WHAT MODEL COMPANIONSHIP CAN SAY ABOUT THE CONTINUUM PROBLEM

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Abstract. We present recent results on the model companions of set theory, placing them in the context of the current debate in the philosophy of mathematics. We start by describing the dependence of the notion of model companionship on the signature, and then we analyze this dependence in the specific case of set theory. We argue that the most natural model companions of set theory describe (as the signature in which we axiomatize set theory varies) theories of H_{κ^+}, as κ ranges among the infinite cardinals. We also single out $2^{\aleph_0} = \aleph_2$ as the unique solution of the Continuum problem which can (and does) belong to some model companion of set theory (enriched with large cardinal axioms). Finally this model-theoretic approach to set-theoretic validities is explained and justified in terms of a form of maximality inspired by Hilbert's axiom of completeness.

1. Introduction

Without doubt the Continuum problem is one of the driving forces of set theory. The attempts to determine the cardinality of the Continuum has accompanied the history of set theory (from its very beginning to the present day) and motivated many of its most significant advances. Cantor's definition of the Perfect Set Property was prompted by a partial solution to the Continuum Problem and more in general the initial developments of descriptive set theory were also driven by an attempt to confirm the Continuum Hypothesis (CH: $2^{\aleph_0} = \aleph_1$) at least for the definable subsets of $\mathbb{R}$. In more recent times, Gödel's constructible universe and Cohen's method of forcing were devised to show the independence of CH from ZFC (the standard first order axiomatization of set theory).

The techniques developed by Gödel and Cohen clarified the intrinsic limitations of the axiomatic approach to the Continuum problem, and profoundly influenced the subsequent development of set theory. As a matter of fact, after the Sixties, the many independence results obtained combining the methods of inner models and forcing partially shifted the main focus of set theory: set theorists progressively devoted more and more efforts to the study of the models of set theory, and only derivatively to the study of sets (understood as autonomous mathematical objects).

The present paper presents a new approach to the Continuum problem which stems from a model-theoretic perspective on set theory and discusses the philosophical import of the results appearing in [31,34]. As is standard, we recognize the intrinsic limitation of ZFC in capturing set-theoretic validities, but instead of proposing new axioms to extend ZFC we suggest to enforce new model-theoretic properties on the models of set theory. Consequently, the sought solution to the Continuum problem is not only motivated by the adoption of a specific axiom, but mainly by the nice model-theoretic properties displayed by the models of set theory enriched with large cardinal axioms.

The central notion we borrow from model-theory and that motivates the present approach is that of model companionship. This concept has been developed by Abraham Robinson in the Sixties and it is meant to capture, in an abstract setting, the closure properties that algebraically closed fields display with respect to commutative rings with

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no zero divisors. As algebraically closed fields contain solutions for all Diophantine equations, the models of the companions of set theory will have solutions for all "simple" set theoretical problems.

A fundamental property of model companionship is its dependence on the signature.[1] Loosely speaking: there can be distinct first order axiomatizations of a mathematical theory T, one in signature σ and the other in signature τ, such that T admits a model companion when axiomatized according to σ, but does not when axiomatized in signature τ. This peculiar aspect of model companionship will therefore motivate a detailed discussion aimed to single out the relevant signatures for set theory. We start by clarifying some fundamental aspects of set-theoretical practice. In particular we will explain why bounded formulae (i.e. Δ_0-formulae) express set-theoretic concepts of low complexity and therefore should be included in any reasonably rich signature for set theory. We will then show how a careful choice of the signatures for set theory allows to classify the complexity of set-theoretic concepts and to clarify the informal notion of "simple" set-theoretic problem. Our analysis will provide precise criteria able to link the logic complexity of a signature for set theory to forcing invariance and to the notion of simplicity for a set-theoretic concept. By varying the signatures, we will show that the corresponding model companions of set theory describe possible (first order) theories of the structures H_{κ^+}, for κ an infinite cardinal.[2] Moreover, given an appropriate signature τ_κ for set theory (relative to the cardinal κ), the τ_κ-theory of H_{κ^+} will maximize the Π_2-sentences that are consistent with the Π_1-fragment of set theory (where all complexities are computed according to τ_κ).

This feature is peculiar to the model companions of set theory and isolates a notion which is strictly stronger than that of model companionship. We will call it absolute model companionship (AMC). Absolute model companionship describes those τ-theories S for which a model companion exists and is axiomatized by the Π_2-sentences (for τ) which are consistent with the Π_1-fragment of any completion of S.[3] The sought solution to the Continuum problem will be motivated by reckoning that:

(i) $\neg\mathsf{CH}$ is a Π_2-formula in any signature for set theory which contains the Δ_0-properties among its atomic formulae and has a parameter for ω_1;
(ii) for each infinite (and definable) cardinal κ there is at least one signature τ_κ containing the Δ_0-properties among its atomic formulae, a constant to interpret the cardinal κ, and such that set theory axiomatized in the corresponding τ_κ admits an AMC;
(iii) any signature τ for set theory, containing the Δ_0-properties among its atomic formulae and such that the corresponding τ-set theory admits an AMC will not have CH among the truths of this absolute model companion;
(iv) there is at least one signature τ^* for set theory which contains the Δ_0-properties among its atomic formulae and such that the corresponding τ^*-set theory (enriched with large cardinals) admits an AMC that contains $\neg\mathsf{CH}$ among its axioms.

