PRESENTATION_IDEALS_AND_BANACH_SPACES

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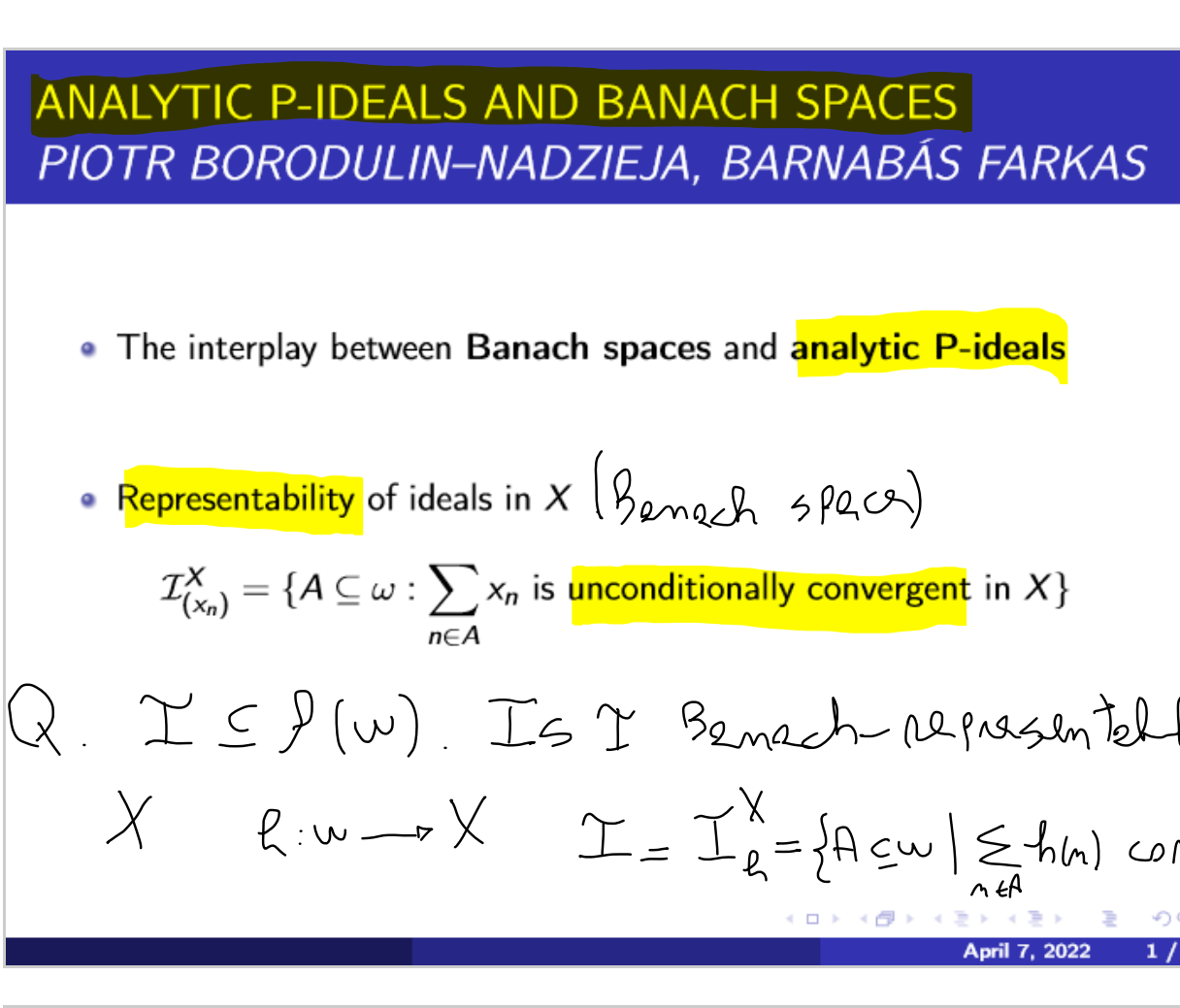
PRESENTATION_IDEAL...

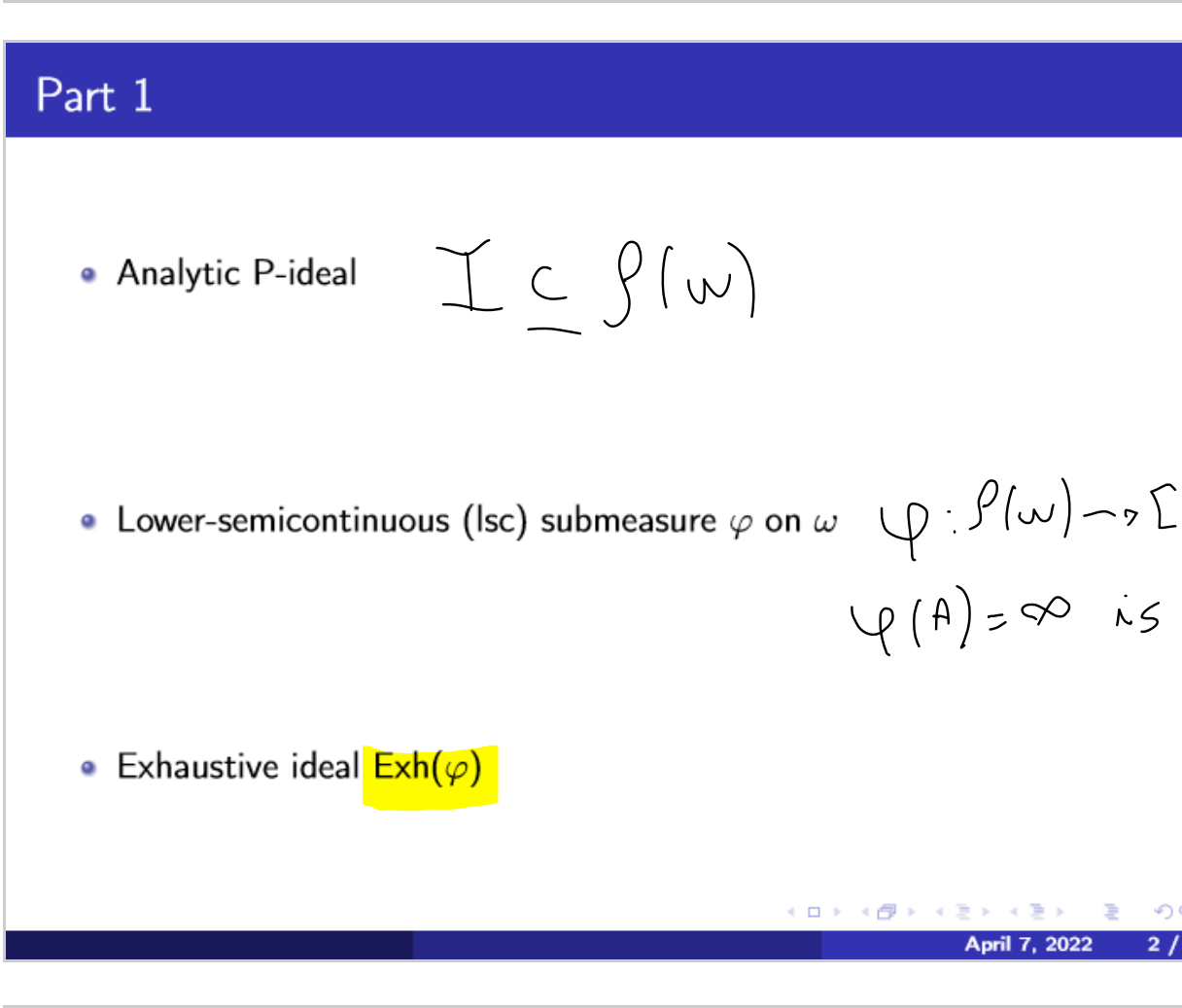


ANALYTIC P-IDEALS AND BANACH SPACES
PIOTR BORODULIN–NADZIEJA, BARNABÁS FARKAS

- The interplay between **Banach spaces** and **analytic P-ideals**
- Representability of ideals in X (Banach space)
  $\mathcal{I}^X_{(x_n)} = \{A\subseteq\omega : \sum_{n\in A} x_n$ is unconditionally convergent in $X\}$

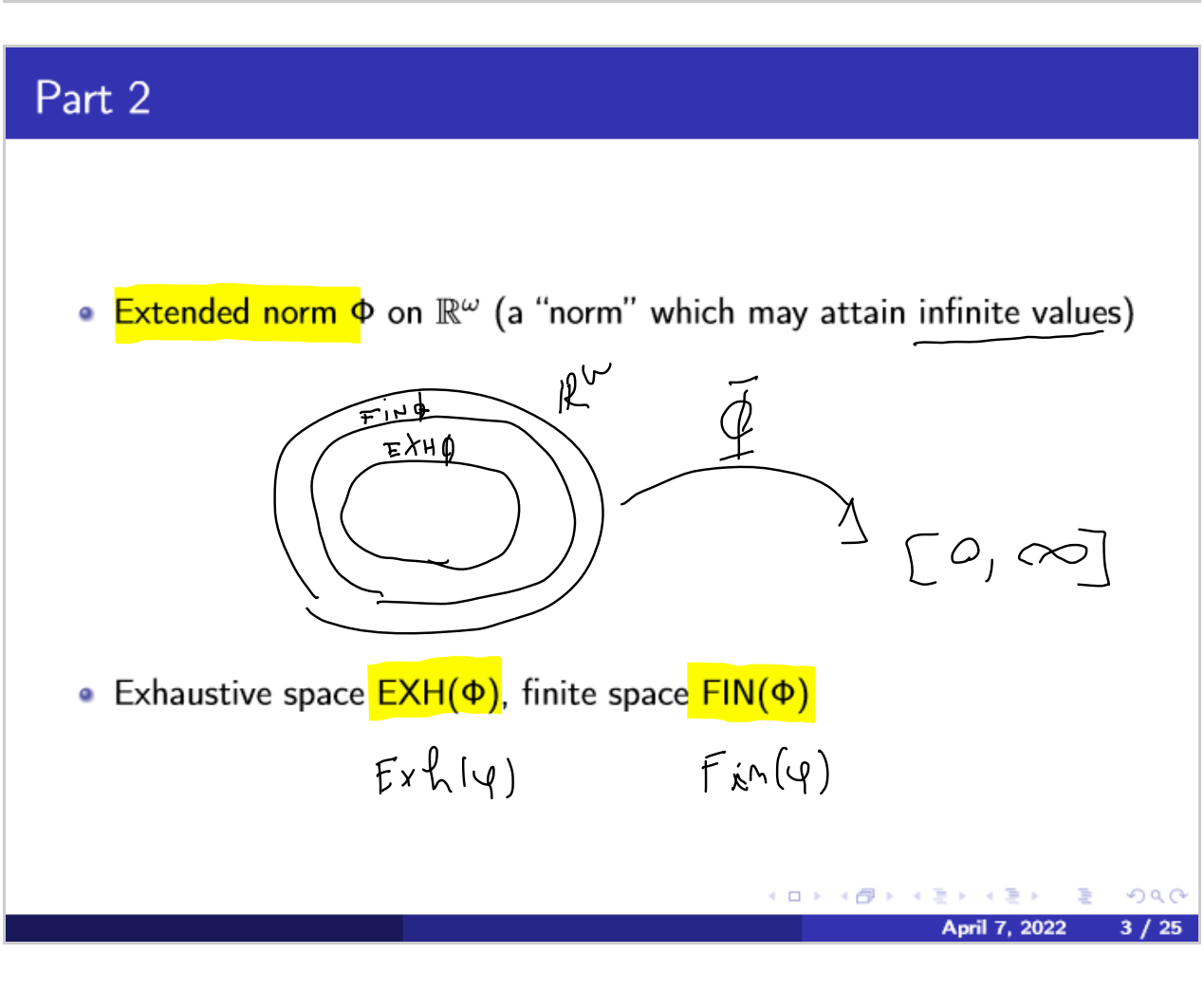
Q. $\mathcal{I}\subseteq\mathcal{P}(\omega)$. Is $\mathcal{I}$ Banach-representable?

X $h:\omega\to X$ $\mathcal{I}=\mathcal{I}^X_h=\{A\subseteq\omega \mid \sum_{n\in A} h(n)$ conv. unc. in $X\}$



Part 1

- Analytic P-ideal $\mathcal{I}\subseteq\mathcal{P}(\omega)$
- Lower-semicontinuous (lsc) submeasure φ on ω $\varphi:\mathcal{P}(\omega)\to[0,\infty]$
  $\varphi(A)=\infty$ is possible
- Exhaustive ideal Exh(φ)

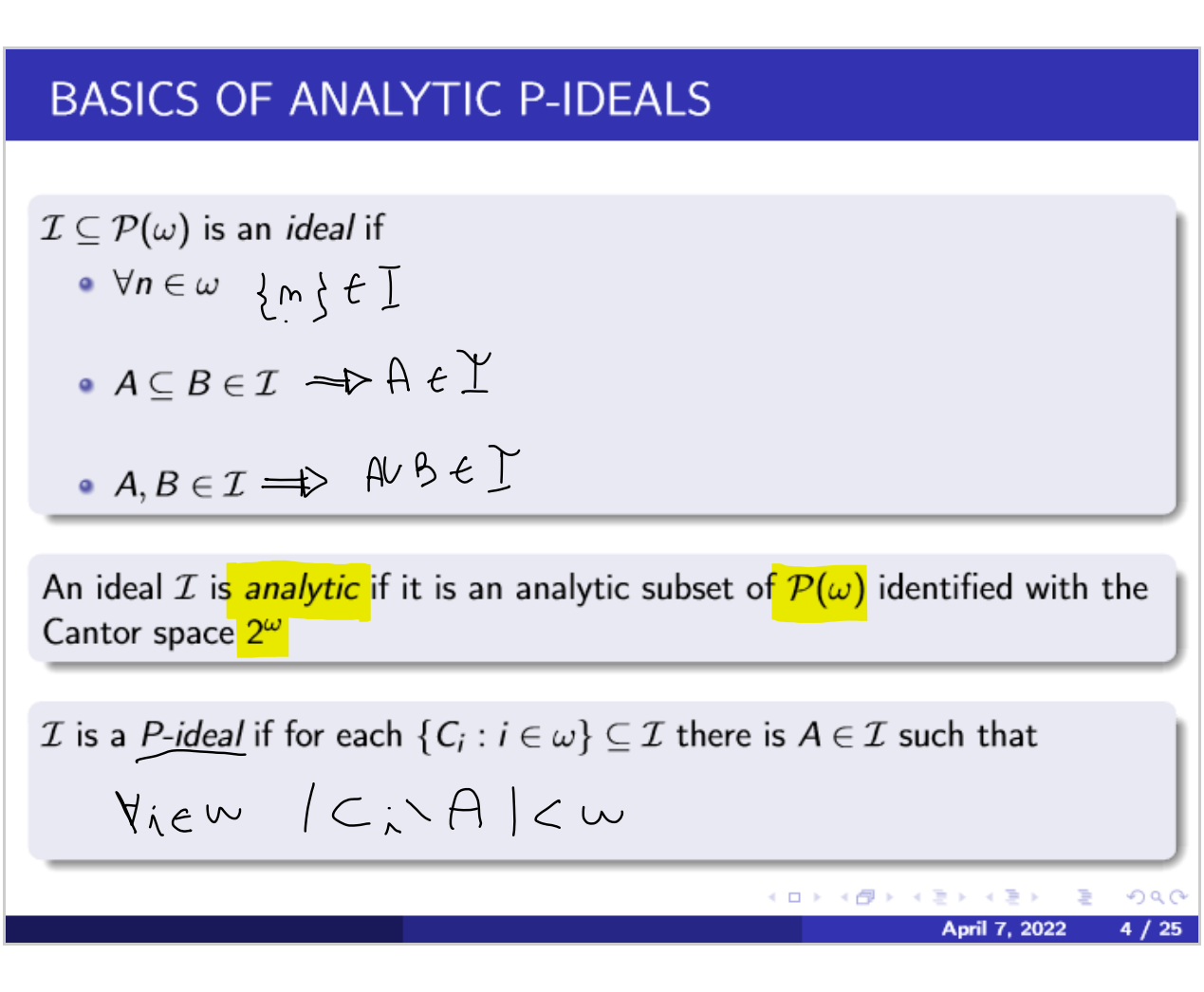


Part 2

- Extended norm Φ on $\mathbb{R}^\omega$ (a "norm" which may attain infinite values)

