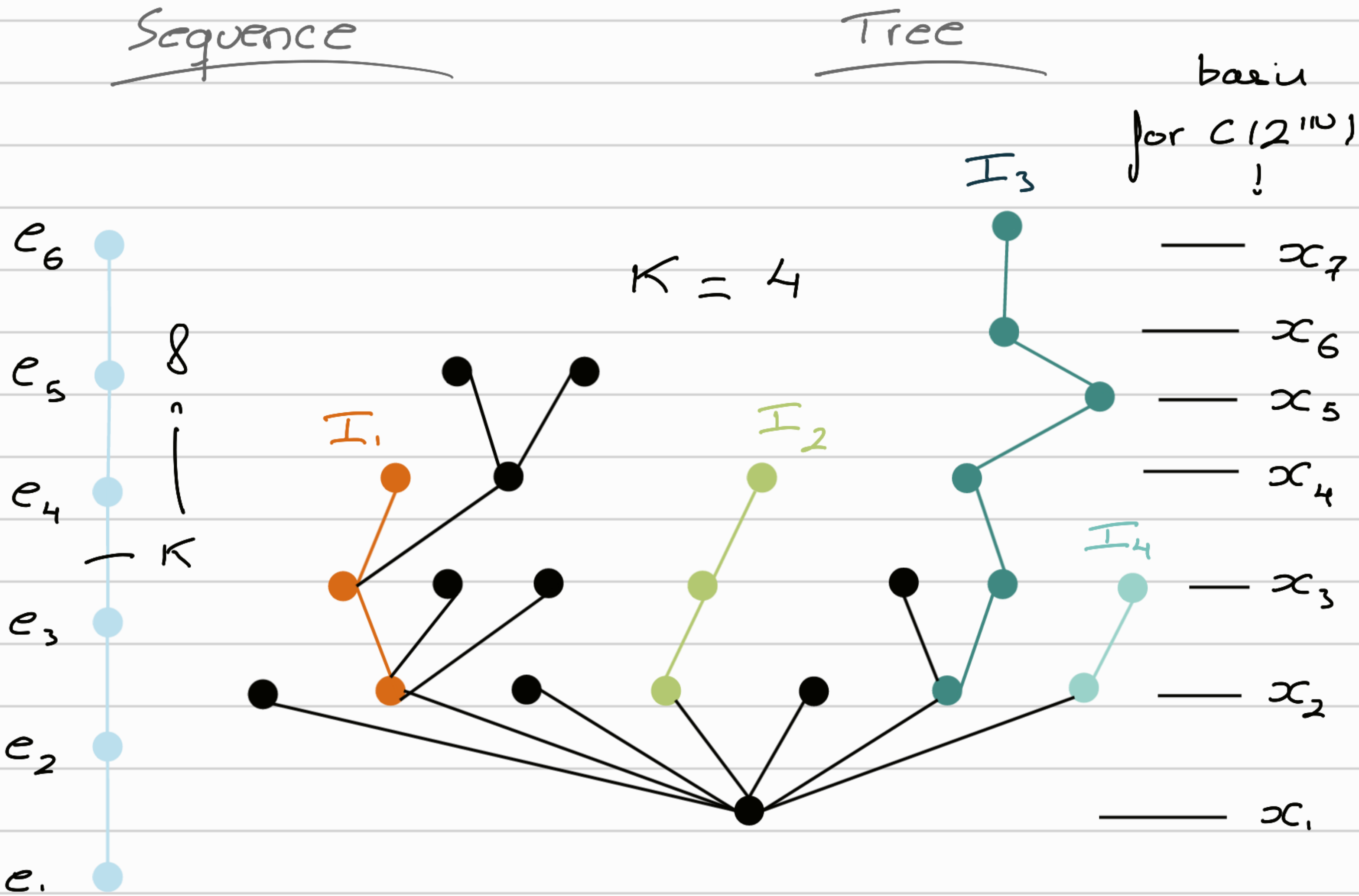
Consider the norm

$$\boxed{\|y\|_2 = \sup_{\substack{K\in\mathbb{N} \\ (I_j : j\le K) \text{ a.c.i.}}} \left(\sum_{j=1}^{K} \Big\| \sum_{s\in I_j} y(s)\, x_{(s)} \Big\|^2 \right)^{1/2}}$$

Lets compare this with the usual norm $\|\cdot\|_2$.

$$\|y\|_2 = \Big(\sum_{i}^{\infty} |y(i)|^2\Big)^{1/2} = \Big(\sum_{i}^{\infty} \|y(i)\, e_i\|^2\Big)^{1/2}$$

$\sup_{K \in \mathbb{N}} \left(\sum_i^K \| y(i) e_i \|^2 \right)^{1/2}$, so here the sup over $K \in \mathbb{N}$ tells us go far we go in the sequence, while in the Tree space it tells us how many branches we consider and the intervals tell us how much of those branches we consider.