A similar but more delicate argument (since $\neg(2^{\aleph_0} > \aleph_2)$ is a Σ_2-sentence in parameter ω_2) will show that the above results hold also if one replaces CH by $2^{\aleph_0} > \aleph_2$ in items (ii)–(iv) (but with the same τ^* working for item (iv)). Therefore, these results single out $2^{\aleph_0} = \aleph_2$ as the unique solution of the Continuum problem which can fall in at least one absolute

[1]See Section 4 for a precise formulation of this dependence.

[2]Recall that H_{κ^+} is the collection of sets whose transitive closure has size at most κ.

[3]This apparently technical property will motivate the definition of a proper strenghtening of Robinson's notion of model companionship: that of absolute model companionship (AMC), see Section 5. AMC seems to be the correct notion of model companionship to apply to set theory. For example it is unknown to the authors whether there can be a signature for which the axiomatization of set theory in it admits a model companion which is not its AMC.

model companion of set theory enriched with large cardinal axioms (e.g. the AMC of set theory enriched with large cardinals with respect to signature τ^*).

The present model-theoretic approach to set theoretic validities is not incompatible with the standard strategy of producing axiomatic extensions of ZFC. On the contrary it can be seen as a practical realization of Gödel's program: a step by step extension of the axioms of set theory. Indeed, given the stratification of V in terms of the H_{κ^+}, we can interpret the closure properties of the absolute model companions of set theory as gradually closing-off the universe of sets with respect to all "simple" set theoretical problems. In this sense, the nice model-theoretic properties displayed by the theories of H_{κ^+} (as the model companions of set theory) realize a form of maximality that we dare to call Hilbertian Completeness. Indeed, as Hilbert's axiom of completeness was meant to maximize the objects of geometry, absolute model companions maximize witnesses for Σ_1-properties (formalized in the appropriate signature). Moreover, important axiomatic extensions of ZFC (considered in the last decades as candidates for fulfilling Gödel's program[4]) as are large cardinals and forcing axioms find, in the present approach, another justification. As a matter of fact, the generic absoluteness results that we can obtain from large cardinals hypotheses are now of pivotal importance in detecting the signature expansions for the language of set theory with respect to which set theory admits a(n absolute) model companion. It also occurs that the theory of $H_{\aleph_2}$ in models of strong forcing axioms decribes the model companion of ZFC + *large cardinals* (as expressed in a natural signature τ_κ for κ the first uncountable cardinal).

The paper is structured as follows: in §2 and 3 we gently analyze the role that signatures can play in outlining the properties of a mathematical theory T (with a focus on how the morphisms between models of T can be used to detect the right signatures for T). In §4 we present a few standard notions from model theory (i.e. existentially closed structures, model completeness, and model companionship), which are discussed and generalized in §5, where we define the notions of partial Morleyzation, absolute model companionship, and (absolute) model companionship spectrum. From §6 onwards we devote our attention to set theory. We first present precise mathematical criteria to detect what are the "simple" set-theoretical concepts. Then we analyze various signatures for set theory in §6.1. In §6.2 we show that the intended models of the model companions of set theory are structures of the form H_{κ^+}, for κ an infinite cardinal. Finally, in §7 we present the main results regarding the model companions of set theory and discuss the information they convey on the cardinality of the Continuum. We conclude, in §8, by comparing our approach with the various notions of maximality we find in the literature and with the debate on the justification of new axioms in set theory.

This paper aims to reach scholars interested at the crossroads of philosophy and mathematics. We tried for this reason to minimize the mathematical and philosophical prerequisites needed to follow it. Those who made through this introduction should encounter no serious obstacles to read the remainder of this article. The model companionship results we discuss in the present paper are presented without proofs. The reader interested in them is referred to [31, 34].

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[4] A clear justification of why large cardinal axioms are a partial realization of Gödel's program can be found in [20]. An analysis of why (strong) forcing axioms can also be seen as a realization of this program can be found in [6] or in the introduction of [32]. See also [36–38] where it is proposed to realize Gödel's program by introducing new axioms to describe the theories $H_{\aleph_1}$ and $H_{\aleph_2}$.

and do not engage in any way those who had the patience to give some advice to improve our presentation.

2. What is the right signature for a mathematical theory?

One of the great successes of mathematical logic consists in providing an efficient formalization of mathematics: by means of first order logic it is possible to render mathematical theories the objects of a mathematical investigation. In this way logic is able to produce unexpected and non-trivial mathematical results as well as novel insights on a variety of mathematical fields. It is a matter of facts that there can be many distinct first order formalizations of a mathematical theory: varying the linguistic presentation of a theory (i.e. its *signature*) we obtain different axiomatic presentations of the same set of theorems. In order to appreciate the variety of possibilities we can encounter, let us consider the concrete case of group theory. This will help us to gently introduce one of the main themes of the present present: the role of the signature in detecting the properties of a theory formalized in it.