FINΦ
EXHΦ
$\mathbb{R}^\omega$
Φ
$[0,\infty]$

- Exhaustive space EXH(Φ), finite space FIN(Φ)
  Exh(φ) Fin(φ)



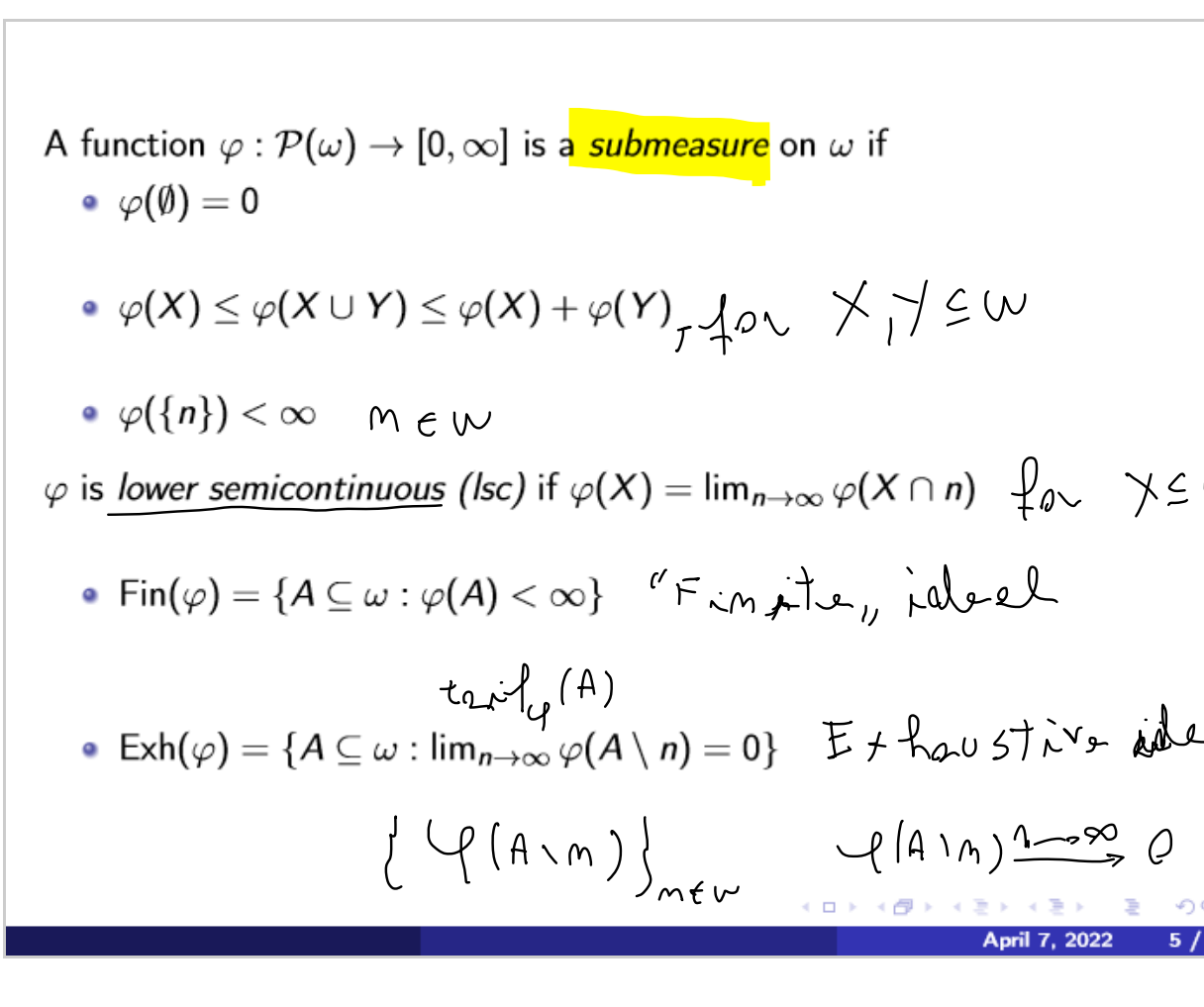
BASICS OF ANALYTIC P-IDEALS

$\mathcal{I}\subseteq\mathcal{P}(\omega)$ is an *ideal* if
- $\forall n\in\omega$ $\{n\}\in\mathcal{I}$
- $A\subseteq B\in\mathcal{I}$ ⇒ $A\in\mathcal{I}$
- $A,B\in\mathcal{I}$ ⇒ $A\cup B\in\mathcal{I}$

An ideal $\mathcal{I}$ is *analytic* if it is an analytic subset of $\mathcal{P}(\omega)$ identified with the Cantor space 2^ω

$\mathcal{I}$ is a *P-ideal* if for each $\{C_i : i\in\omega\}\subseteq\mathcal{I}$ there is $A\in\mathcal{I}$ such that

$\forall i\in\omega$ $|C_i\setminus A|<\omega$

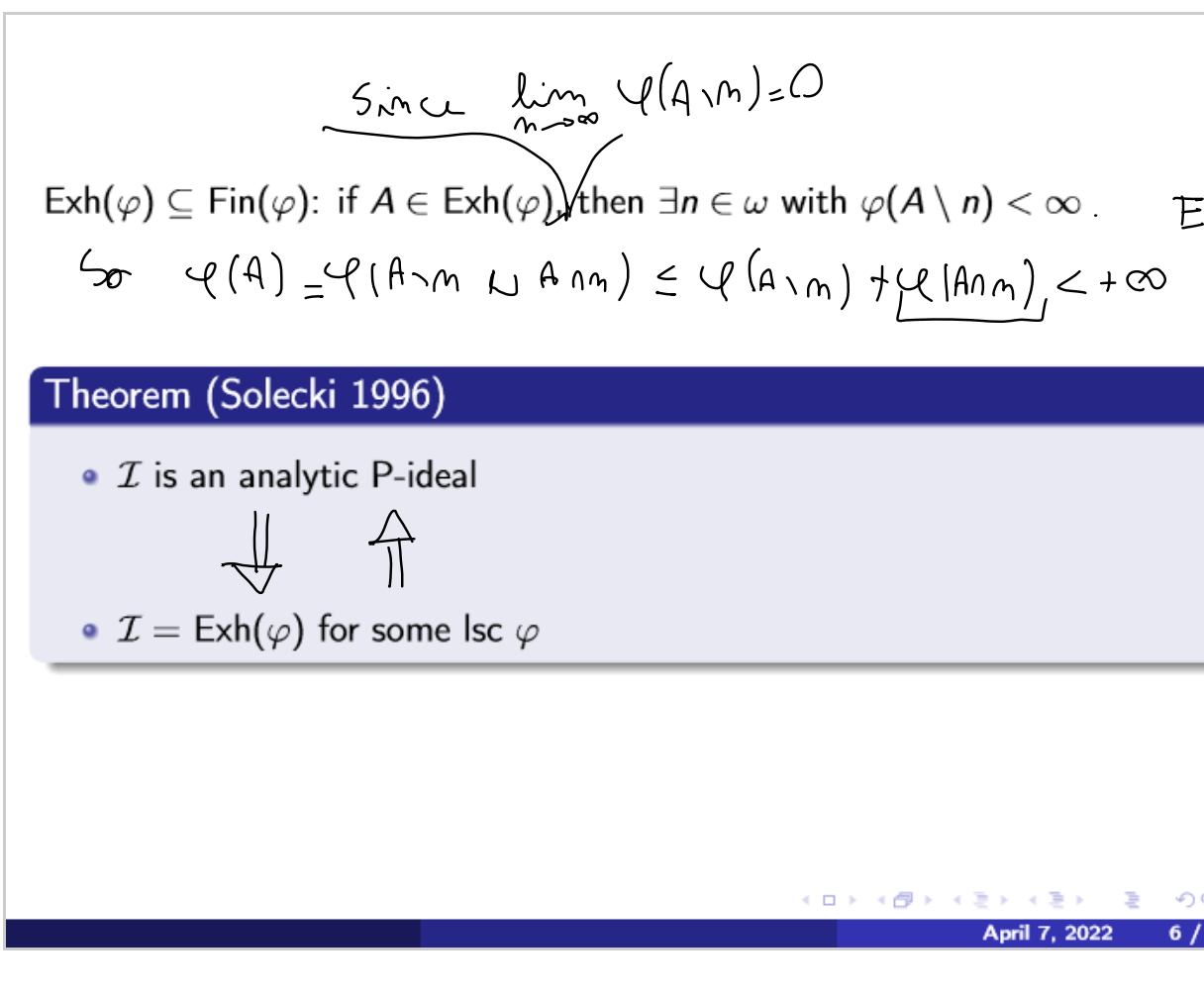


A function $\varphi:\mathcal{P}(\omega)\to[0,\infty]$ is a *submeasure* on ω if
- $\varphi(\emptyset)=0$
- $\varphi(X)\le\varphi(X\cup Y)\le\varphi(X)+\varphi(Y)$, for $X,Y\subseteq\omega$
- $\varphi(\{n\})<\infty$ $n\in\omega$

φ is *lower semicontinuous* (lsc) if $\varphi(X)=\lim_{n\to\infty}\varphi(X\cap n)$ for $X\subseteq\omega$
- $\mathrm{Fin}(\varphi)=\{A\subseteq\omega:\varphi(A)<\infty\}$ "Finite" ideal
- $\mathrm{Exh}(\varphi)=\{A\subseteq\omega:\lim_{n\to\infty}\overset{\mathrm{tail}_\varphi(A)}{\varphi(A\setminus n)}=0\}$ Exhaustive ideal
  $\{\varphi(A\setminus n)\}_{n\in\omega}$ $\varphi(A\setminus n)\xrightarrow{n\to\infty}0$

$n=\{0,1,\dots n-1\}$

$A\setminus n=A\setminus\{0,1,\dots n-1\}$



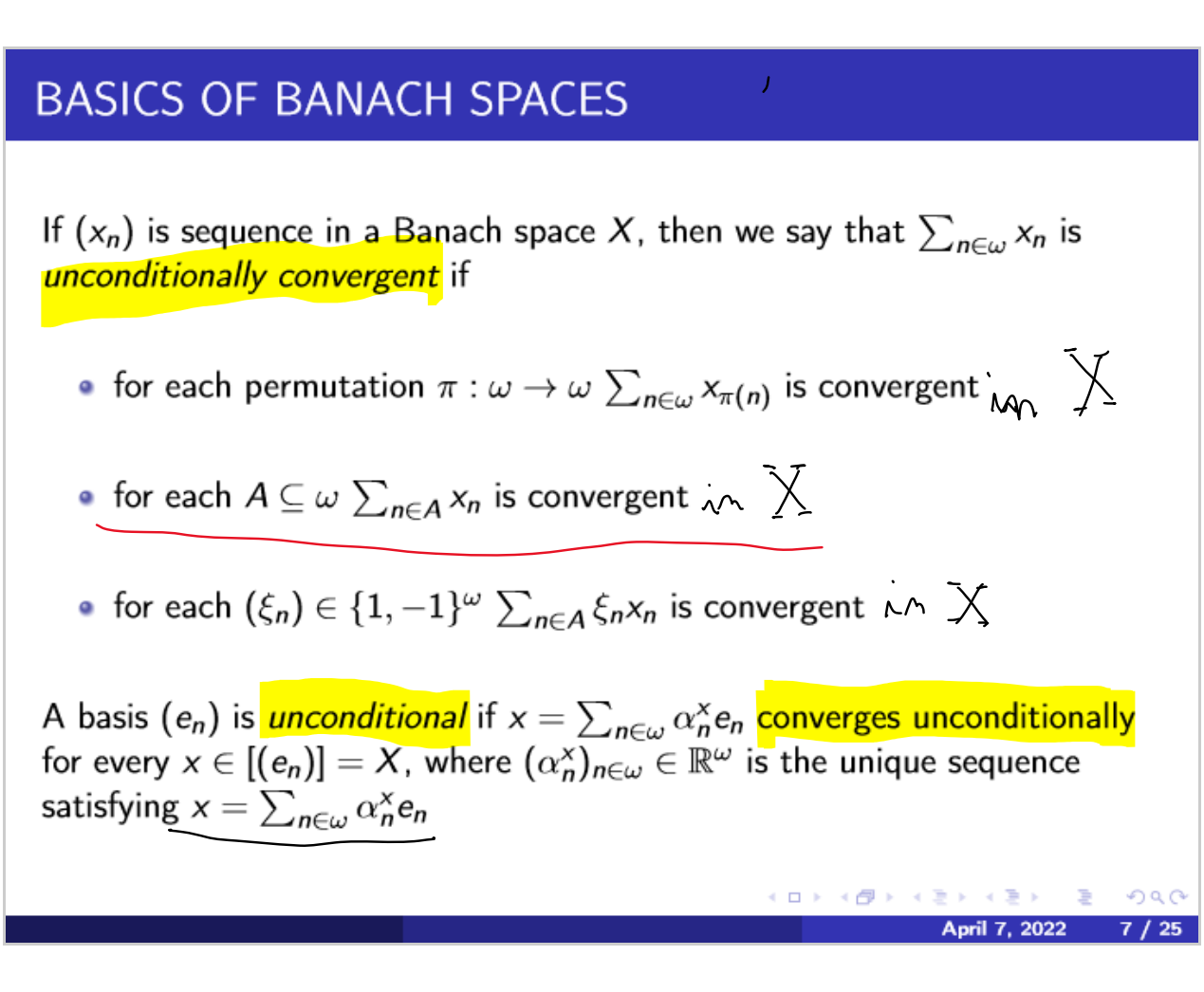
Since $\lim_{n\to\infty}\varphi(A\setminus n)=0$

Exh(φ) $\subseteq$ Fin(φ): if $A\in\mathrm{Exh}(\varphi)$ then $\exists n\in\omega$ with $\varphi(A\setminus n)<\infty$.

So $\varphi(A)=\varphi(A\cap n\cup A\setminus n)\le\varphi(A\cap n)+\varphi(A\setminus n)<+\infty$

Theorem (Solecki 1996)
- $\mathcal{I}$ is an analytic P-ideal
  ⇓ ⇑
- $\mathcal{I}=\mathrm{Exh}(\varphi)$ for some lsc φ

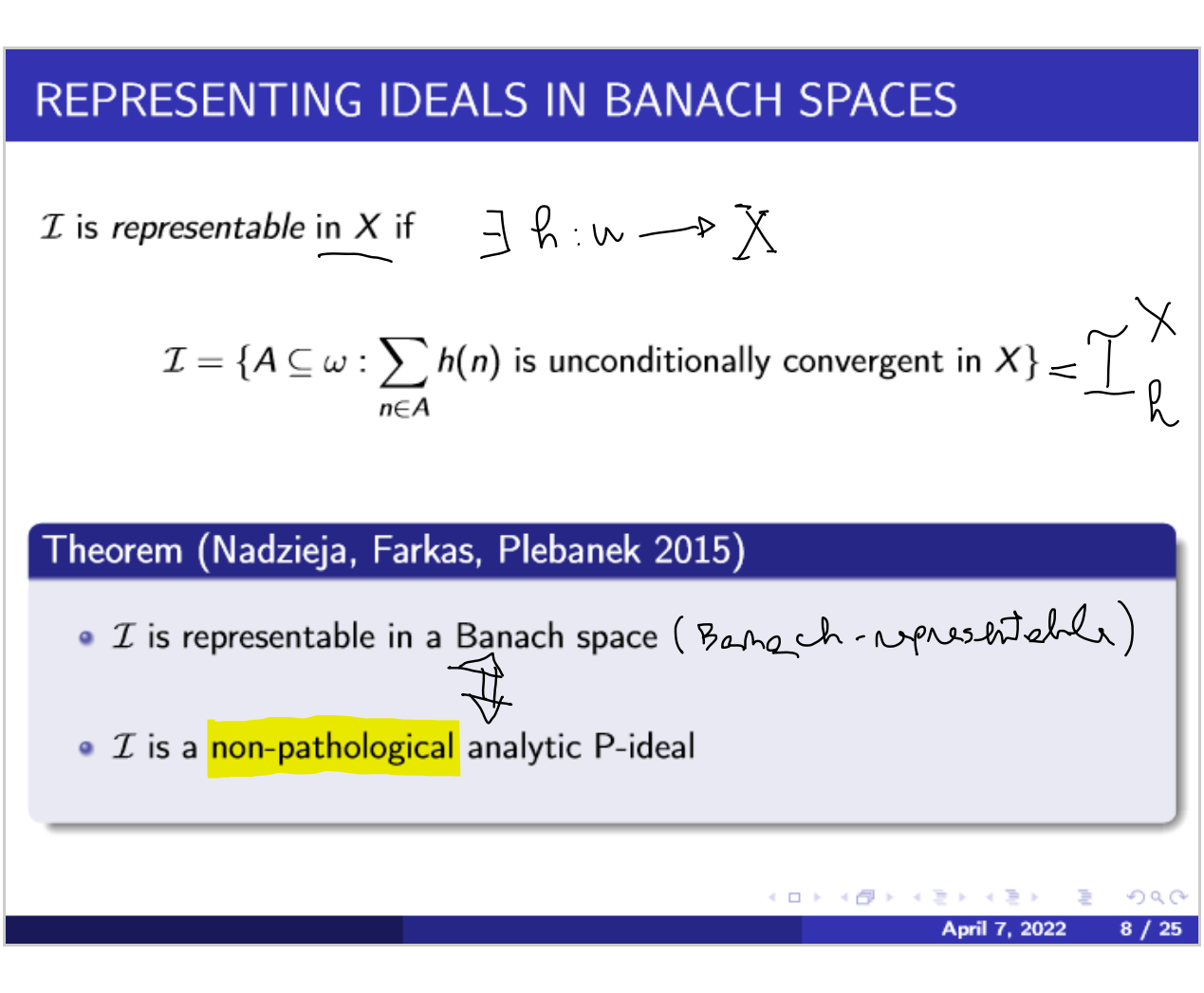
Exh(φ) $\subseteq$ Fin(φ)