For each $A \subset \mathbb{N}^{<\mathbb{N}}$ we define $\Phi(A)$ the $\|\cdot\|_2$ - completion of

$$\operatorname{span}(\{ \chi_s : s \in A \}).$$

We want to proof that for $T \in Tr$ we have $\Phi(T)$ reflexive when $T \in WF$ and universal when $T \in IF$. We need to prove three lemmas.

Lemma. For any $A \subset \mathbb{N}^{<\mathbb{N}}$ the set $\{x_s : s \in A\}$ is a basis in $\Phi(A)$.

Lemma. If $b = \{b_j : j \in \mathbb{N}\} \subset \mathbb{N}^{<\mathbb{N}}$ is a branch, then $(x_{b_j} : j \in \mathbb{N})$ is equivalent to $(x_n)_{n \in \mathbb{N}}$.

This two lemmas imply that for any $T \in IF$ the space $\Phi(T)$ contains $C(2^{\mathbb{N}})$. For the other direction we need the following.

Lemma. Let $(A_i)_{i \in \omega}$ be subsets of $\mathbb{N}^{<\mathbb{N}}$ such that any branch intersects at most one A_i. Then

$$\Phi\Big(\bigcup_{i \in \omega} A_i\Big) \simeq \bigoplus^2_{i \in \omega} \Phi(A_i).$$

Fix two bijections $\mathbb{N} \to \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, $i \to s_i$, $s \to \underline{s}$ such that $s \sqsubset t \to \underline{s} \le \underline{t}$.

- <u>Proof of lemma 1.</u> Let $(\lambda_i)_{i \le M}$ and $N < M$.

we need to prove $\|\sum_i^N \lambda_i x_{s_i}\|_2 \le K \|\sum_i^M \lambda_i$

$x_{s_i} \|_2$. K will be the constant of the basis $(x_n)_{n\in\mathbb{N}}$. Let I be an interval.

$$\Big\| \sum_{s\in I} \Big(\sum_i^N \lambda_i \chi_{s_i} \Big)(s)\, x_{(s)} \Big\|^{(2)} = \Big\| \sum_{\substack{s\in I \\ \underline{s}\leq N}} \lambda_{\underline{s}}\, x_{(s)} \Big\|$$

$$\leq K \Big\| \sum_{\substack{s\in I \\ \underline{s}\leq M}} \lambda_{\underline{s}}\, x_{(s)} \Big\| = K \Big\|^{(2)} \sum_{s\in I} \Big(\sum_i^M \lambda_i \chi_{s_i} \Big)(s)$$

$x_{(s)} \|$. Then for any a.c.i. $(I_j : j \leq K)$:

$$\sum_j^K \Big\| \sum_{s\in I_j} \Big(\sum_i^N \lambda_i \chi_{s_i} \Big)(s)\, x_{(s)} \Big\|^2 \leq K^2 \cdot$$

$$\sum_j^K \Big\| \sum_{s\in I_j} \Big(\sum_i^M \lambda_i \chi_{s_i} \Big)(s)\, x_{(s)} \Big\|^2.$$ Since

this holds for any a.c.i. we can take the sup and conclude.

- <u>Proof of lemma 2.</u> We need to prove that $(x_n)_{n\in\mathbb{N}}$ and $(\chi_{b_j} : j\in\mathbb{N})$ are equivalent. Take $(\lambda_i)_i^n \subset \mathbb{R}$. We want