We can formalize group theory in first order logic using the signature $\{\cdot\}$, which simply consists of a binary function symbol and the following axioms:

$$\begin{aligned}
&\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)],\\
&\exists x \forall y\,(x \cdot y = y \wedge y \cdot x = y),\\
&\forall x \exists y \forall z\,[(x \cdot y) \cdot z = z \wedge (y \cdot x) \cdot z = z \wedge z \cdot (x \cdot y) = z \wedge z \cdot (y \cdot x) = z].
\end{aligned}$$

Notice that the third axiom, which expresses the existence of a multiplicative inverse, represents a rather complicated assertion, both from the point of view of its syntactic readability and its Lévy complexity (being a Π_3-sentence). The reason is that in this basic signature we lack a constant symbol to denote the neutral element of a group. Enriching the language to $\{\cdot, e\}$, with e a constant symbol, we can now formalize group theory with a simpler set of axioms:

$$\begin{aligned}
&\forall x, y, z\,(x \cdot y) \cdot z = x \cdot (y \cdot z),\\
&\forall y\,(e \cdot y = y \wedge y \cdot e = y),\\
&\forall x \exists y\,[x \cdot y = e \wedge y \cdot x = e].
\end{aligned}$$

We increased readability and decreased (Lévy) complexity, as we are now dealing with a Π_2-axiomatization. But we can do better. Indeed, if we consider the signature $\left\{\cdot, e, {}^{-1}\right\}$, which further adds a unary operation symbol for the inverse operation, we can axiomatize group theory with a set of universal equations (Π_1-sentences):

$$\begin{aligned}
&\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)],\\
&\forall x\,(x \cdot e = x \wedge e \cdot x = x),\\
&\forall x\,[x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e].
\end{aligned}$$

On the other hand, we could follow a completely different route and axiomatize group theory avoiding the use of function symbols. For example we can consider the signature $\{R, e\}$ consisting of a ternary relation symbol R and a constant symbol e and produce the following axiomatization for group theory:

$$\begin{aligned}
&\forall x, y \exists! z\, R(x, y, z),\\
&\forall x, y, z, w, t\,[((R(x, y, w) \wedge R(y, z, t)) \to \exists u\,(R(x, t, u) \wedge R(w, z, u))],\\
&\forall y\,[R(e, y, y) \wedge R(y, e, y)],\\
&\forall x \exists y\,[R(x, y, e) \wedge R(y, x, e)].
\end{aligned}$$

At the cost of further complicating our axiomatization, we could even drop the use of the constant symbol e and formalize group theory in the signature $\{R\}$. The latter is clearly an artificial solution; moreover, the minimality of the signature does not help the perspicuity of the axiomatization.

Of all the above formalizations, the one mathematicians use more frequently is certainly the one in signature $\{\cdot, e, ^{-1}\}$. On the contrary, to recognize that the system in signature $\{R, e\}$ is an axiomatization of group theory would surely require some logical training

These considerations suggest that among the many possible signatures in which we can formalize a mathematical theory some are better than others. Our aim is to unfold criteria which allow us to detect the best signatures τ for the formalization of a mathematical theory T. To do so we appeal to two sets of arguments.

On the one hand, we can select τ on the basis of specific considerations internal to T. For example, by checking the adherence of a formalization to the standard informal presentation of the theory (e.g. the signature $\{\cdot, e, ^{-1}\}$ clearly gives the best presentation of group theory in terms of its basic operations). On the other hand, we can give abstract criteria for the choice of τ based on the structural properties that the τ-axiomatization of T displays, disregarding any consideration on the adherence of the τ-axiomatization of T with the informal one.

It occurs that for several theories T what allows us to identify a signature τ as good according to the first set of arguments have corresponding procedures which also allow us to validate τ with respect to the second set of arguments; and conversely. Furthermore it is clear that the second set of arguments is prone to a clearer mathematical formulation (e.g. we need to define precisely what structural properties are preferable and for a given theory T accept the signatures σ for which the σ-axiomatization of T has the preferred property). It is less transparent how to give a precise mathematical formulation of the first set of arguments: which mathematical criterion allow us to recognize $\{\cdot, e\}$ as a better signature than $\{R, e\}$ for the formalization of group theory? For example note that both give a Π_2-axiomatization of group theory. It is clear that any mathematician would regard the $\{\cdot, e\}$-axiomatization of group theory as more natural than the one given in $\{R, e\}$. Can we turn this qualitative preference into a precise mathematical criterion?