BASICS OF BANACH SPACES

If (x_n) is sequence in a Banach space X, then we say that $\sum_{n\in\omega}x_n$ is *unconditionally convergent* if
- for each permutation $\pi:\omega\to\omega$ $\sum_{n\in\omega}x_{\pi(n)}$ is convergent in X
- for each $A\subseteq\omega$ $\sum_{n\in A}x_n$ is convergent in X
- for each $(\xi_n)\in\{1,-1\}^\omega$ $\sum_{n\in A}\xi_nx_n$ is convergent in X

A basis (e_n) is *unconditional* if $x=\sum_{n\in\omega}\alpha^x_ne_n$ converges unconditionally for every $x\in[(e_n)]=X$, where $(\alpha^x_n)_{n\in\omega}\in\mathbb{R}^\omega$ is the unique sequence satisfying $x=\sum_{n\in\omega}\alpha^x_ne_n$



REPRESENTING IDEALS IN BANACH SPACES

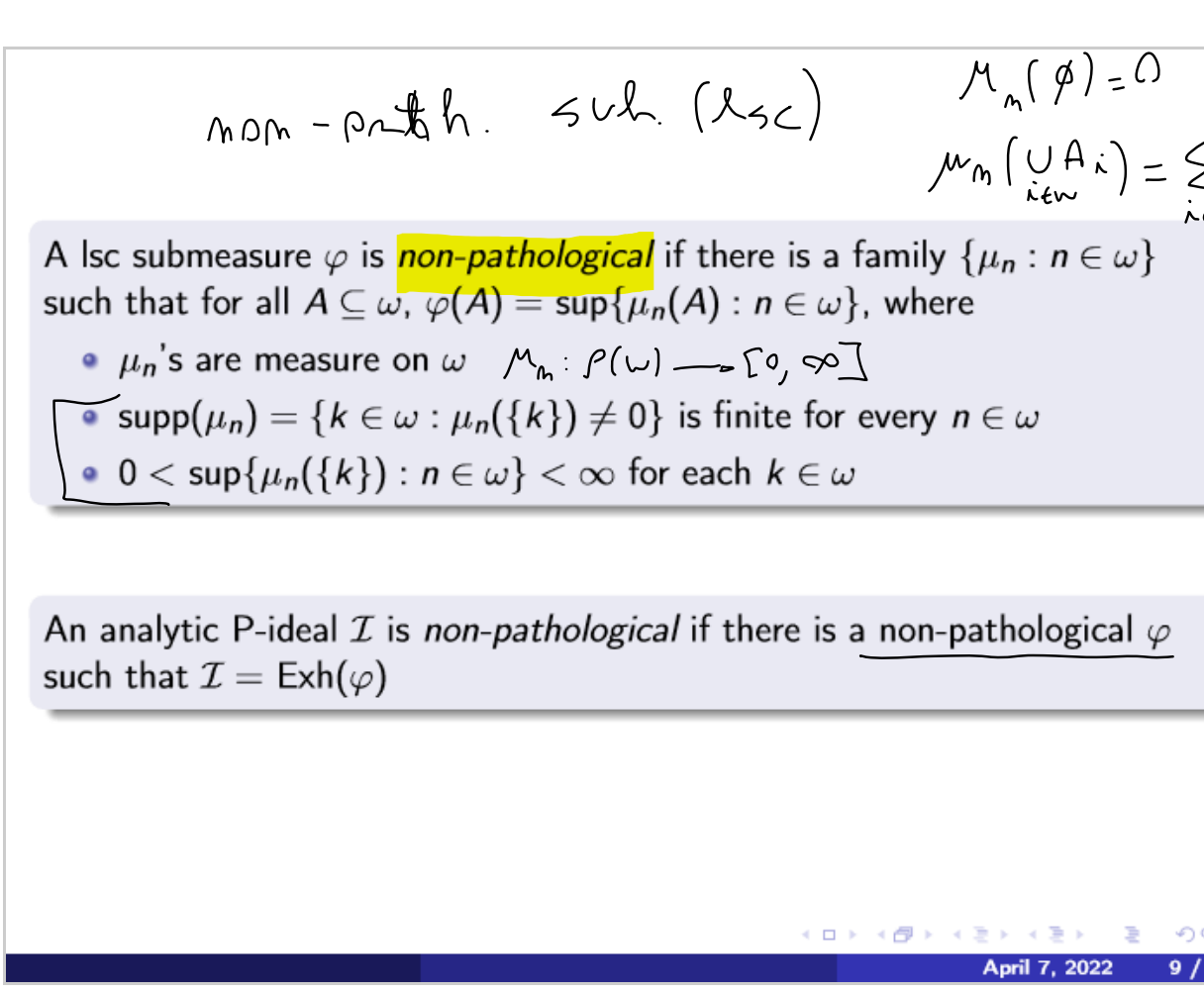
$\mathcal{I}$ is *representable* in X if $\exists h:\omega\to X$

$$\mathcal{I}=\{A\subseteq\omega:\sum_{n\in A}h(n)\text{ is unconditionally convergent in }X\}=\mathcal{I}^X_h$$

Theorem (Nadzieja, Farkas, Plebanek 2015)
- $\mathcal{I}$ is representable in a Banach space (Banach-representable)
  ⇕
- $\mathcal{I}$ is a non-pathological analytic P-ideal

$\mathcal{I}_{1/n}:=\{A\subseteq\omega \mid \sum_{n\in A}\frac{1}{n+1}<+\infty\}$ sum ideal

$\mathcal{I}_{1/n}$ is representable in each Bach space



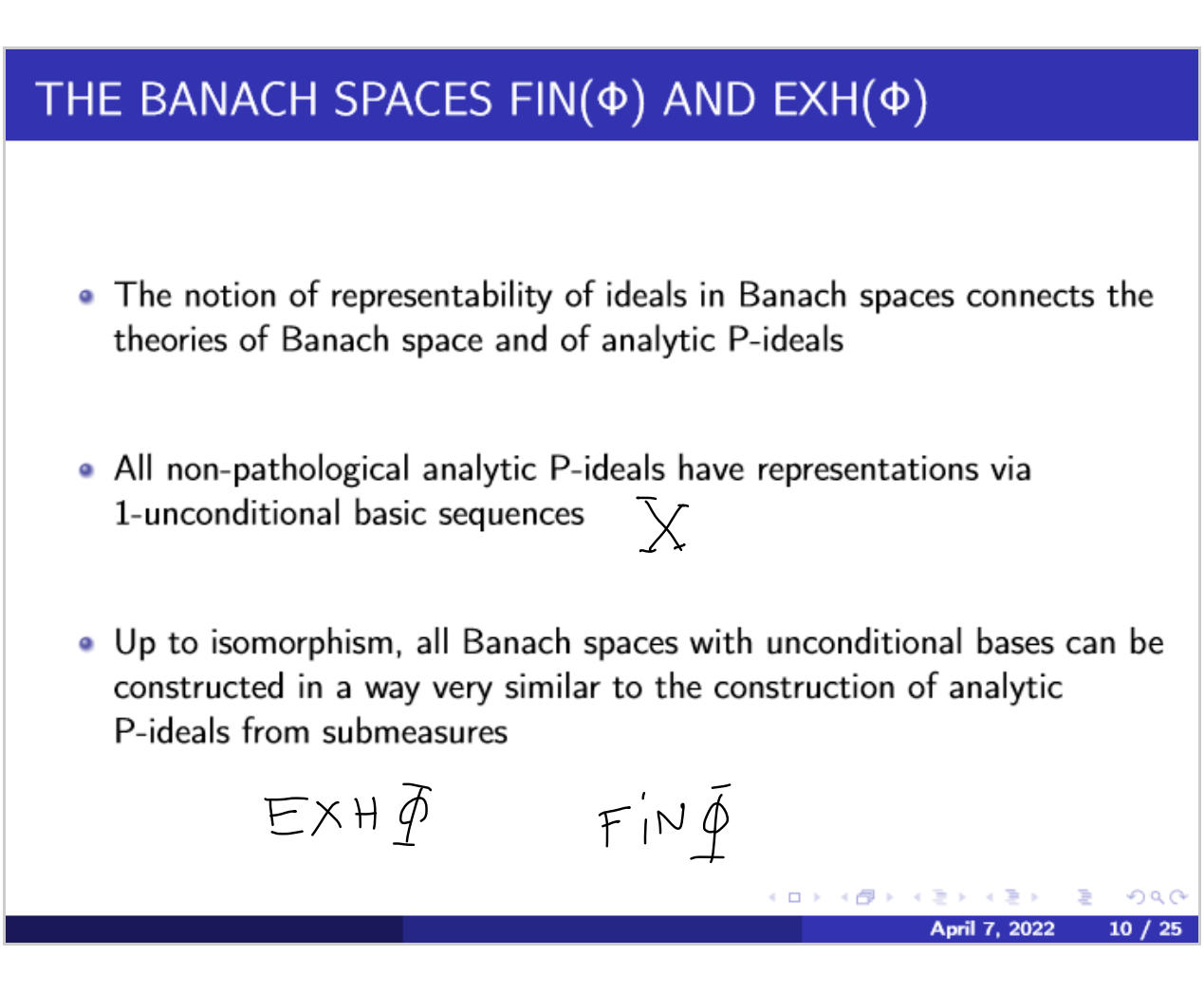
non-path. sub. (lsc)

$\mu_n(\emptyset)=0$

$\mu_n(\bigcup_{i\in\omega}A_i)=\sum_{i\in\omega}\mu_n(A_i)$

A lsc submeasure φ is *non-pathological* if there is a family $\{\mu_n:n\in\omega\}$ such that for all $A\subseteq\omega$, $\varphi(A)=\sup\{\mu_n(A):n\in\omega\}$, where
- μ_n's are measure on ω $\mu_n:\mathcal{P}(\omega)\to[0,\infty]$
- $\mathrm{supp}(\mu_n)=\{k\in\omega:\mu_n(\{k\})\neq0\}$ is finite for every $n\in\omega$
- $0<\sup\{\mu_n(\{k\}):n\in\omega\}<\infty$ for each $k\in\omega$

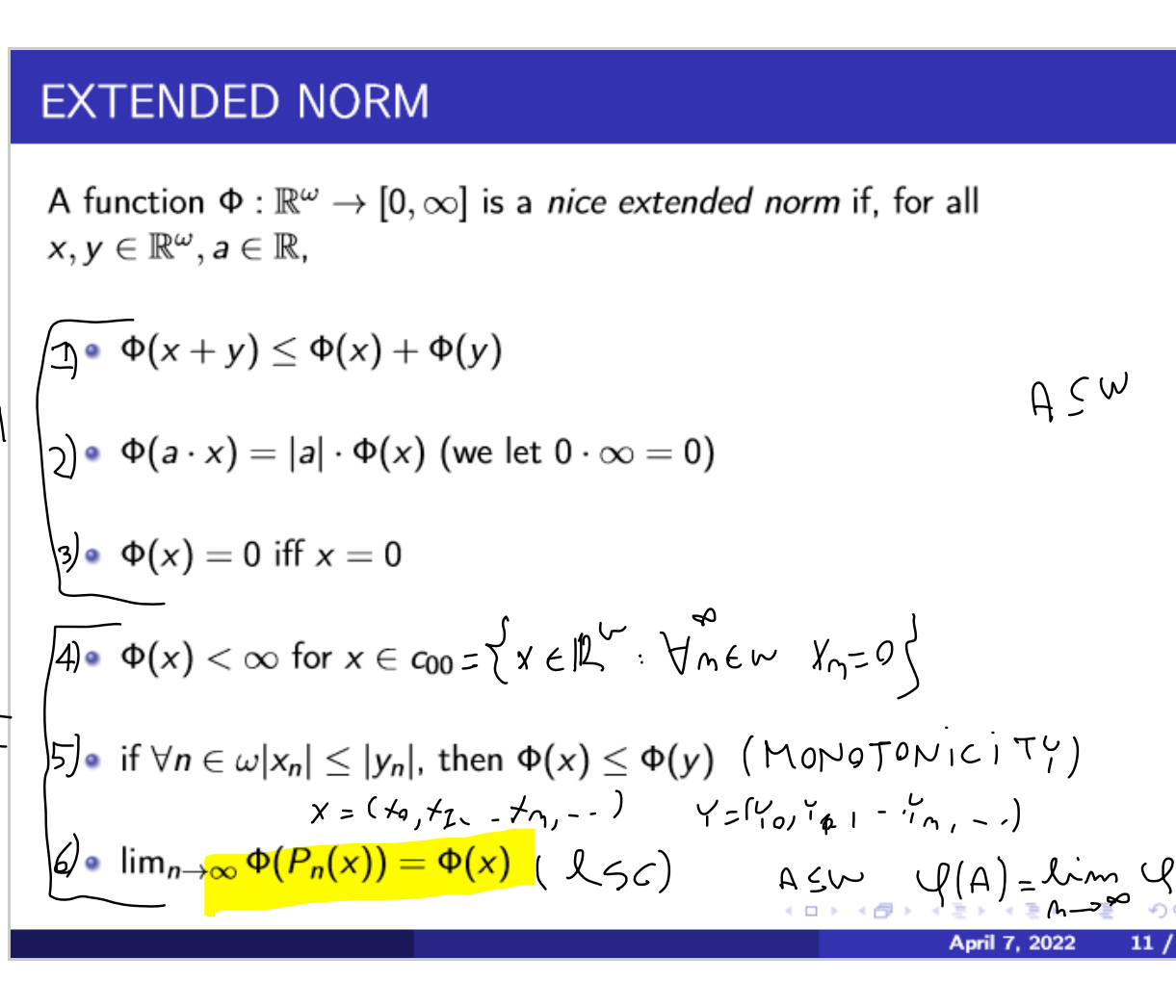
An analytic P-ideal $\mathcal{I}$ is *non-pathological* if there is a non-pathological φ such that $\mathcal{I}=\mathrm{Exh}(\varphi)$



THE BANACH SPACES FIN(Φ) AND EXH(Φ)

- The notion of representability of ideals in Banach spaces connects the theories of Banach space and of analytic P-ideals
- All non-pathological analytic P-ideals have representations via 1-unconditional basic sequences X
- Up to isomorphism, all Banach spaces with unconditional bases can be constructed in a way very similar to the construction of analytic P-ideals from submeasures

EXHΦ FINΦ



EXTENDED NORM

A function $\Phi:\mathbb{R}^\omega\to[0,\infty]$ is a *nice extended norm* if, for all $x,y\in\mathbb{R}^\omega$, $a\in\mathbb{R}$,

NORM
1) $\Phi(x+y)\le\Phi(x)+\Phi(y)$
2) $\Phi(a\cdot x)=|a|\cdot\Phi(x)$ (we let $0\cdot\infty=0$)
3) $\Phi(x)=0$ iff $x=0$