To check $\|\sum_i^n \lambda_i x_i\| \leq \|\sum_i^n \lambda_i x_{b_i}\| \leq 2K \|\sum_i^n \lambda_i x_i\|$. We have

$$\|\sum \lambda_i x_{b_i}\| =$$

$$\sup\{\|\sum_{s\in I} (\sum_i^n \lambda_i x_{b_i})(s) x_{|s|}\| : I \subset b \text{ int}\} =$$

$$\sup\{\|\sum_{j=m}^{l} (\sum_i^n \lambda_i x_{b_i})(b_j) x_{|b_j|}\| : 1 \leq m \leq l \leq n\} =$$

$$\sup\{\|\sum_{j=m}^{l} \lambda_j x_j\| : 1 \leq m \leq l \leq n\},$$ proving

the first inequality. For the other

$$\|\sum_i^n \lambda_i x_{b_i}\| = \|\sum_{j=m}^{l} \lambda_j x_j\| = \|P_{\{1,\dots,l\}}(\sum_{i=1}^n \lambda_i x_i) - P_{\{1,\dots,m\}}(\sum_{i=1}^n \lambda_i x_i)\| = \|P_{\{1,\dots,l\}} - P_{\{1,\dots,m\}}(\sum_{i=1}^n \lambda_i x_i)\| \leq \|P_{\{1,\dots,l\}} - P_{\{1,\dots,m\}}\| \cdot \|\sum_{i=1}^n \lambda_i x_i\| \leq (\|P_{\{1,\dots,l\}}\| + \|P_{\{1,\dots,m\}}\|) \|\dots\|$$

$$= 2K \|\sum_{i=1}^n \lambda_i x_i\|.$$

Proof of lemma 3

We define a linear isometry on

$$J : sp(\{x_s : s \in \bigcup_{i \in \omega} A_i\}) \to \bigoplus_{i\in\omega} \Phi(A_i).$$

Since both spaces are dense this extends to an isometry on $\Phi(\bigcup A_i) \to \bigoplus^2_{i\in\omega} \Phi(A_i)$. (Use continuity of the norm).

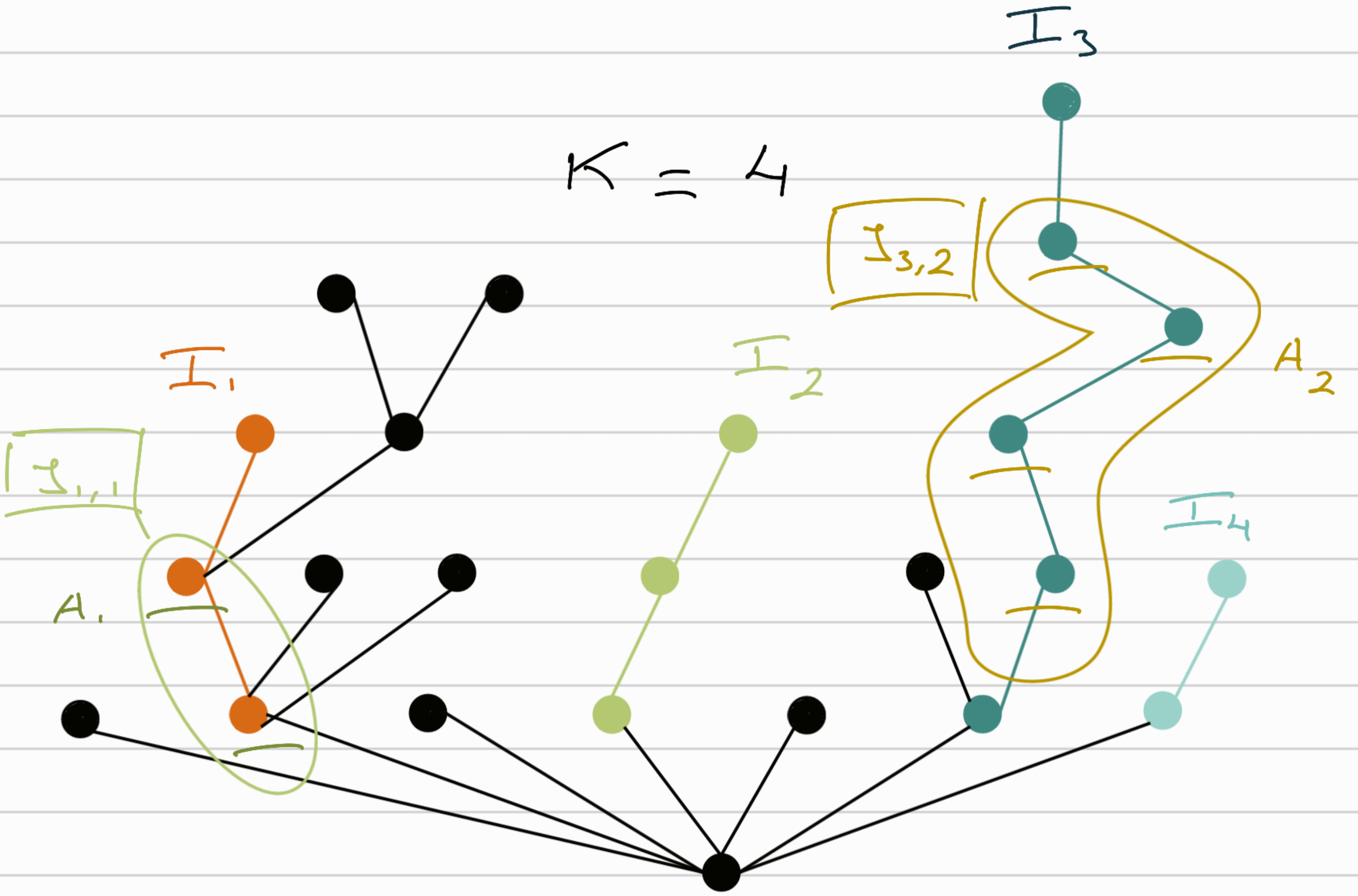
If $y \in sp(\{x_s : s \in \bigcup_{i\in\omega} A_i\})$ $\exists m \in \mathbb{N}$ such that $y(s) = 0$ $\forall s \in A_n$, $n \geq m$. We can decompose $y = \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} \sum_{s\in A_i} y(s)\, x_s$ providing the definition