3. Simple and complicated concepts for a mathematical theory

When dealing with a first order theory, the formulae that compose it naturally suggest a notion of complexity which, derivatively, can be attributed to the concepts of the theory formalized by these formulae. This is the well-known notion of Lévy complexity which connects the number and patterns of the quantifiers that appear in a formula with its conceptual complexity. Given a theory T expressed in a signature τ, the Lévy hierarchy of the τ-formulae with respect to T-equivalence stratifies the concepts of T as follows: the basic concepts are those formalized by a boolean combination of atomic τ-formulae. The complexity of a concept then increases according to the number of alternations of $\forall, \exists$-quantifiers that a formula ϕ formalizing it displays when expressed in a prenex normal form. In assigning a Lévy complexity to a concept of T, we consider, among all the τ-formulae which are T-equivalent and formalize the given concept, those in prenex normal form with the least number of quantifier alternations. In this way we can assign complexity Π_n, Σ_n or Δ_n to the concepts of T that are expressible by means of τ-formulas. More precisely, $\Pi_0 = \Sigma_0 = \Delta_0$ is the complexity of concepts formalized by boolean combinations of atomic formulae; Π_{n+1} represents the complexity of those concepts formalized by a formula of type $\forall \vec{x}\, \psi(\vec{x}, \vec{y})$ with $\psi(\vec{x}, \vec{y})$ Σ_n; Σ_{n+1} represents the complexity of those concepts formalized by a formula of type $\exists \vec{x}\, \phi(\vec{x}, \vec{y})$ with $\phi(\vec{x}, \vec{y})$ Π_n, and Δ_n represents the complexity of the concepts whose complexity lays in $\Pi_n \cap \Sigma_n$. However, the measure of complexity does not always match with the intuitive notion of complexity we attribute in practice to some concepts. For example the process of Morleyization (see [28, Section 3.2] or Def. 5.5 below) produces a mathematically equivalent axiomatization T^* of a first order τ-theory T in a new signature $\tau^* \supseteq \tau$, such that every τ^*-formula is T^*-equivalent to an atomic formula; hence all concepts of T becomes of the same logical complexity when expressed in signature τ^* according to the mathematically equivalent theory T^*.

To better understand the relevance of the signature in describing the correct logical complexity of the concepts of a theory, let us examine the case of the theory of commutative semi-rings with no zero-divisors. Consider the signature $\{+, \cdot, 0, 1\}$ which is standardly used to axiomatize rings and fields. In this signature we can provide the axioms of *commutative semi-rings with no zero-divisors* by means of the following universal sentences:

$$
\begin{aligned}
&\forall x, y\, (x \cdot y = y \cdot x),\\
&\forall x, y, z\, [(x \cdot y) \cdot z = x \cdot (y \cdot z)],\\
&\forall x\, (x \cdot 1 = x \wedge 1 \cdot x = x),\\
\\
&\forall x, y\, (x + y = y + x),\\
&\forall x, y, z\, [(x + y) + z = x + (y + z)],\\
&\forall y\, (x + 0 = x \wedge 0 + x = x),\\
\\
&\forall x, y, z\, [(x + y) \cdot z = (x \cdot y) + (x \cdot z)],\\
\\
&\forall x, y\, [x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)].
\end{aligned} \tag{1}
$$

By further extending the above list we can obtain the theory of *commutative rings with no zero-divisors* adding the Π_2-axiom

$$\forall x \exists y\, (x + y = 0), \tag{2}$$

and the theory of *fields* by further adding the Π_2-axiom

$$\forall x\, [x \neq 0 \rightarrow \exists y\, (x \cdot y = 1)], \tag{3}$$

and finally the theory of *algebraically closed fields* by supplementing the above axioms with the following Π_2-sentences[5] for all $n \in \mathbb{N}$

$$\forall x_0 \ldots x_n \exists y \sum x_i \cdot y^i = 0. \tag{4}$$

A common trait of the above set of Π_2-axioms is that they assert the existence of solutions to certain basic equations of the theory expressible in terms of sum, multiplication, 0, 1, e.g. the atomic formulae of $\{\cdot, +, 0, 1\}$. Indeed, any commutative semi-ring $\mathcal{M}$ without zero divisors can be extended to an algebraically closed field simply by extending it with solutions to the basic polynomial equations with parameters in $\mathcal{M}$ (and which may not exist in $\mathcal{M}$).

In the signature $\{+, \cdot, 0, 1\}$ we have as basic operations $+, \cdot$. Using this signature in commutative rings we can also define the additive inverse, $-$, using the atomic formula $(x + z = 0)$, while in fields we can add the multiplicative inverse, $^{-1}$, using the boolean combination of atomic formulae $(x = 0 \wedge z = 0) \vee (x \neq 0 \wedge x \cdot z = 1)$. Notice that even when subtraction and division are partially defined on a semiring, we can still meaningfully interpret these two operations in it, just by adding two unary function symbols $-$, $^{-1}$

[5]Notice that any model of Axioms (1) and (4) is automatically a field.

for them and by adopting the convention that the corresponding inverse operations are trivially defined on the non invertible elements; this is captured for example by the axioms

$$\forall x\,[(\exists y\,(x\cdot y=1)\wedge x\cdot x^{-1}=1)\vee(\neg\exists y\,(x\cdot y=1)\wedge x^{-1}=0)]. \tag{5}$$

to interpret $^{-1}$ and

$$\forall x\,[(\exists y\,(x+y=0)\wedge x+(-x)=0)\vee(\neg\exists y\,(x+y=0)\wedge(-x)=0)] \tag{6}$$

to interpret $-$.

It is clear that the class of $\{+,\cdot,0,1,-,{}^{-1}\}$-structures satisfying axioms (1), (5), (6) are exactly the commutative semi-rings with no zero-divisors, however in doing so we perturbed the notion of morphism between these structures. To see this, notice that the inclusion of $\mathbb{N}$ into $\mathbb{Z}$ is a $\{+,\cdot,0,1\}$-morphism but not a $\{+,\cdot,0,1,-,{}^{-1}\}$-morphism as $(-2)=0$ when computed in $\mathbb{N}$ seen as a model of (1), (5), (6), while $(-2)=-2$ when computed in $\mathbb{Z}$ seen as a model of (1), (5), (6). Similarly the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$ is a $\{+,\cdot,0,1,-\}$-morphism but not a $\{+,\cdot,0,1,-,{}^{-1}\}$-morphism.