NICE
4) $\Phi(x)<\infty$ for $x\in c_{00}=\{x\in\mathbb{R}^\omega:\overset{\infty}{\forall}n\in\omega\ x_n=0\}$
5) if $\forall n\in\omega|x_n|\le|y_n|$, then $\Phi(x)\le\Phi(y)$ (MONOTONICITY)
   $x=(x_0,x_1,\dots x_n,\dots)$ $y=(y_0,y_1,\dots y_n,\dots)$
6) $\lim_{n\to\infty}\Phi(P_n(x))=\Phi(x)$ (lsc) $A\subseteq\omega$ $\varphi(A)=\lim_{n\to\infty}\varphi(A\cap n)$

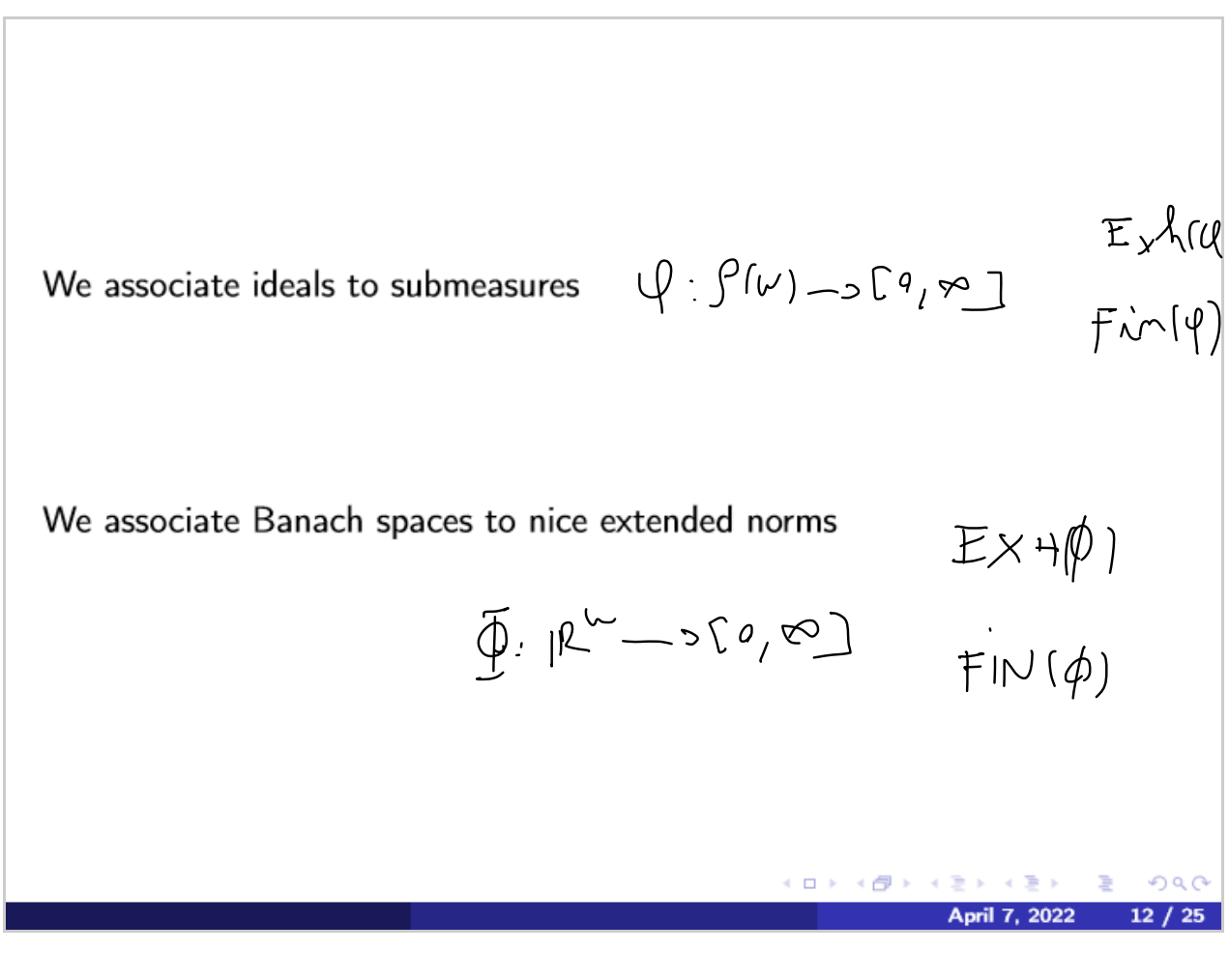
$A\subseteq\omega$

$n\in\omega$ $n=\{0,1,\dots n-1\}\subseteq\omega$

$P_A:\mathbb{R}^\omega\to\mathbb{R}^\omega$

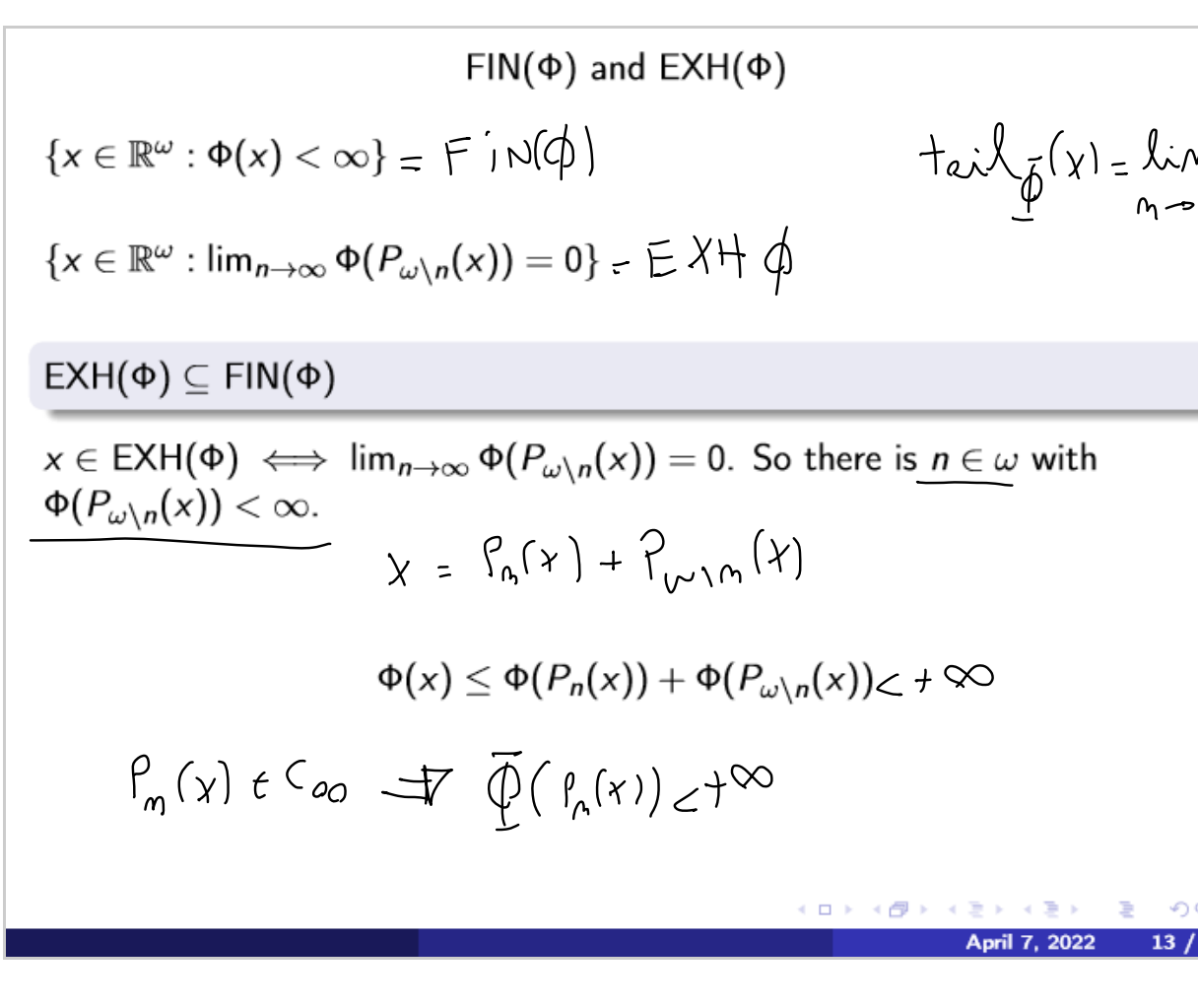
$P_A(x)(n)=\begin{cases}0 & n\notin A\\ x_n & n\in A\end{cases}$ $x=(x_0,x_1,\dots x_n,x_{n+1},\dots)\in\mathbb{R}^\omega$

$P_A(x)=(\underset{}{0},\dots\underset{i\in A}{x_i},0\dots,\underset{j\in A}{x_j},\dots0,\underset{k\in A}{x_k},\dots0)$



We associate ideals to submeasures $\varphi:\mathcal{P}(\omega)\to[0,\infty]$ Exh(φ) Fin(φ)

We associate Banach spaces to nice extended norms $\Phi:\mathbb{R}^\omega\to[0,\infty]$ EXH(Φ) FIN(Φ)



FIN(Φ) and EXH(Φ)

$\{x\in\mathbb{R}^\omega:\Phi(x)<\infty\}=\mathrm{FIN}(\Phi)$

$\{x\in\mathbb{R}^\omega:\lim_{n\to\infty}\Phi(P_{\omega\setminus n}(x))=0\}=\mathrm{EXH}\,\Phi$

$\mathrm{tail}_\Phi(x)=\lim_{n\to\infty}\Phi(P_{\omega\setminus n}(x))$

EXH(Φ) $\subseteq$ FIN(Φ)

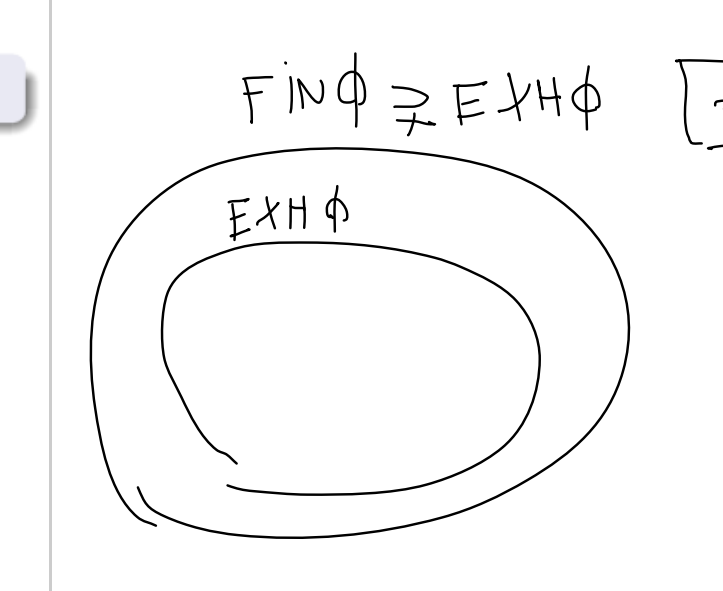
$x\in\mathrm{EXH}(\Phi)\iff\lim_{n\to\infty}\Phi(P_{\omega\setminus n}(x))=0$. So there is $n\in\omega$ with $\Phi(P_{\omega\setminus n}(x))<\infty$.