$$J(y) = \sum_{i=1}^{m} y_i .$$

Now we prove $\|y\|_2^2 = \sum_{i=1}^{m} \|y_i\|^2$.

Consider $\{I_j : j \leq K\}$ a.c.i. For each y_i denote $M_i = \{j \leq K : I_j \cap A_i \neq \phi\}$ and

For $i \le m$ and $j \in M_i$ let $\mathcal{I}_{i,j} \subset I_j$ be the bigger interval with end-points on A_i.



For each y_i the set $\{\mathcal{I}_{i,j} : j \in M_i\}$ is an a.c.i. Then:

$$\sum_j^K \| \sum_{s \in I_j} y(s) x_{(s)} \|^2 = \sum_i^m \sum_{j \in M_i} \| \sum_{s \in \mathcal{I}_{i,j}} y_i(s) x_{(s)} \|^2 \le \sum_i^m \| y_i \|^2 .$$

For the other inequality let, for each y_i, $\{ I_j^i : j \leq K_i \}$ be an a.c.i. Consider $\mathcal{J}_j^i \subset I_j^i$ the biggest interval with ends in A_i. Then $\{ \mathcal{J}_j^i : j \leq K_i , i \leq m \}$ is an a.c.i.

We have

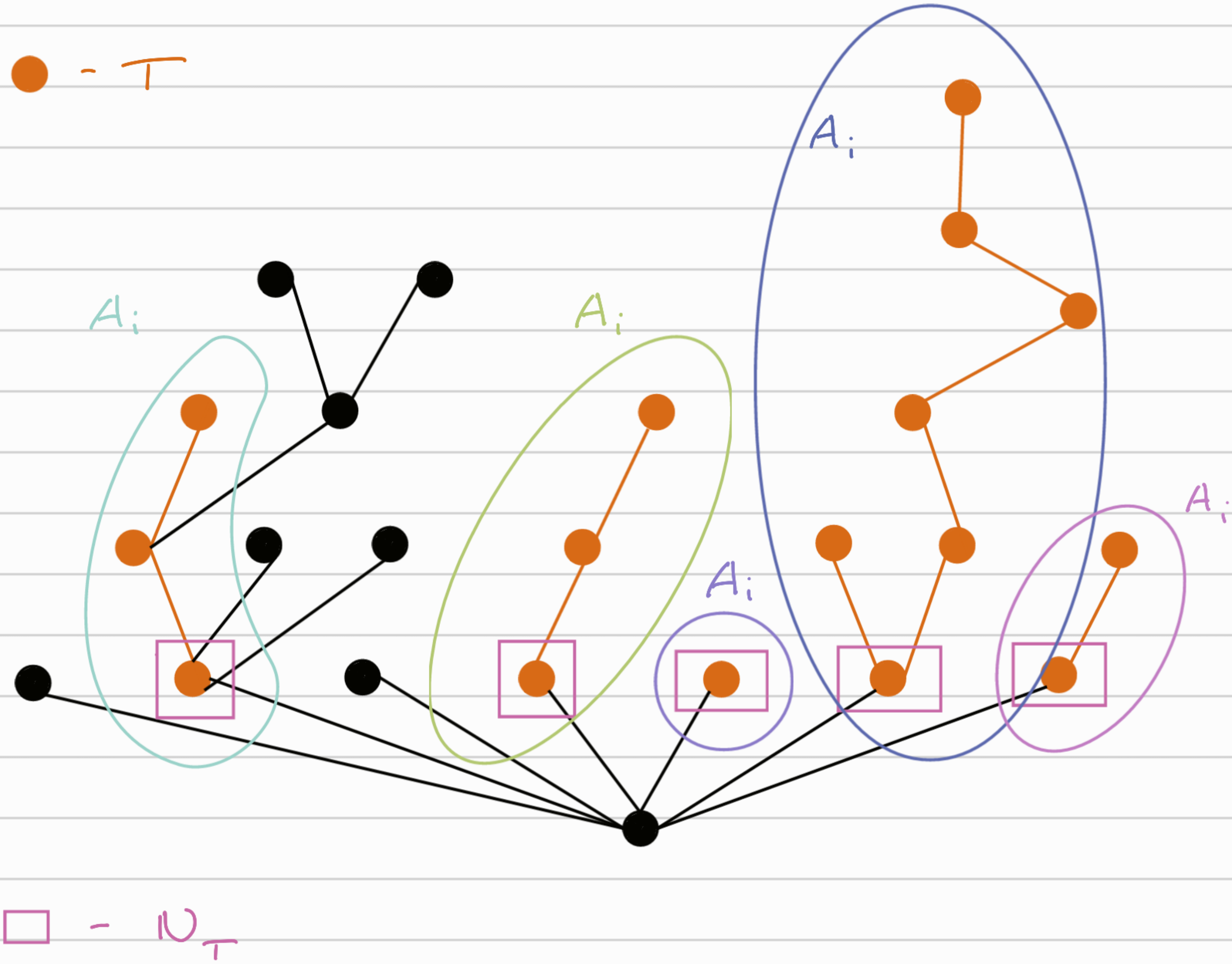
$$\sum_j^{K_i} \Big\| \sum_{s \in I_j^i} y_i(s)\, x_{|s|} \Big\|^2 = \sum_j^{K_i} \Big\| \sum_{s \in \mathcal{J}_j^i} y_i(s) \cdot x_{|s|} \Big\|^2 = \sum_j^{K_i} \Big\| \sum_{s \in \mathcal{J}_j^i} y(s)\, x_{|s|} \Big\| ,$$ then