Moreover, we can observe that in the class of commutative rings with no zero-divisors as formalized in signature $\{+,\cdot,0,1,-\}$ subtraction is axiomatized now by the universal sentence $\forall x\,[x+(-x)=0]$ rather than the Π_2-sentence (6) which defines it in the larger class of commutative semirings. Similarly if we consider the class of $\{+,\cdot,0,1,-,{}^{-1}\}$-structures which are fields, i.e. models of (1), (2), (3), then (5) becomes logically equivalent, modulo the other axioms, to $\forall x\,[x\neq 0\to x\cdot x^{-1}=1]$, which is a universal $\{+,\cdot,0,1,-,{}^{-1}\}$-sentence. On the algebraic side note that the $\{\cdot,+,0,1\}$-morphisms between commutative rings with no zero-divisors naturally extend to $\{\cdot,+,0,1,-\}$-morphisms, as the operation $x\mapsto -x$ is preserved by additive morphisms on rings. Similarly the $\{\cdot,+,0,1\}$-morphisms between fields naturally extends to $\{+,\cdot,0,1,-,{}^{-1}\}$-morphisms, as the operation $x\mapsto x^{-1}$ is preserved by additive and multiplicative morphisms between fields.

More generally when a τ-theory T can define an operation by means of a universal τ-sentence, the τ-morphisms between τ-models of T preserve the operation. In our concrete example, $\{+,\cdot,0,1\}$-morphisms between commutative rings with no zero-divisors naturally extends to $\{+,\cdot,0,1,-\}$-morphisms, while $\{+,\cdot,0,1\}$-morphisms between fields naturally extends to $\{+,\cdot,0,1,-,{}^{-1}\}$-morphisms exactly because the operations $-,{}^{-1}$ are defined by universal axioms in the respective theories.

These examples suggest the following observation: when considering the first order axiomatization of a theory, we should carefully consider not only its class of models, but also the class of morphisms between these models. Dealing with the theory of commutative semi-rings with no zero-divisors, the notion of additive inverse has a complexity which exceeds that of $+$ and $\cdot$. We can detect this by noticing that the class of morphisms between these structures shrinks when we impose the preservation of this operation. On the other hand, in the context of commutative rings, the notion of additive inverse has the same complexity of $+,\cdot$. But, again, this is not the case for the notion of multiplicative inverse: we need to focus on the theory of field in order to regard multiplicative inverse as a concept with the same complexity as addition and multiplication. We can sum up these considerations as follows:

- The signature $\{+,\cdot,0,1\}$ is suitable for commutative (semi)rings with no zero-divisors, and fields;
- The signature $\{+,\cdot,0,1,-\}$ is better suited for commutative rings with no zero-divisors, and fields;
- The signature $\{+,\cdot,0,1,-,{}^{-1}\}$ is better suited (only) for fields.

The algebraic case we discussed is instructive and guides us towards more general considerations on the interplay between the complexity of concepts of a theory T, the first order language τ in which to express them, the properties of the τ-morphisms between models of T. First of all, the Lévy hierarchy gives a potentially useful hierarchy by which we can stratify the complexity of the concepts of a mathematical theory; furthermore our discussion suggests that for a given a signature τ, the basic concepts of a τ-theory T are not only those expressed by (boolean combinations of) atomic τ-formulae, but also those expressed by universal τ-sentences (e.g. the additive inverse is a simple operation for ring theory, being axiomatized in signature $\{+, \cdot, 0, 1, -\}$ by a universal sentence, while it is not for semi-ring theory, being axiomatized in signature $\{+, \cdot, 0, 1, -\}$ by a Π_2-sentence). The reason is that if $\mathcal{M} \sqsubseteq \mathcal{N}$ are τ-structures which model the same τ-theory T, the graph of a relation/operation $R^{\mathcal{M}}$ defined on $\mathcal{M}$ as prescribed by the universal axiom ψ_R of T is exactly the restriction to $\mathcal{M}$ of the graph of that same relation/operation $R^{\mathcal{N}}$ defined now on $\mathcal{N}$ as prescribed by ψ_R (in our example, $-$ is preserved by $\{+, \cdot, 0, 1, -\}$-morphisms of rings but not by $\{+, \cdot, 0, 1, -\}$-morphisms of semi-rings).

Summing up, when dealing with a mathematical theory T for which we have a clear picture of its intended class of models, the choice of a signature τ in which to formalize T can be made on the basis of which morphism between models of T we would like to be the τ-morphisms between the τ-models of T, or —correspondingly— on the basis of which concepts of T we would like to be of low logical complexity in the Lévy hierarchy induced by τ on T. Once this decision is made, this affects the type of signature τ in which the theory can be formalized, what are the τ-morphisms between the T-models, and what are the universal τ-axioms of T. Therefore, we can get different stratifications of the complexity of concepts for a theory by selecting the appropriate class of morphisms between its models and by selecting which concepts of T we require to be axiomatized by universal sentences.