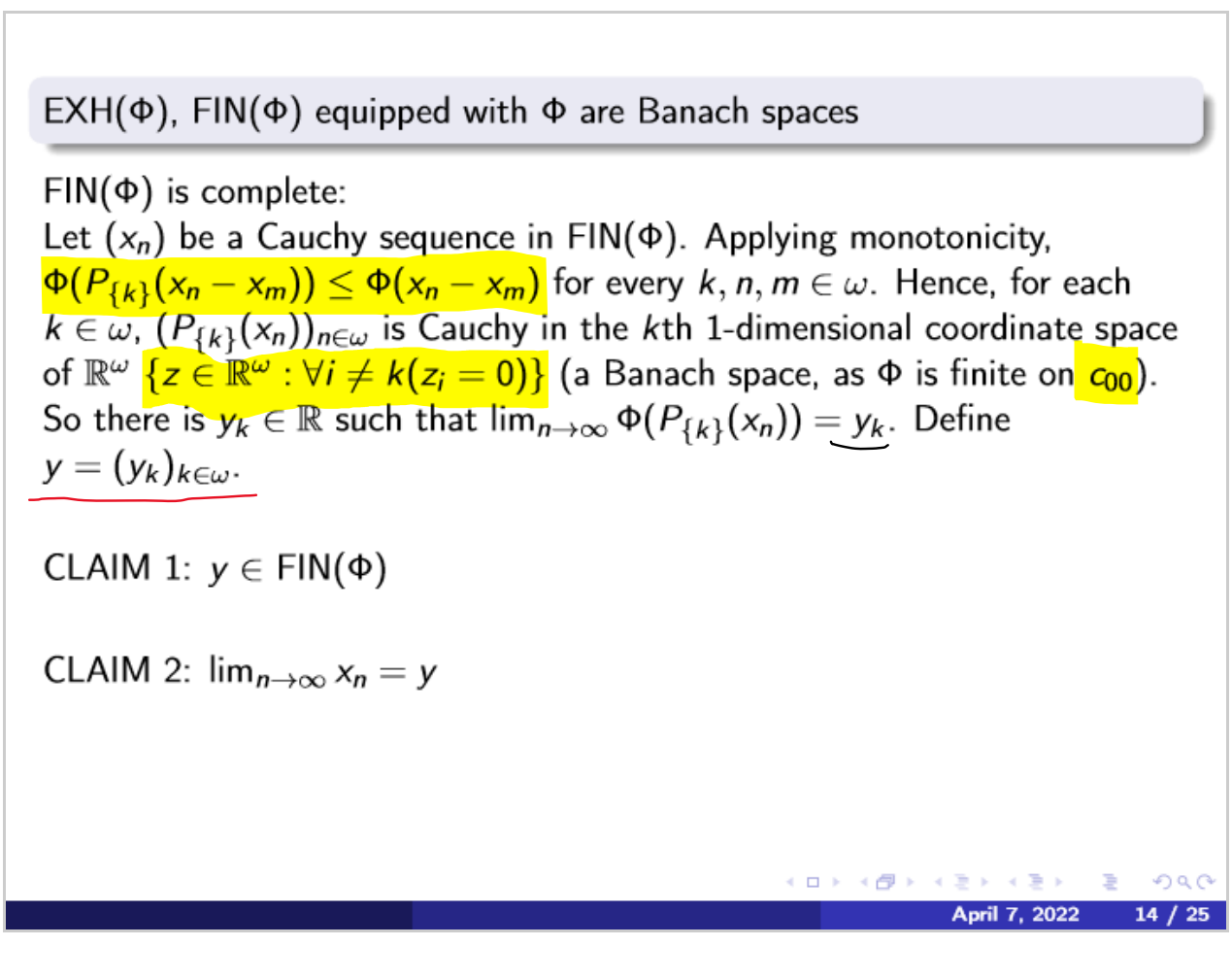
$x=P_n(x)+P_{\omega\setminus n}(x)$

$\Phi(x)\le\Phi(P_n(x))+\Phi(P_{\omega\setminus n}(x))<+\infty$

$P_n(x)\in c_{00}\Rightarrow\Phi(P_n(x))<+\infty$

FINϕ $\supsetneq$ EXHϕ $\boxed{\mathrm{FIN}\phi=\mathrm{EXH}\phi}$





EXH(Φ), FIN(Φ) equipped with Φ are Banach spaces

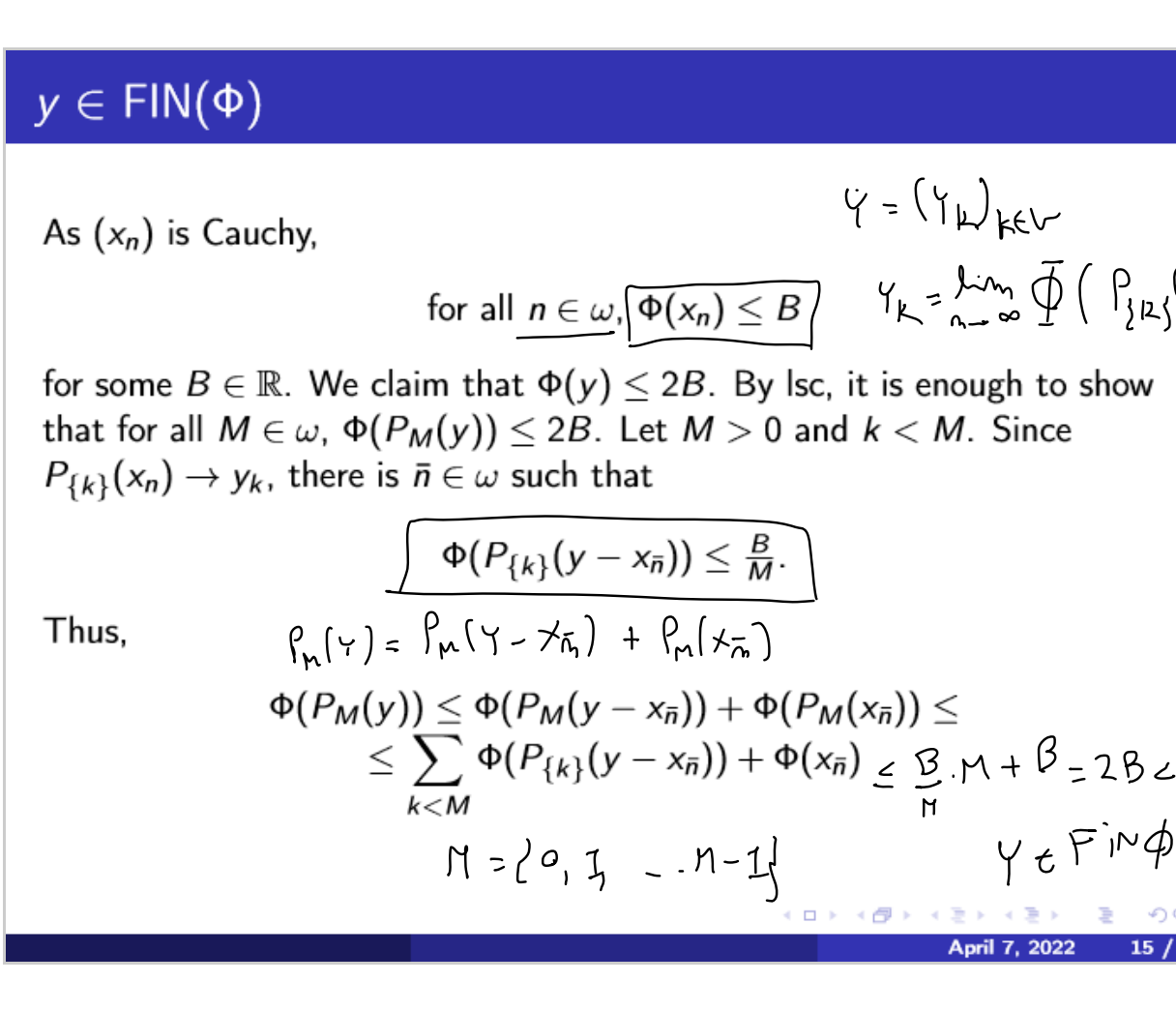
FIN(Φ) is complete:
Let (x_n) be a Cauchy sequence in FIN(Φ). Applying monotonicity, $\Phi(P_{\{k\}}(x_n-x_m))\le\Phi(x_n-x_m)$ for every $k,n,m\in\omega$. Hence, for each $k\in\omega$, $(P_{\{k\}}(x_n))_{n\in\omega}$ is Cauchy in the kth 1-dimensional coordinate space of $\mathbb{R}^\omega$ $\{z\in\mathbb{R}^\omega:\forall i\neq k(z_i=0)\}$ (a Banach space, as Φ is finite on c_{00}). So there is $y_k\in\mathbb{R}$ such that $\lim_{n\to\infty}\Phi(P_{\{k\}}(x_n))=y_k$. Define $y=(y_k)_{k\in\omega}$.

CLAIM 1: $y\in\mathrm{FIN}(\Phi)$

CLAIM 2: $\lim_{n\to\infty}x_n=y$

(FINϕ, $\phi\restriction\mathrm{FIN}\phi$)

$\Phi\restriction\mathrm{FIN}\phi$ is a norm



$y\in\mathrm{FIN}(\Phi)$

As (x_n) is Cauchy, $Y=(Y_k)_{k\in\omega}$ $Y_k=\lim_{n\to\infty}\Phi(P_{\{k\}}(x_n))$

for all $n\in\omega$, $\boxed{\Phi(x_n)\le B}$

for some $B\in\mathbb{R}$. We claim that $\Phi(y)\le2B$. By lsc, it is enough to show that for all $M\in\omega$, $\Phi(P_M(y))\le2B$. Let $M>0$ and $k<M$. Since $P_{\{k\}}(x_n)\to y_k$, there is $\bar n\in\omega$ such that

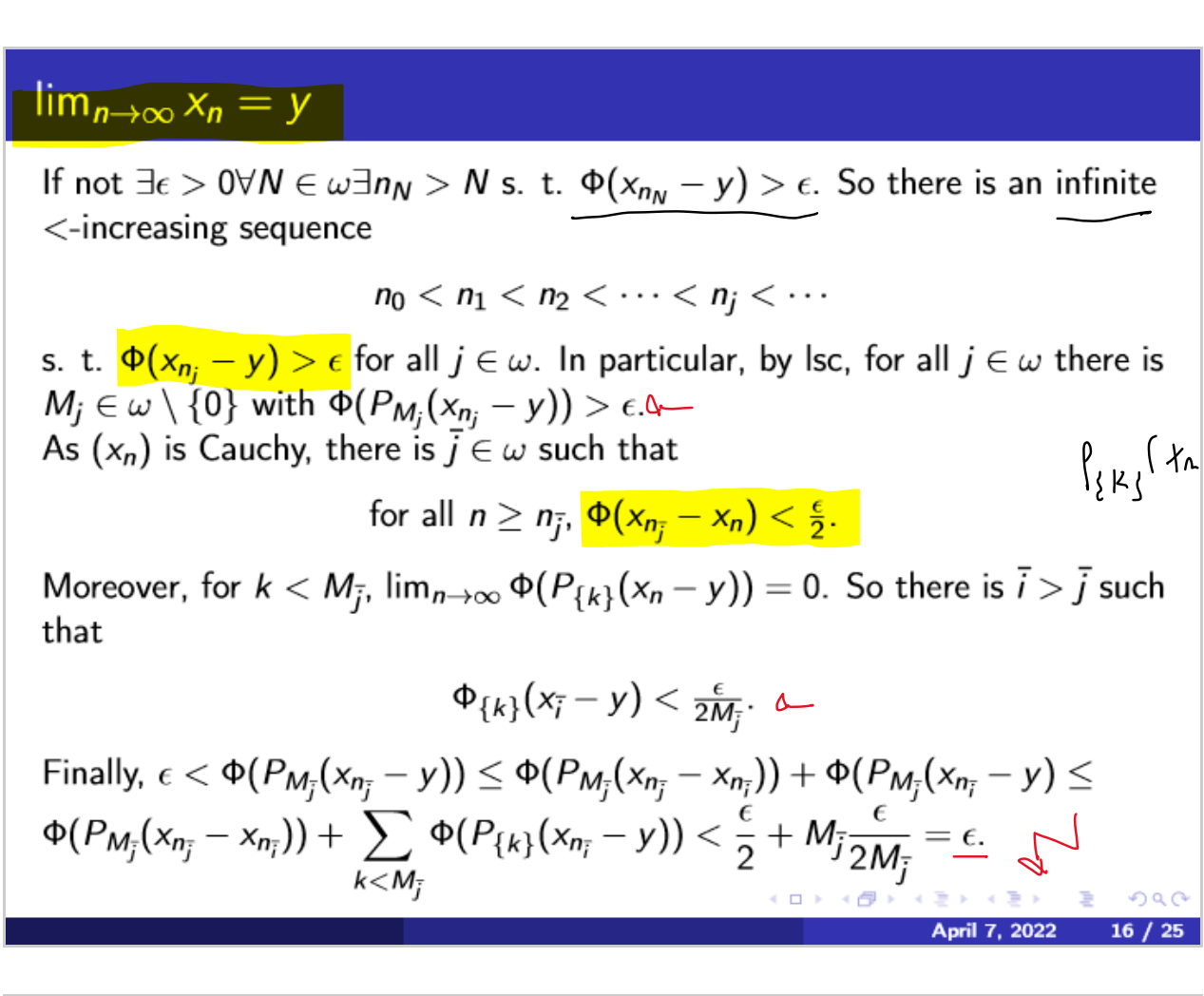
$$\Phi(P_{\{k\}}(y-x_{\bar n}))\le\frac{B}{M}.$$

Thus, $P_M(y)=P_M(y-x_{\bar n})+P_M(x_{\bar n})$

$\Phi(P_M(y))\le\Phi(P_M(y-x_{\bar n}))+\Phi(P_M(x_{\bar n}))\le$
$\le\sum_{k<M}\Phi(P_{\{k\}}(y-x_{\bar n}))+\Phi(x_{\bar n})\le\frac{B}{M}M+B=2B<+\infty$

$M=\{0,1,\dots,n-1\}$ $y\in\mathrm{FIN}\phi$

$\Phi(P_n(x))\to\Phi(x)$



$\lim_{n\to\infty}x_n=y$

If not $\exists\epsilon>0\ \forall N\in\omega\ \exists n_N>N$ s.t. $\Phi(x_{n_N}-y)>\epsilon$. So there is an infinite <-increasing sequence

$$n_0<n_1<n_2<\dots<n_j<\cdots$$

s.t. $\Phi(x_{n_j}-y)>\epsilon$ for all $j\in\omega$. In particular, by lsc, for all $j\in\omega$ there is $M_j\in\omega\setminus\{0\}$ with $\Phi(P_{M_j}(x_{n_j}-y))>\epsilon$.
As (x_n) is Cauchy, there is $\bar j\in\omega$ such that

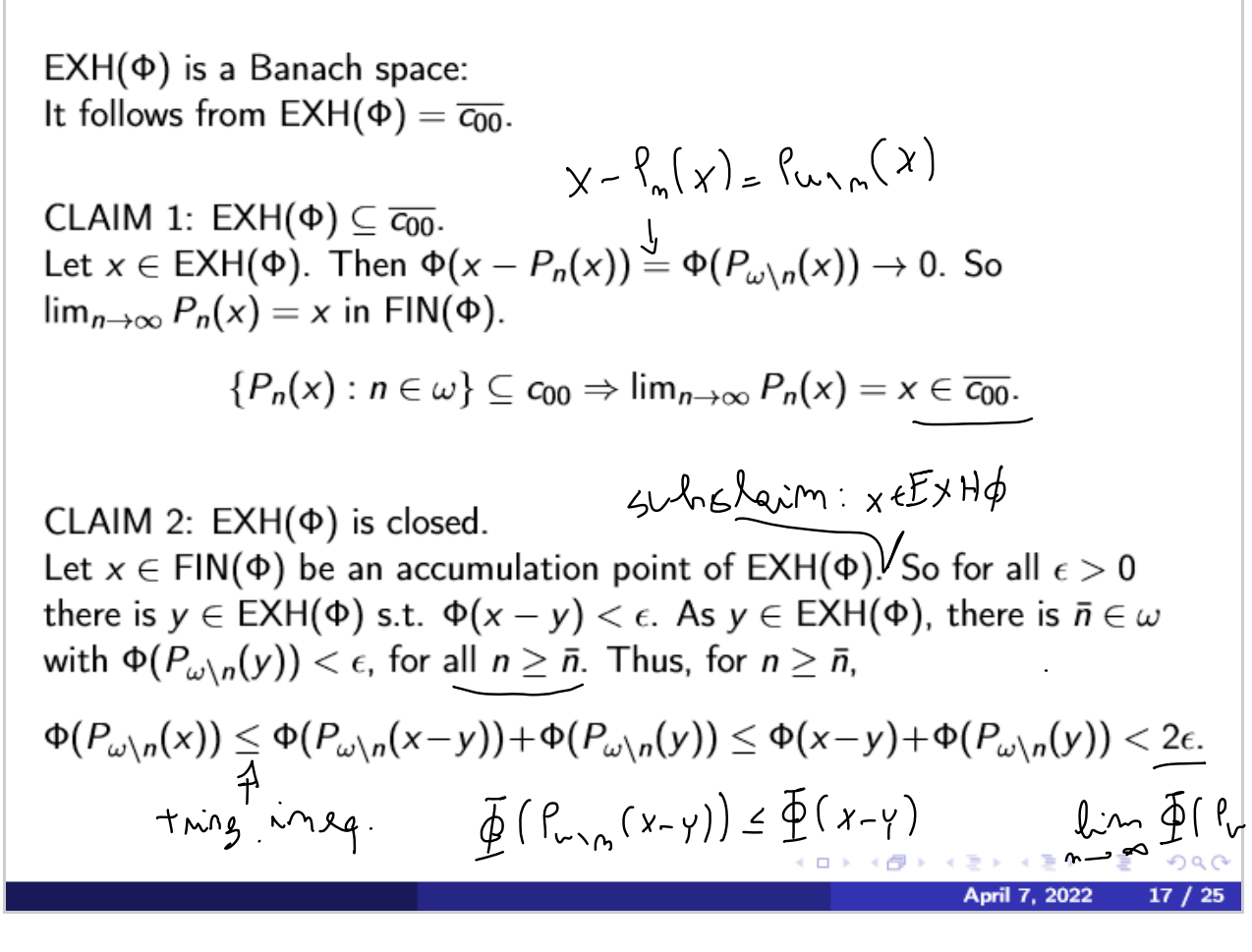
for all $n\ge n_{\bar j}$, $\Phi(x_{n_{\bar j}}-x_n)<\frac{\epsilon}{2}$.

Moreover, for $k<M_{\bar j}$, $\lim_{n\to\infty}\Phi(P_{\{k\}}(x_n-y))=0$. So there is $\bar i>\bar j$ such that

$$\Phi_{\{k\}}(x_{\bar i}-y)<\frac{\epsilon}{2M_{\bar j}}.$$

Finally, $\epsilon<\Phi(P_{M_{\bar j}}(x_{n_{\bar j}}-y))\le\Phi(P_{M_{\bar j}}(x_{n_{\bar j}}-x_{n_{\bar i}}))+\Phi(P_{M_{\bar j}}(x_{n_{\bar i}}-y)\le$
$\Phi(P_{M_{\bar j}}(x_{n_{\bar j}}-x_{n_{\bar i}}))+\sum_{k<M_{\bar j}}\Phi(P_{\{k\}}(x_{n_{\bar i}}-y))<\frac{\epsilon}{2}+M_{\bar j}\frac{\epsilon}{2M_{\bar j}}=\epsilon.$

$P_{\{k\}}(x_n)\to Y(k)=Y_k$



EXH(Φ) is a Banach space:
It follows from EXH(Φ) $=\overline{c_{00}}$.

CLAIM 1: EXH(Φ) $\subseteq\overline{c_{00}}$.
Let $x\in\mathrm{EXH}(\Phi)$. Then $\Phi(x-P_n(x))\overset{\downarrow}{=}\Phi(P_{\omega\setminus n}(x))\to0$. So $\lim_{n\to\infty}P_n(x)=x$ in FIN(Φ).