$$\sum_i^m \sum_j^{K_i} \Big\| \sum_{s \in I_j^i} y_i(s)\, x_{|s|} \Big\|^2 = \sum_i^m \sum_j^{K_i} \Big\| \sum_{s \in \mathcal{J}_j^i} y(s) \cdot x_{|s|} \Big\|^2 \leq \| y \|_2^2 .$$

Taking the sup over all a.c.i. $\displaystyle\sum_i^m \| y_i \|^2 \leq \| y \|_2^2 .$

Proof of $T \in WF \to \Phi(T)$ reflexive.

By induction on the height of the tree, suppose that result holds for all trees of height $< \alpha$ and take $ht(T) = \alpha$. Consider $N_T = \{ i \in \omega : (i) \in T \}$ and $A_i = \{ t \in T : t(0) = (i) \}$. Note that $T = \bigcup_{i \in \omega} A_i$ and $\Phi(A_i)$ has $\{ \chi_{\{i\}} \} \cup \{ \chi_s : s \in A_i \}$ as a basis, in particular $\Phi(A_i) \simeq \mathbb{R} \times \Phi(A_i \setminus \{(i)\})$. Since $ht(\Phi(A_i \setminus \{(i)\}) < \alpha$, it is reflexive and so is $\Phi(A_i)$. Then so is $\bigoplus^2_{i \in \omega} \Phi(A_i) \cong \Phi(\bigcup_{i \in N_T} A_i) = \Phi(T)$.



- T
A_i
A_i
A_i
A_i
A_i
- U_T