4. Existentially closed models, model completeness, and model companionship

It is now the time to introduce the notion of existentially closed model and of model companionship[6]. These model-theoretic notions will be the structural properties of first order theories we will use to select good signatures for set theory. Before giving the precise mathematical definitions of these notions, let us start by introducing these concepts in the context of (semi-)ring theory.

A common trait of the set of axioms (2), (3), (4) is that they assert the existence of solutions to certain basic equations that are expressible by atomic formulae in the $\{+, \cdot, 0, 1\}$-theory given by axioms (1). Indeed, any commutative (semi-)ring $\mathcal{M}$ without zero divisors can be enlarged to an algebraically closed field simply by extending it with solutions to the basic polynomial equations with parameters in $\mathcal{M}$ (which in principle may not exist in $\mathcal{M}$). Notice that the process of adding new solutions to a commutative (semi-)ring can be seen as closing it with respect to its basic operations. It is here that the choice of a signature τ for the formalization of a theory T becomes relevant: the richer is τ, the more complex are the concepts expressible by atomic τ-formulae (expressing the basic properties) and universal τ-sentence (expressing the basic T-definable operations). Therefore, the richer is τ, the more closed-off are the models of a τ-theory that are closed with respect to the basic T-operations.

This closing-off process is one of the driving forces of mathematics. For example it is by adding new solutions to basic equations that commutative semi-rings with no zero-divisors (like $\mathbb{N}$) have been extended to commutative rings (like $\mathbb{Z}$, which contains all additive inverses), and then to fields (like $\mathbb{Q}$, which contains all multiplicative inverses), and finally to algebraically closed fields (like $\mathbb{C}$, in which all diophantine equations have a solution).

[6]Reference texts for existentially closed models, model completeness, and model companionship are [10] and the second author's notes [35].

Of course, the choice of the signature τ in which a theory T is formalized affects the outcome of this closing-off process. A key request is that the τ-theory T' describing the validities in the closed-off structures, should agree with T at least with respect to the Π_1-consequences of T in signature τ (as the latter express the properties of the basic concepts and operations of T according to τ). Indeed, only in this case we can say that the closing-off process has been performed with respect to the basic concepts and operations of T. The notion of existentially closed model defines precisely which structures are the outcome of this closing-off process.

Definition 4.1. Let τ be a first order signature and T be a τ-theory. A τ-structure $\mathcal{M}$ is T-ec if:

- There is some $\mathcal{N}$ τ-superstructure of $\mathcal{M}$ which models T.
- $\mathcal{M}$ is a Σ_1-substructure of $\mathcal{P}$ whenever $\mathcal{P}$ is a τ-superstructure of $\mathcal{M}$ which models T.

A caveat is in order: while a mathematical theory T can be unambiguosly defined regardless of any of its possible first order axiomatizations, to define unambiguously the notion of T-ec model we commit ourselves to choose a specific signature τ in which to formalize T.

Definition 4.2. Given a signature τ and a τ-theory T, $T_\forall^\tau$ is the set of universal τ-sentences which follows from T (accordingly we define $T_\exists^\tau$, $T_{\forall\exists}^\tau$, etc.).

While the set of logical consequences of T is independent of the signature in which we axiomatize it, the set $T_\forall^\tau$ heavily depends on the choice of τ. For example if T is the theory of rings, subtraction is axiomatized by an axiom in $T_\forall^\tau$ for $\tau = \{\cdot, +, 0, 1, -\}$ but this is not the case when we consider the signature $\sigma = \{\cdot, +, 0, 1\}$.

Notice that any T-ec τ-structure is a model of $T_\forall^\tau$. A less trivial observation is the following.

Fact 4.3. *[35, Prop. 1.9(2)] A τ-structure $\mathcal{M}$ is T-ec if and only if it is $T_\forall^\tau$-ec. Hence a τ-structure $\mathcal{M}$ which is T-ec is also S-ec for any τ-theory S such that $S_\forall^\tau = T_\forall^\tau$.*

It is not hard to see that algebraically closed fields, in the signature $\sigma = \{+, \cdot, 0, 1\}$, are S-ec structures for S the σ-theory of commutative (semi-)rings with no zero-divisors. Furthermore, notice that existentially closed models can be such with respect to different theories. For example, consider the two σ-theories U and R, respectively, of fields and of commutative rings with no zero-divisors that are not fields. Then, any σ-structure $\mathcal{M}$ which is an algebraically closed field is automatically S-ec, U-ec, and R-ec at the same time (since $R_\forall^\sigma = S_\forall^\sigma = U_\forall^\sigma$).

In general, given a τ-theory T, the class of T-ec models might not be elementary. In the specific case of algebraically closed fields this is the case, since this class of structures is axiomatized by the $\{+, \cdot, 0, 1\}$-theory ACF given by axioms (1) and (4); but this is a rather peculiar case, which also depends on the specific choice of the signature. On the contrary, if we consider the $\left\{\cdot, e, ^{-1}\right\}$-theory T of groups, then the T-ec $\left\{\cdot, e, ^{-1}\right\}$-structures do not form an elementary class[7] [10, Example 3.5.16]. Abraham Robinson introduced the notion of model companionship to describe exactly when the collection of T-ec τ-models form an elementary class:

Definition 4.4. Let τ be a signature.