$x-P_n(x)=P_{\omega\setminus n}(x)$

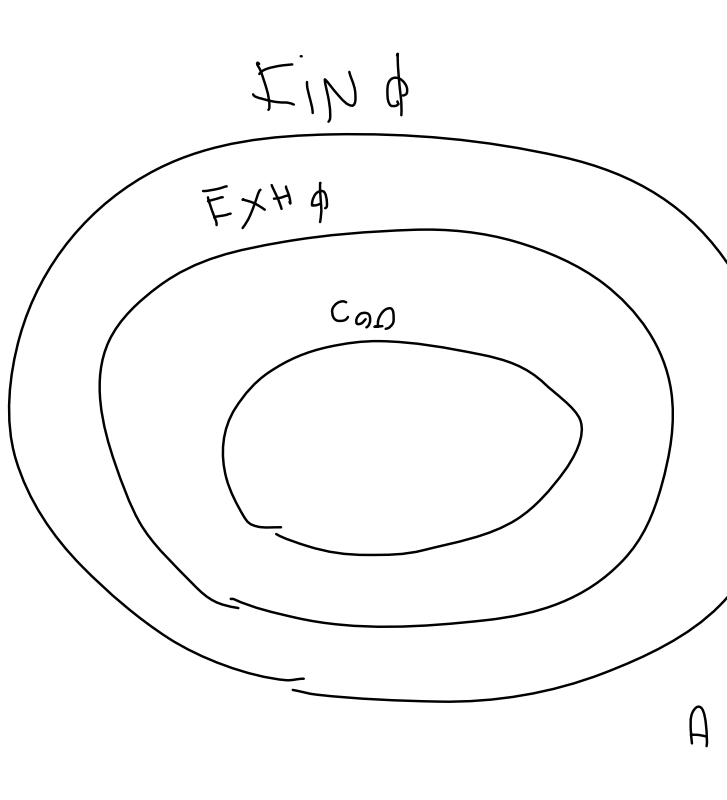
$$\{P_n(x):n\in\omega\}\subseteq c_{00}\Rightarrow\lim_{n\to\infty}P_n(x)=x\in\overline{c_{00}}.$$

CLAIM 2: EXH(Φ) is closed. subclaim: $x\in\mathrm{EXH}\phi$
Let $x\in\mathrm{FIN}(\Phi)$ be an accumulation point of EXH(Φ). So for all $\epsilon>0$ there is $y\in\mathrm{EXH}(\Phi)$ s.t. $\Phi(x-y)<\epsilon$. As $y\in\mathrm{EXH}(\Phi)$, there is $\bar n\in\omega$ with $\Phi(P_{\omega\setminus n}(y))<\epsilon$, for all $n\ge\bar n$. Thus, for $n\ge\bar n$,

$\Phi(P_{\omega\setminus n}(x))\le\Phi(P_{\omega\setminus n}(x-y))+\Phi(P_{\omega\setminus n}(y))\le\Phi(x-y)+\Phi(P_{\omega\setminus n}(y))<2\epsilon.$

triang. ineq. $\Phi(P_{\omega\setminus n}(x-y))\le\Phi(x-y)$ $\lim_{n\to\infty}\Phi(P_{\omega\setminus n}(x))=0$

$\overline{c_{00}}$
$\|$
$x\in\mathrm{EXH}\phi$
$\Uparrow$

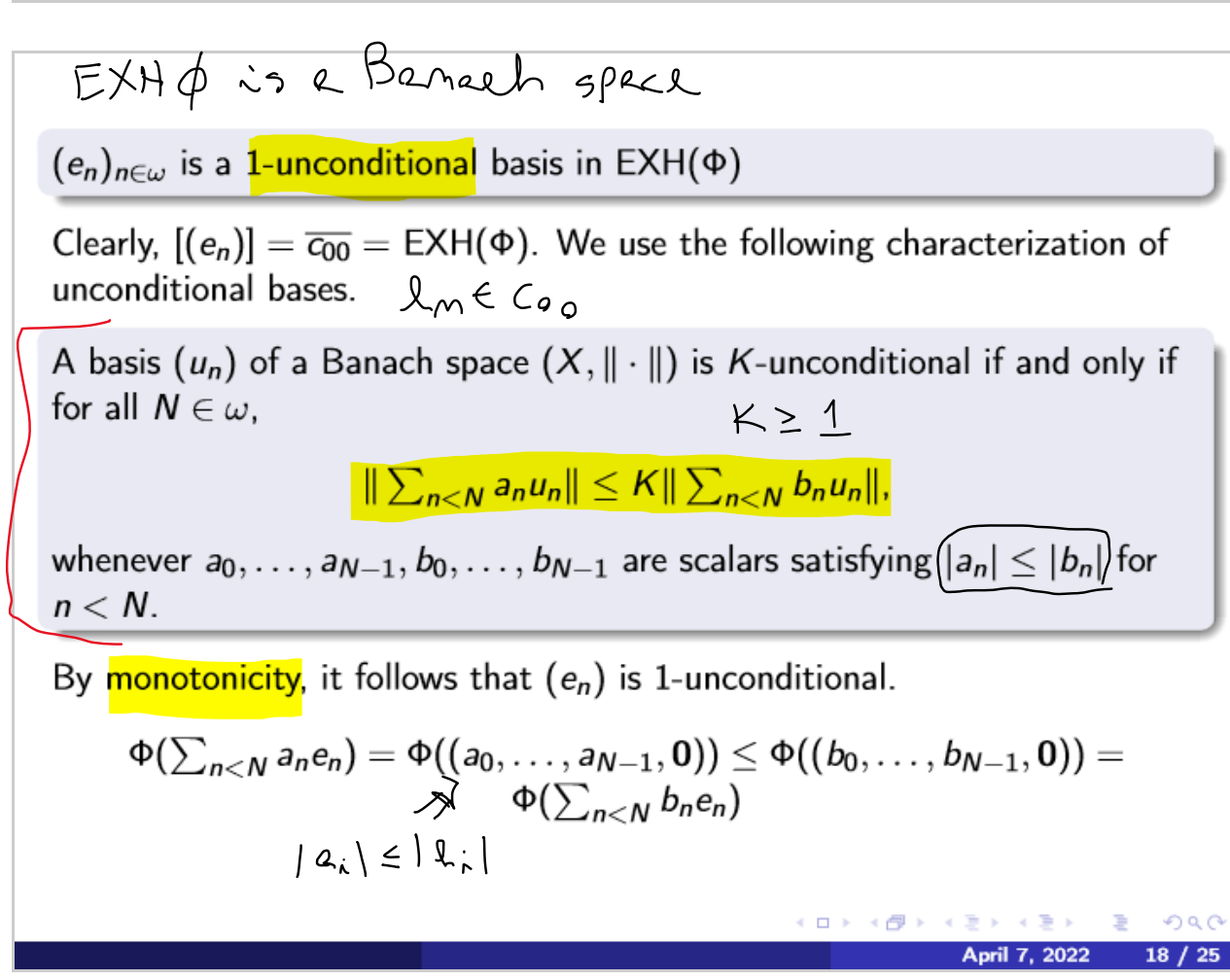


EXHϕ is closed

$\overline{c_{00}}=\mathrm{EXH}\phi$
⇓
EXHϕ is Bach space

(FINϕ, $\phi\restriction\mathrm{FIN}\phi$)

$A\subseteq\mathrm{FIN}\phi\subseteq\mathbb{R}^\omega$



EXHϕ is a Banach space

$(e_n)_{n\in\omega}$ is a 1-unconditional basis in EXH(Φ)

Clearly, $[(e_n)]=\overline{c_{00}}=\mathrm{EXH}(\Phi)$. We use the following characterization of unconditional bases. $e_n\in c_{00}$

A basis (u_n) of a Banach space $(X,\|\cdot\|)$ is K-unconditional if and only if for all $N\in\omega$, $K\ge1$

$$\|\sum_{n<N}a_nu_n\|\le K\|\sum_{n<N}b_nu_n\|,$$

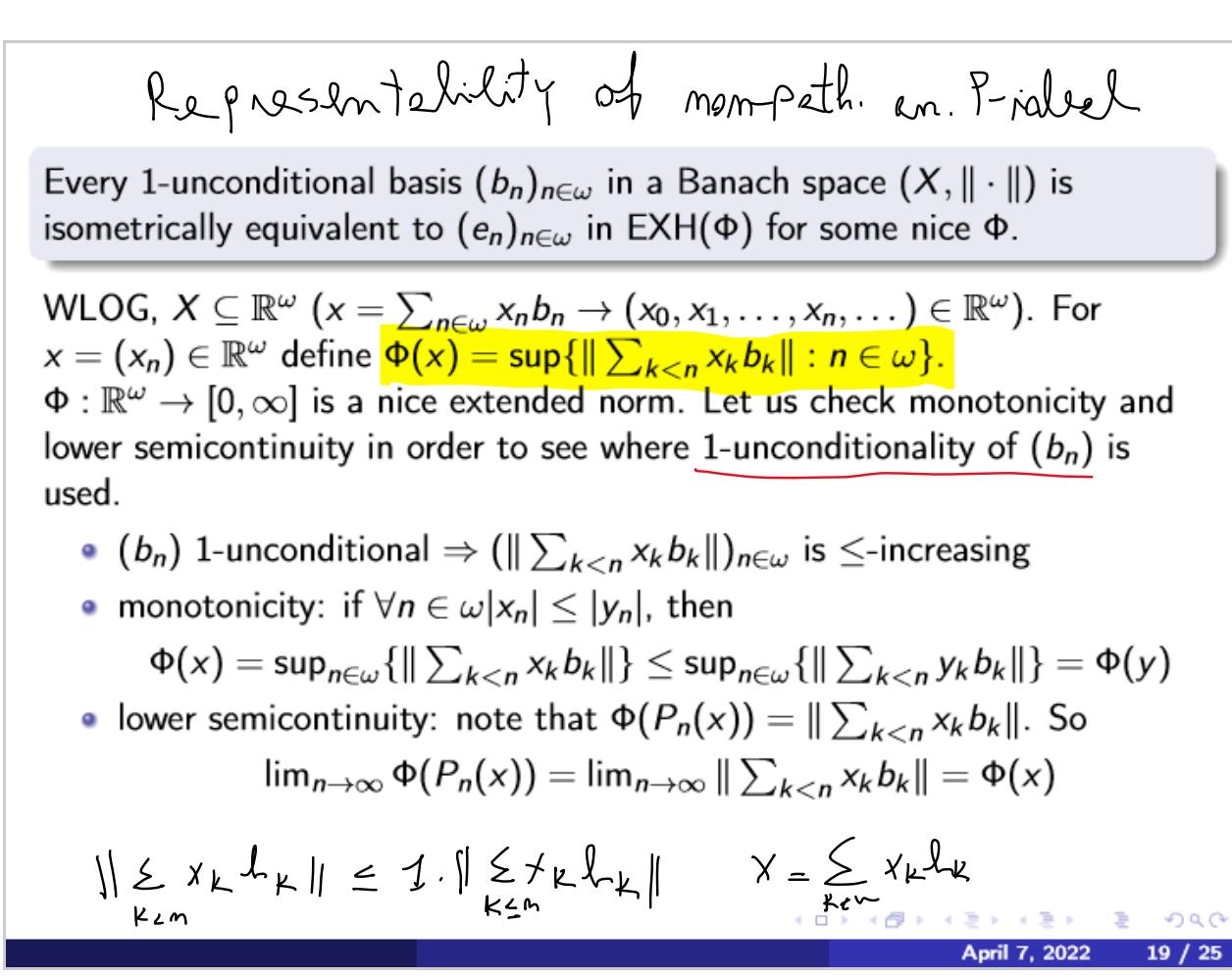
whenever $a_0,\dots,a_{N-1},b_0,\dots,b_{N-1}$ are scalars satisfying $|a_n|\le|b_n|$ for $n<N$.

By monotonicity, it follows that (e_n) is 1-unconditional.

$$\Phi(\sum_{n<N}a_ne_n)=\Phi((a_0,\dots,a_{N-1},0))\le\Phi((b_0,\dots,b_{N-1},0))=\Phi(\sum_{n<N}b_ne_n)$$

$|a_i|\le|b_i|$

$e_0=(1,0,0\dots\dots0,\dots0)$
$e_1=(0,1,0\dots\dots,0\dots0)$
$e_n=(0,0,0,\dots\underset{\underset{n}{\uparrow}}{1},0\dots\dots0)$



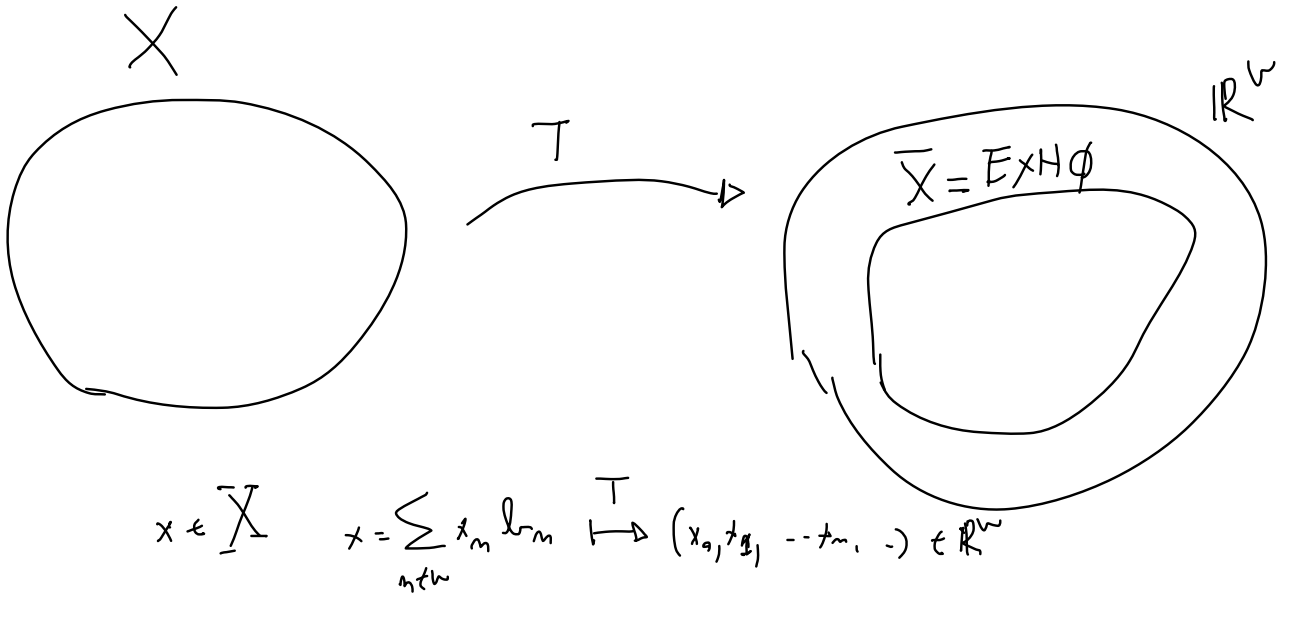
Representability of non-path. an. P-ideal

Every 1-unconditional basis $(b_n)_{n\in\omega}$ in a Banach space $(X,\|\cdot\|)$ is isometrically equivalent to $(e_n)_{n\in\omega}$ in EXH(Φ) for some nice Φ.