[7]Moreover, none of the signatures for group theory that we presented in Section 2 have existentially closed models which give rise to an elementary classe

- A τ-theory T is *model complete*[8] if for any τ-structure $\mathcal{M}$

$$\mathcal{M} \models T \text{ if and only if } \mathcal{M} \text{ is } T_\forall^\tau\text{-ec.}$$

- A τ-theory R is the *model companion* of a τ-theory T if:
  (i) T and R are jointly consistent: i.e. $T_\forall^\tau = R_\forall^\tau$, (or equivalently —by [34, Lemma 2.1.1] or [10, Remark 3.5.6(2)]— every model of T embeds in a model of T^* and vice versa);
  (ii) R is model complete.

It is useful to recall that model complete theories are axiomatized by their Π_2-consequence (see [10, Prop. 3.5.10]), and that the model companion of a τ-theory T (if it exists) is unique (by [10, Prop. 3.5.13]).

We also note (as is the case for existentially closed structures) that a model complete τ-theory T can be the model companion of different τ-theories S: it suffices that $S_\forall^\tau = T_\forall^\tau$. For example let S_0 be the $\{\cdot, +, 0, 1\}$-theory of commutative rings with no zero-divisors that are not fields (i.e. the models of axioms (1), (2) and of the negation of (5)) and let S_1 be the $\{\cdot, +, 0, 1\}$-theory of fields. Then the model complete $\{\cdot, +, 0, 1\}$-theory ACF is the model companion of both S_0 and S_1. Another interesting observation is that no algebraically closed field is a model of S_0 and conversely. As a matter of fact, the notion of model companionship for a τ-theory S does not isolate a proper subclass of the models of S, but rather the models that are closed-off with respect to the basic operations and relations definable in S by universal τ-sentences.

Given the connection between model companions and existentially closed structures, it should be no surprise that the notion of model companionship is sensible to the choice of the signature. For example: consider the language $\{+, \cdot, 0, 1, ^{-1}\}$ and the theory S_0' which results by adding axiom (5) to axioms (1), (2). We obtain that S_0' is still the theory of commutative rings with no zero-divisors, now formalized in the signature $\{+, \cdot, 0, 1, ^{-1}\}$. However, the $\{+, \cdot, 0, 1, ^{-1}\}$-theory of algebraically closed fields ACF' given by axioms (1), (4), (5) is not the model companion of the $\{+, \cdot, 0, 1, ^{-1}\}$-theory S_0'. In order to see this, let $\mathcal{M}$ be the field of complex numbers and let $\mathcal{N}$ be $\mathbb{C}[X]$: the ring of polynomials in complex coefficients and variable X (both seen as $\{+, \cdot, 0, 1, ^{-1}\}$-structures). Then it is the case that $\mathcal{M} \sqsubseteq \mathcal{N}$, but now $\mathcal{N}$ models the Σ_1-sentence for $\{+, \cdot, 0, 1, ^{-1}\}$ $\exists y\, (y \neq 0 \wedge y^{-1} = 0)$ (as witnessed by the polynomial X), while $\mathcal{M}$ does not. In particular, $\mathcal{M}$ is not S_0'-ec, as $\mathcal{M}$ is not a Σ_1-substructure of $\mathcal{N}$, which is a model of S_0'.

Given that the existence of a model companion for a theory T depends so directly on the choice of the signature, we can use the notion of model companionship to select signatures:

Model-theoretic criterion for selecting signatures: Given a mathematical theory T and two signatures τ and σ in which T can be axiomatized, τ is preferable to σ for T if the τ-axiomatization of T admits a model companion, while its σ-axiomatization does not.

We will put to use this criterion in the case of set theory, while also outlining that the existence of a model companion for set theory in a given signature provides important information on the properties of the universe of sets.

[8]Usually model completeness is defined by requiring that the substructure relation overlaps with the elementary substructure relation (see [10, Def. on page 186]). Our definition is equivalent in view of [10, Prop. 3.5.15] combined with [34, Lemma 2.1.1].

5. Partial Morleyizations, absolute model companionship, and the AMC-spectrum of a theory

We can now give a more formal treatment of the ideas presented in the previous sections, by introducing the central concepts of (partial) Morleyization and absolute model companionship.

5.1. Partial Morleyizations.

Notation 5.1. Given a signature τ, let $\phi(x_0,\dots,x_n)$ be a τ-formula.

We let:

- R_ϕ be a new $n+1$-ary relation symbols,
- f_ϕ be a new n-ary function symbols[9]
- c_τ be a new constant symbol.

We also let:

$$\mathsf{AX}^0_\phi := \forall\vec{x}[\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x})],$$

$$\begin{aligned}\mathsf{AX}^1_\phi :=& \forall x_1,\dots,x_n \\ &[(\exists! y\phi(y,x_1,\dots,x_n) \to \phi(f_\phi(x_1,\dots,x_n),x_1,\dots,x_n))\wedge \\ &\wedge (\neg\exists! y\phi(y,x_1,\dots,x_n) \to f_\phi(x_1,\dots,x_n) = c_\tau)]\end{aligned}$$

for $\phi(x_0,\dots,x_n)$ having at least two free variables, and

$$\mathsf{AX}^1_\phi := [(\exists! y\phi(y)) \to \phi(f_\phi)] \wedge [(\neg\exists! y\phi(y)) \to c_\tau = f_\phi]\,.$$

for $\phi(x)$ having exactly one free variable.