WLOG, $X\subseteq\mathbb{R}^\omega$ $(x=\sum_{n\in\omega}x_nb_n\to(x_0,x_1,\dots,x_n,\dots)\in\mathbb{R}^\omega)$. For $x=(x_n)\in\mathbb{R}^\omega$ define $\Phi(x)=\sup\{\|\sum_{k<n}x_kb_k\|:n\in\omega\}$.
$\Phi:\mathbb{R}^\omega\to[0,\infty]$ is a nice extended norm. Let us check monotonicity and lower semicontinuity in order to see where 1-unconditionality of (b_n) is used.
- (b_n) 1-unconditional $\Rightarrow(\|\sum_{k<n}x_kb_k\|)_{n\in\omega}$ is $\le$-increasing
- monotonicity: if $\forall n\in\omega|x_n|\le|y_n|$, then
  $\Phi(x)=\sup_{n\in\omega}\{\|\sum_{k<n}x_kb_k\|\}\le\sup_{n\in\omega}\{\|\sum_{k<n}y_kb_k\|\}=\Phi(y)$
- lower semicontinuity: note that $\Phi(P_n(x))=\|\sum_{k<n}x_kb_k\|$. So
  $\lim_{n\to\infty}\Phi(P_n(x))=\lim_{n\to\infty}\|\sum_{k<n}x_kb_k\|=\Phi(x)$

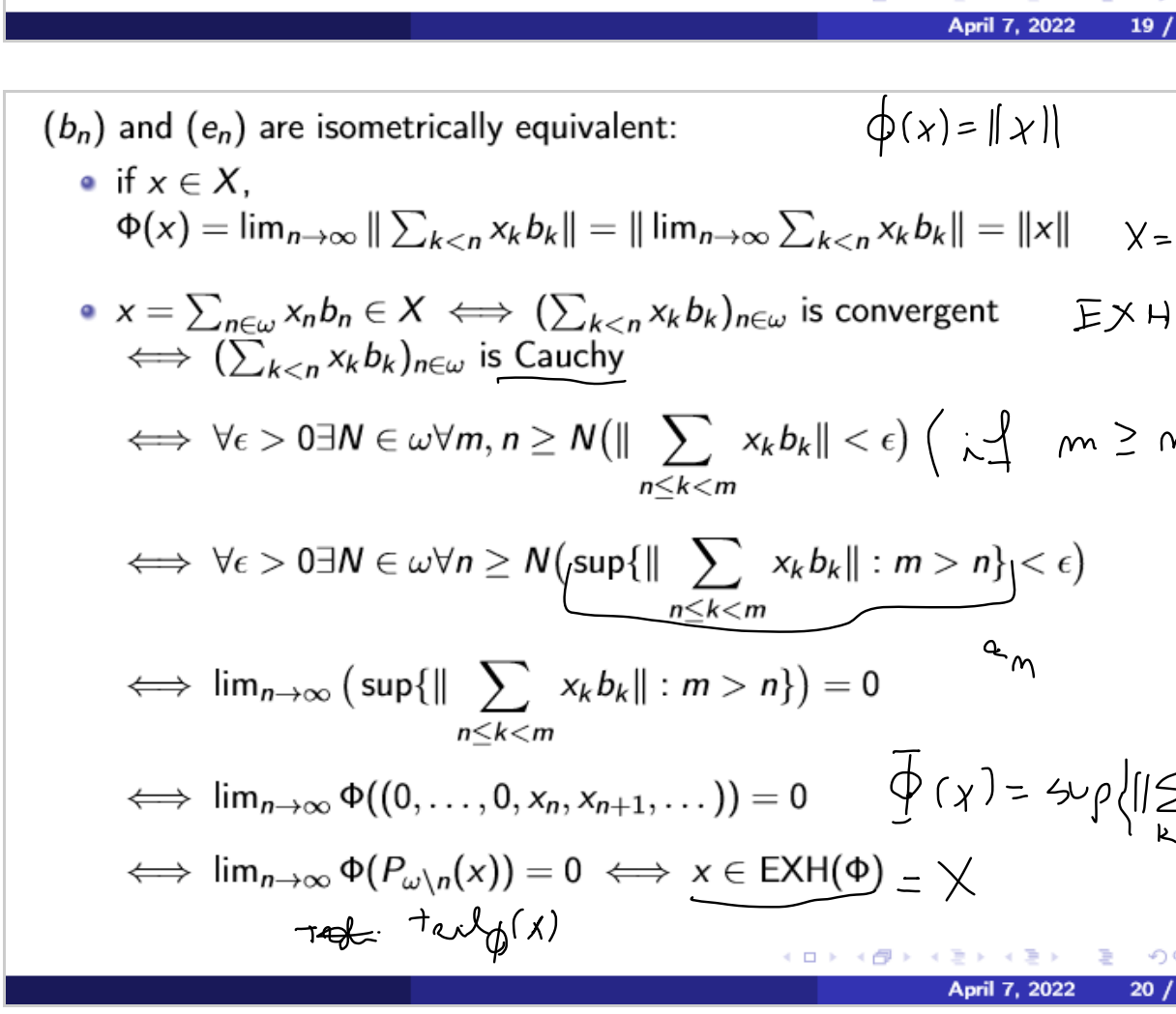
$\|\sum_{k<m}x_kb_k\|\le1\cdot\|\sum_{k\le m}x_kb_k\|$ $x=\sum_{k\in\omega}x_kb_k$

EXHϕ: X 1-uncond. $\Phi:\mathbb{R}^\omega\to[0,\infty]$

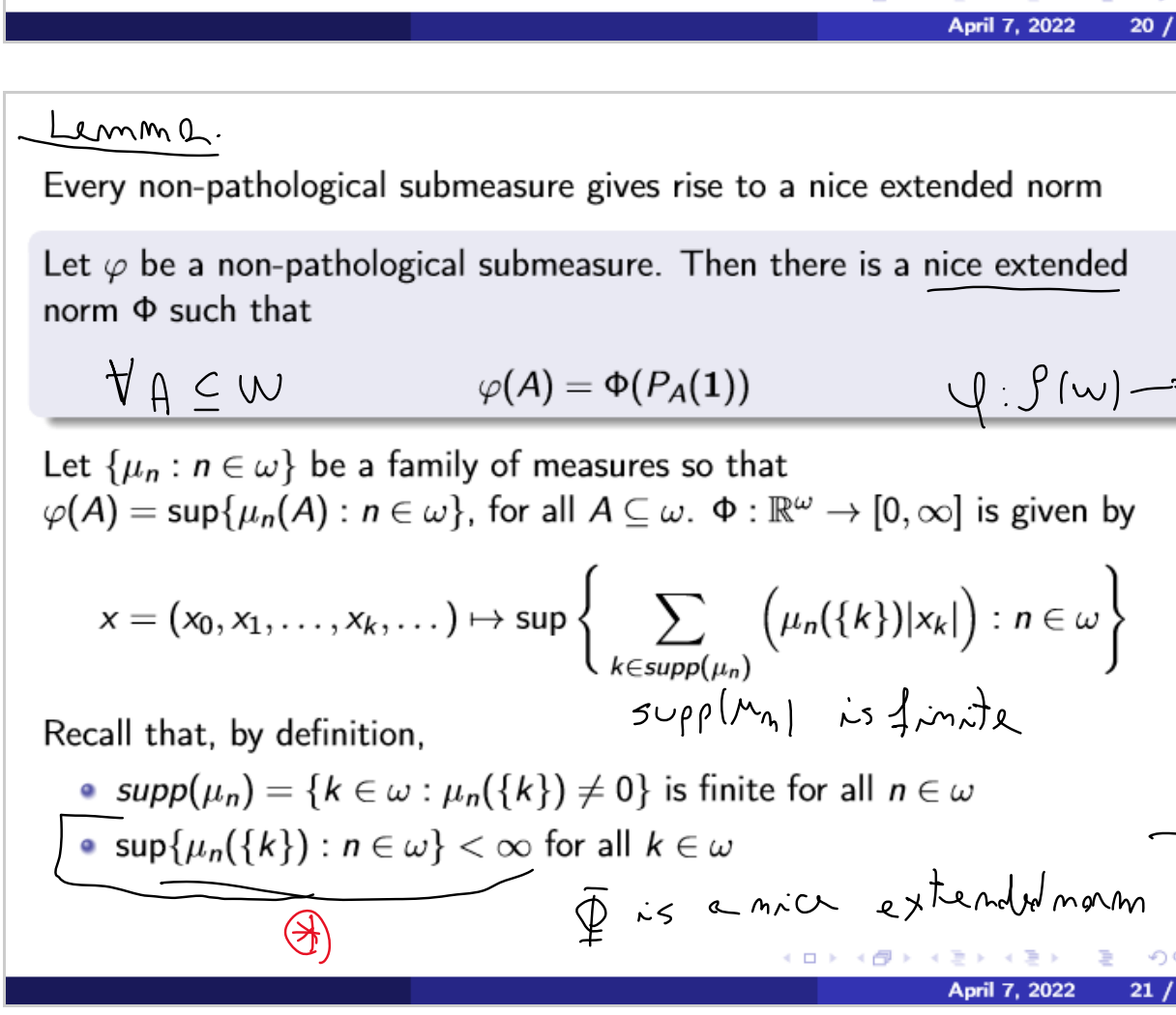


$x\in X$ $x=\sum_{n\in\omega}x_nb_n\overset{T}{\mapsto}(x_0,x_1,\dots x_n,\dots)\in\mathbb{R}^\omega$

$\bar X=T[X]=\{T(x):x\in X\}$



(b_n) and (e_n) are isometrically equivalent: $\phi(x)=\|x\|$
- if $x\in X$,
  $\Phi(x)=\lim_{n\to\infty}\|\sum_{k<n}x_kb_k\|=\|\lim_{n\to\infty}\sum_{k<n}x_kb_k\|=\|x\|$ $x=\sum_k x_kb_k$
- $x=\sum_{n\in\omega}x_nb_n\in X\iff(\sum_{k<n}x_kb_k)_{n\in\omega}$ is convergent $\iff(\sum_{k<n}x_kb_k)_{n\in\omega}$ is Cauchy EXH$\phi=X$ $(=\bar X)$
  $\iff\forall\epsilon>0\exists N\in\omega\forall m,n\ge N(\|\sum_{n\le k<m}x_kb_k\|<\epsilon)$ (if $m\ge n$)
  $\iff\forall\epsilon>0\exists N\in\omega\forall n\ge N(\sup\{\|\sum_{n\le k<m}x_kb_k\|:m>n\}<\epsilon)$
  $\iff\lim_{n\to\infty}(\sup\{\|\sum_{n\le k<m}x_kb_k\|:m>n\})=0$
  $\iff\lim_{n\to\infty}\Phi((0,\dots,0,x_n,x_{n+1},\dots))=0$ $\Phi(x)=\sup\{\|\sum_{k<n}x_kb_k\|:n\in\omega\}$
  $\iff\lim_{n\to\infty}\Phi(P_{\omega\setminus n}(x))=0\iff x\in\mathrm{EXH}(\Phi)=X$
  $\mathrm{tail}_\Phi(x)$



Lemma.

Every non-pathological submeasure gives rise to a nice extended norm

Let φ be a non-pathological submeasure. Then there is a nice extended norm Φ such that

$\forall A\subseteq\omega$ $\varphi(A)=\Phi(P_A(\mathbf{1}))$ $\varphi:\mathcal{P}(\omega)\to[0,\infty]$

Proof. Let $\{\mu_n:n\in\omega\}$ be a family of measures so that $\varphi(A)=\sup\{\mu_n(A):n\in\omega\}$, for all $A\subseteq\omega$. Then $\Phi:\mathbb{R}^\omega\to[0,\infty]$ is given by

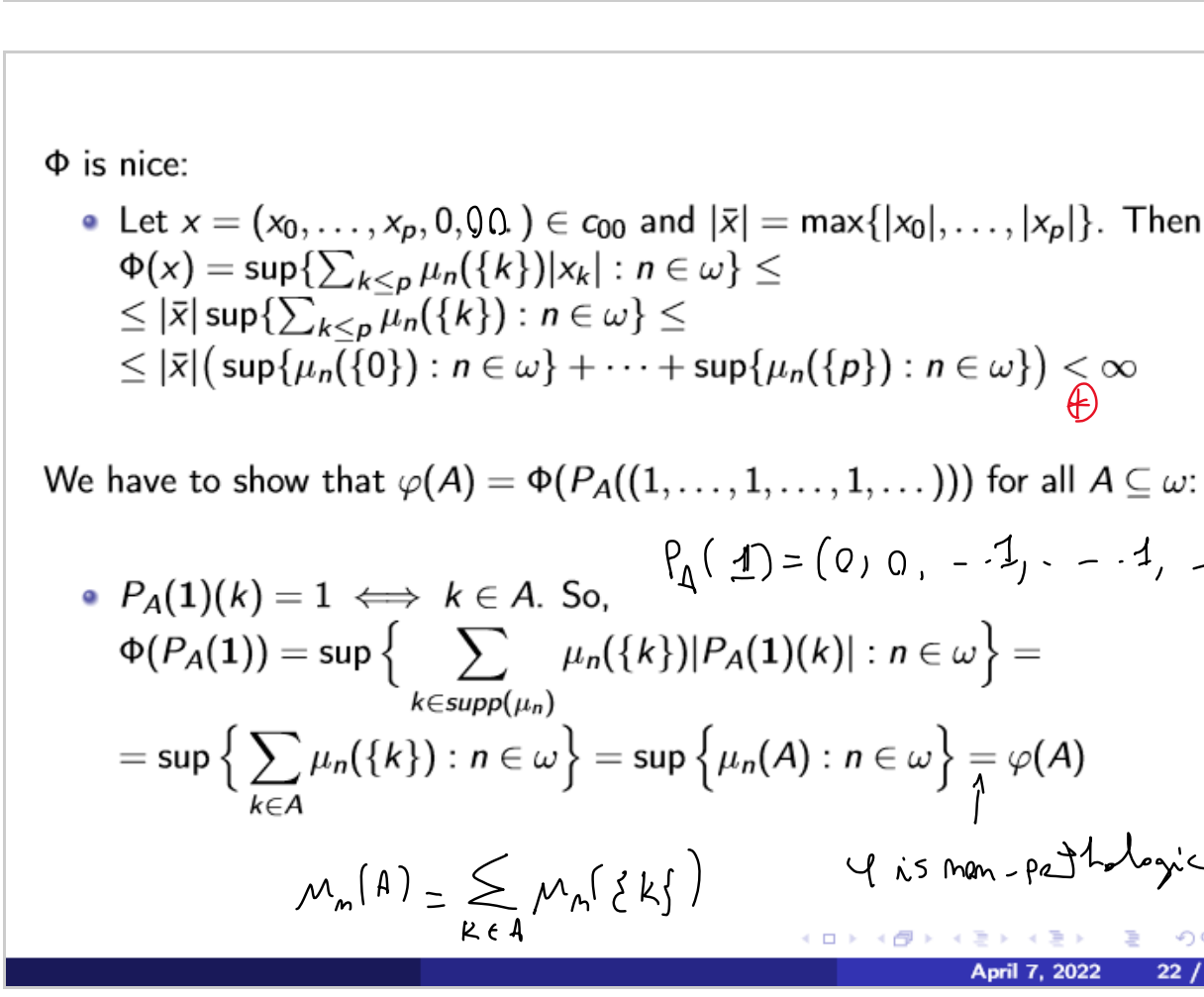
$$x=(x_0,x_1,\dots,x_k,\dots)\mapsto\sup\Big\{\sum_{k\in\mathrm{supp}(\mu_n)}(\mu_n(\{k\})|x_k|):n\in\omega\Big\}$$

$\mathrm{supp}(\mu_n)$ is finite

Recall that, by definition,
- $\mathrm{supp}(\mu_n)=\{k\in\omega:\mu_n(\{k\})\neq0\}$ is finite for all $n\in\omega$
- $\sup\{\mu_n(\{k\}):n\in\omega\}<\infty$ for all $k\in\omega$

③ Φ is a nice extended norm

$\forall x\in c_{00}$ $\Phi(x)<+\infty$ ✓

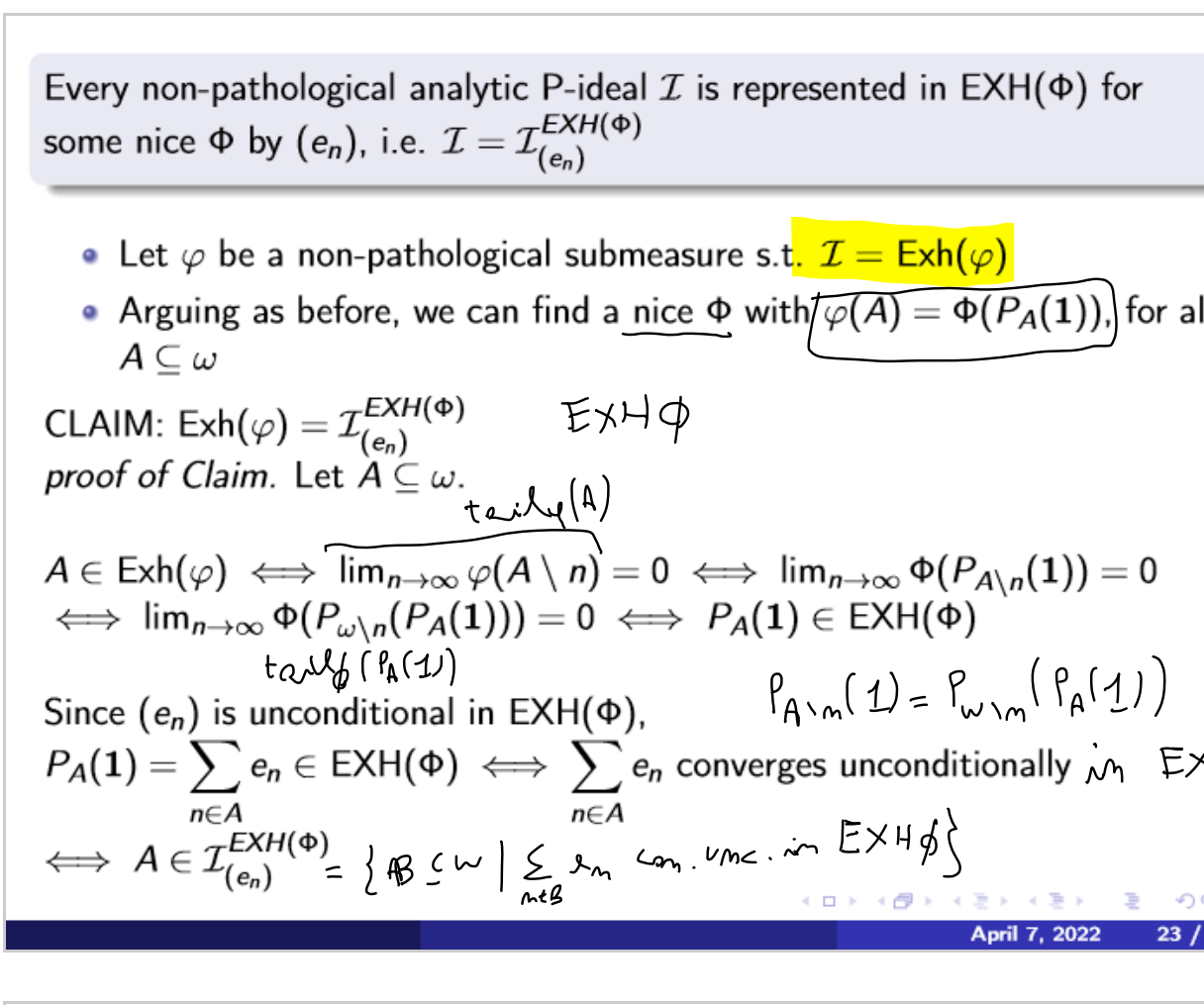


Φ is nice:
- Let $x=(x_0,\dots,x_p,0,0,0\dots)\in c_{00}$ and $|\bar x|=\max\{|x_0|,\dots,|x_p|\}$. Then
  $\Phi(x)=\sup\{\sum_{k\le p}\mu_n(\{k\})|x_k|:n\in\omega\}\le$
  $\le|\bar x|\sup\{\sum_{k\le p}\mu_n(\{k\}):n\in\omega\}\le$
  $\le|\bar x|(\sup\{\mu_n(\{0\}):n\in\omega\}+\dots+\sup\{\mu_n(\{p\}):n\in\omega\})<\infty$ ④

We have to show that $\varphi(A)=\Phi(P_A((1,\dots,1,\dots,1,\dots)))$ for all $A\subseteq\omega$: $n\in A$

$P_A(\mathbf{1})=(0,0,\dots1,\dots,1,0,\dots)$ $n\in A$
- $P_A(\mathbf{1})(k)=1\iff k\in A$. So,
  $\Phi(P_A(\mathbf{1}))=\sup\Big\{\sum_{k\in\mathrm{supp}(\mu_n)}\mu_n(\{k\})P_A(\mathbf{1})(k):n\in\omega\Big\}=$
  $=\sup\Big\{\sum_{k\in A}\mu_n(\{k\}):n\in\omega\Big\}=\sup\{\mu_n(A):n\in\omega\}=\varphi(A)$

$\mu_n(A)=\sum_{k\in A}\mu_n(\{k\})$ φ is non-pathological



Every non-pathological analytic P-ideal $\mathcal{I}$ is represented in EXH(Φ) for some nice Φ by (e_n), i.e. $\mathcal{I}=\mathcal{I}^{EXH(\Phi)}_{(e_n)}$
- Let φ be a non-pathological submeasure s.t. $\mathcal{I}=\mathrm{Exh}(\varphi)$
- Arguing as before, we can find a nice Φ with $\varphi(A)=\Phi(P_A(\mathbf{1}))$, for all $A\subseteq\omega$

CLAIM: $\mathrm{Exh}(\varphi)=\mathcal{I}^{EXH(\Phi)}_{(e_n)}$ EXHΦ
proof of Claim. Let $A\subseteq\omega$.

$A\in\mathrm{Exh}(\varphi)\iff\overbrace{\lim_{n\to\infty}\varphi(A\setminus n)}^{\mathrm{tail}_\varphi(A)}=0\iff\lim_{n\to\infty}\Phi(P_{A\setminus n}(\mathbf{1}))=0$
$\iff\lim_{n\to\infty}\Phi(P_{\omega\setminus n}(P_A(\mathbf{1})))=0\iff P_A(\mathbf{1})\in\mathrm{EXH}(\Phi)$
$\mathrm{tail}_\Phi(P_A(\mathbf{1}))$ $P_{A\setminus n}(\mathbf{1})=P_{\omega\setminus n}(P_A(\mathbf{1}))$

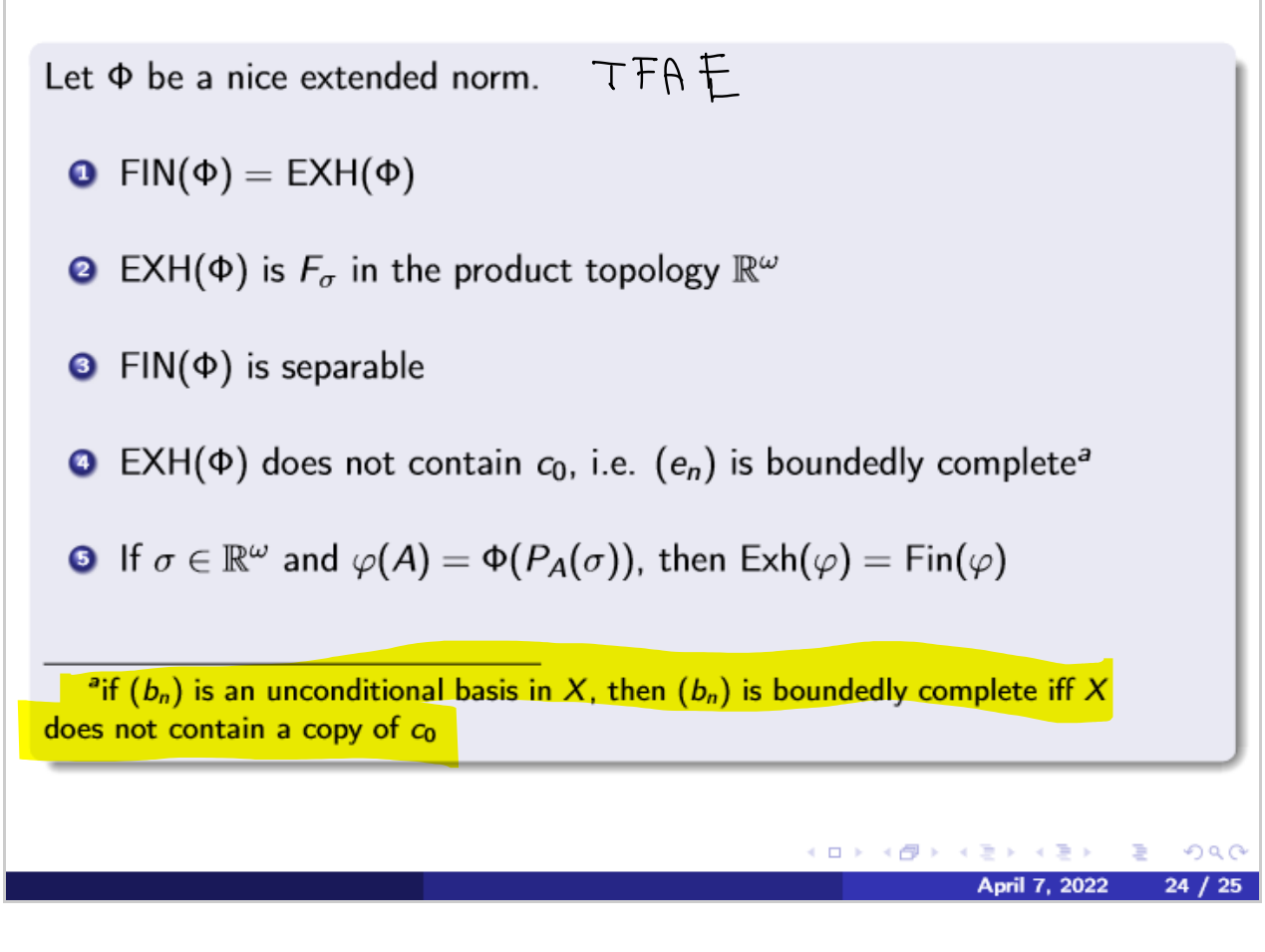
Since (e_n) is unconditional in EXH(Φ),
$P_A(\mathbf{1})=\sum_{n\in A}e_n\in\mathrm{EXH}(\Phi)\iff\sum_{n\in A}e_n$ converges unconditionally in EXHΦ
$\iff A\in\mathcal{I}^{EXH(\Phi)}_{(e_n)}=\{B\subseteq\omega \mid \sum_{n\in B}e_n$ con. unc. in EXH$\phi\}$

$\mathcal{I}=\mathrm{Exh}(\varphi)$ φ non-path.

Φ

$\mathcal{I}$ is repr EXHϕ FINϕ = EXHϕ

$\mathrm{Exh}(\varphi)=\mathcal{I}^{EXH\phi}_{(e_n)}=\{A\subseteq\omega \mid \sum_{n\in A}e_n\overset{unc.}{\text{conv.}}$ in EXH$\phi\}$



Let Φ be a nice extended norm. TFAE
1. FIN(Φ) = EXH(Φ)
2. EXH(Φ) is F_σ in the product topology $\mathbb{R}^\omega$
3. FIN(Φ) is separable
4. EXH(Φ) does not contain c_0, i.e. (e_n) is boundedly complete[a]
5. If $\sigma\in\mathbb{R}^\omega$ and $\varphi(A)=\Phi(P_A(\sigma))$, then $\mathrm{Exh}(\varphi)=\mathrm{Fin}(\varphi)$

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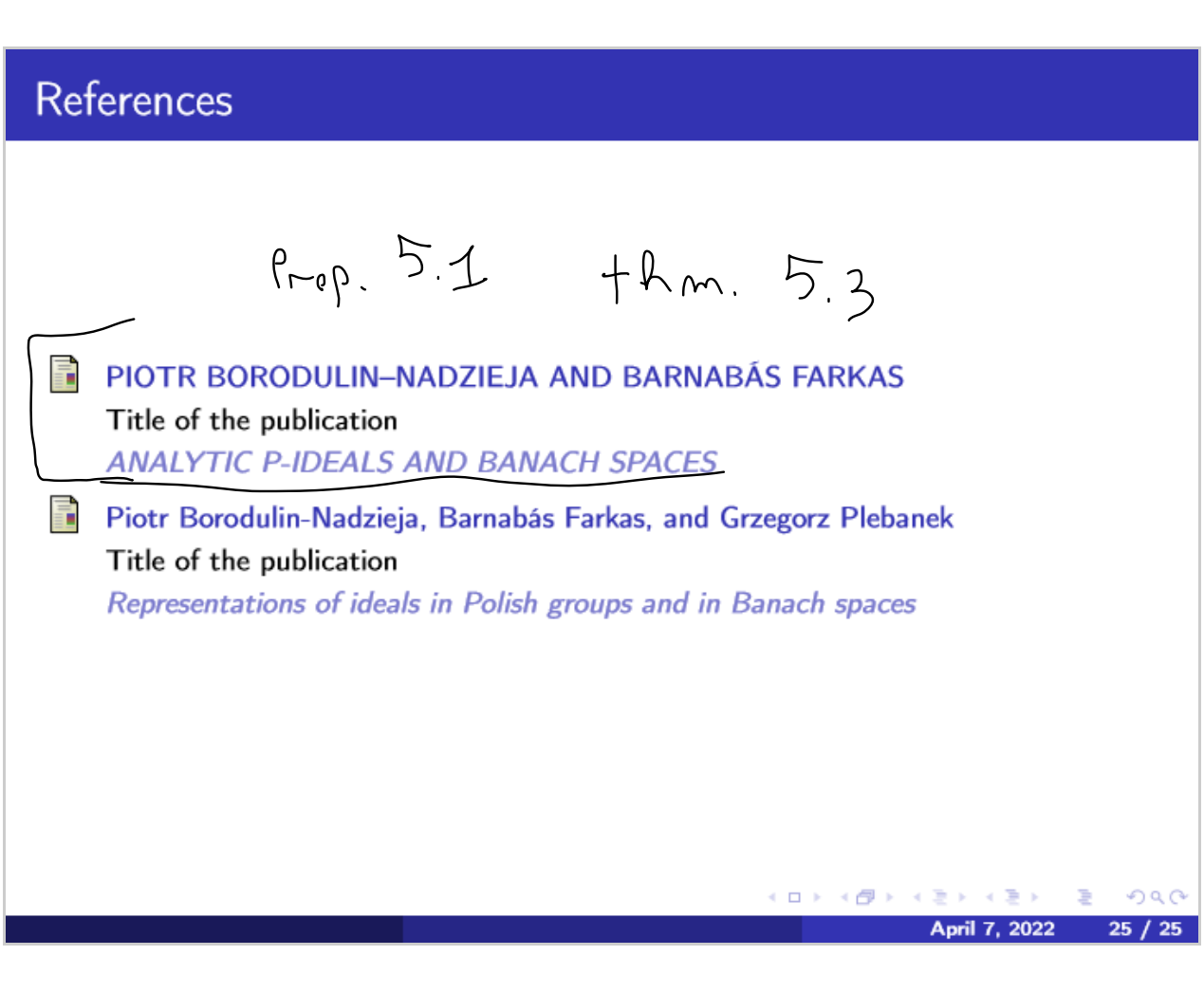
[a] if (b_n) is an unconditional basis in X, then (b_n) is boundedly complete iff X does not contain a copy of c_0



References

Prop. 5.1 thm. 5.3

PIOTR BORODULIN–NADZIEJA AND BARNABÁS FARKAS
Title of the publication
ANALYTIC P-IDEALS AND BANACH SPACES

Piotr Borodulin-Nadzieja, Barnabás Farkas, and Grzegorz Plebanek
Title of the publication
Representations of ideals in Polish groups and in Banach spaces

FINϕ = EXHϕ ⟺ (e_n) is bound. complete (b.c.)

Def. A basis (b_n) in $(X,\|\|)$ is b.c. if $\forall a=(a_0,\dots a_n\dots)\in\mathbb{R}^\omega$

$\sup\{\|\sum_{k<n}a_kb_k\|:n\in\omega\}<+\infty\Rightarrow\sum_{n\in\omega}a_nb_n$ conv. in X

Remark. (e_n) 1-unconditional $\Rightarrow(\Phi(\sum_{k<n}a_ke_k))_{n\in\omega}$ is $\le$-increasing $(a=(a_0,a_1,\dots a_n,\dots)\in\mathbb{R}^\omega)$

so $\sup\{\Phi(\sum_{k<n}a_ke_k):n\in\omega\}=\lim_{n\to\infty}\Phi(\sum_{k<n}a_ke_k)$

FINϕ = EXHϕ ⇒ (e_n) is b.c.

$a\in\mathbb{R}^\omega$ $\lim_{n\to\infty}\Phi(\sum_{k<n}a_ke_k)<\infty$. $\sum_{k\in\omega}a_ke_k=a\in\mathrm{EXH}\phi$.

$\Phi(a)=\lim_{n\to\infty}\Phi(P_n(a))=\lim_{n\to\infty}\Phi(\sum_{k<n}e_ka_k)<+\infty\Rightarrow a\in\mathrm{FIN}\phi$

$\Rightarrow a\in\mathrm{EXH}\phi\Rightarrow\lim_{n\to\infty}\Phi(P_{\omega\setminus n}(a))=0$. Let $\varepsilon>0$

$\exists N\forall n\ge N$ $\Phi(P_{\omega\setminus n}(a))<\varepsilon$. But $P_{\omega\setminus n}(a)=a-\sum_{k<n}a_ke_k$

So $\Phi(a-\sum_{k<n}a_ke_k)<\varepsilon\Rightarrow\sum_{k<n}a_ke_k\xrightarrow{n\to\infty}a$

(⟸) FINϕ $\subseteq$ EXHϕ. Let $x\in\mathrm{FIN}\phi$. So

$\lim_{n\to\infty}\Phi(\sum_{k<n}x_ke_k)=\lim_{n\to\infty}\Phi(P_n(x))=\Phi(x)<\infty$.

(e_n) is b.d. ⇒ $\sum_{k\in\omega}x_ke_k$ in EXHϕ. So $x=\sum_{k\in\omega}x_ke_k\in\mathrm{EXH}\phi$

FINϕ $\subseteq$ EXHϕ