Let Form_τ denotes the set of τ-formulae. For $A \subseteq \mathsf{Form}_\tau \times 2$

- τ_A is the signature obtained by adding to τ all relation symbols R_ϕ, for $(\phi,0)\in A$, and all function symbols f_ϕ, for $(\phi,1)\in A$ (together with the special symbol c_τ if at least one $(\phi,1)$ is in A).
- $T_{\tau,A}$ is the τ_A-theory having as axioms the sentences AX^i_ϕ for $(\phi,i)\in A$.

We let $\tau^* = \tau_C$ for $C = \mathsf{Form}_\tau \times \{0\}$ and T^* be the τ^*-theory $T_{\tau,C}$.

With respect to the theories in signature $\sigma = \{\cdot,+,0,1\}$ discussed in the previous section, Axiom (5) is (logically equivalent to) AX^1_ϕ for $\phi(x,y)$ being $x\cdot y = 1$ (assuming c_τ is interpreted by 0). Similarly Axiom (6) is (logically equivalent to) AX^1_ψ for $\psi(x,y)$ being $x+y=0$.

Remark 5.2. For any τ-theory T, T^* is a τ^*-theory admitting quantifier elimination (the Morleyization of T)[10].

Remark 5.3. Given a τ-formula ϕ, a τ-structure $\mathcal{M}$ with domain M, and a $c\in M$, there is exactly one extension of $\mathcal{M}$ to a $\tau_{\{\phi\}\times\{1\}}$-structures which interprets the value of the special constant c_τ as c and models AX^1_ϕ.

In what follows we want to analyze what happens when the Morleyization process is performed on arbitrary subsets of $\mathsf{Form}_\tau \times 2$.

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[9] As usual we confuse 0-ary function symbols with constants.

[10] See [28, Section 3.2, pp. 31-32].

5.2. **Absolute model companionship.** The following properties will bring us to introduce the notion of absolute model companionship.

- A τ-structure $\mathcal{M}$ is T-ec if and only if it is $T_\forall^\tau$-ec (by [35, Prop. 1.9(2)]).
- If T is *complete*, a T-ec structure $\mathcal{M}$ realizes any Π_2-sentence which holds true in some τ-model of $T_\forall^\tau$ (by [35, Prop. 1.9(5)]).
- If a τ-theory S is the model companion of a τ-theory T, S is axiomatized by its Π_2-consequences for signature τ (by [35, Thm. 1.20]).

Combining these results, one might wonder whether the model companion S of a τ-theory T (whenever it exists) could be axiomatized by the family of all Π_2-sentences for τ which holds in some model of $T_\forall^\tau$. This is not alway the case and, in fact, holds only when T is a *complete* model companionable theory. The standard counterexample is given by the $\{+,\cdot,0,1\}$-theory Fields_0 of fields of characteristic 0, given by axioms (1), (2), (3) and the infinitely many axioms granting that the characteristic of its models is 0 (Fields_0 is not a complete theory), and its model companion ACF_0 which adds (4) to the above axioms (ACF_0 is a complete theory). Indeed, the sentence $\forall x \neg(x^2+1=0)$ is a Π_2-sentence (actually Π_1) which is not in ACF_0 but holds in $\mathbb{Q}$, hence is consistent with the universal fragment of Fields_0. Absolute model companionship rules out counterexamples of this kind.

Definition 5.4. [34, Def. 1.1, Thm. 1.3] Given a τ-theory T, we say that a τ-sentence ψ is *strongly T-consistent for* τ whenever $\psi + R^\tau_{\forall\vee\exists}$ is consistent for any τ-theory R which extends T.

A τ-theory S is the *absolute model companion* (AMC) of a τ-theory T if it is the model companion of T and it is axiomatized by the Π_2-sentences (for τ) which are strongly T-consistent for τ.

Equivalently T is the AMC of S if (letting $R^\tau_{\forall\vee\exists}$ denote the set of boolean combinations of universal τ-sentences which follow from a τ-theory R):

- $T^\tau_{\forall\vee\exists} = S^\tau_{\forall\vee\exists}$,
- T is model complete.

A τ-theory S can have an AMC if, to some extent, it has already maximized the family of existential sentences which are consistent with its universal consequences. Let us elaborate more on this point. Given a τ-theory T, the family of strongly T-consistent Π_2-sentences for τ describes a fragment of the Π_2-sentences which hold in a T-ec τ-structure (see [34, Remark 2.2.8]). An AMC describes the set-up in which the choice of τ is such that the family of strongly T-consistent Π_2-sentences axiomatize a model complete theory. When this occurs the process of closing-off a τ-model with respect to the T-operations and relations (axiomatized by $T_\forall^\tau$-sentences) gives rise to models that are axiomatized by the Π_2-sentences which are strongly T-consistent for τ.

As we already mentioned, complete first order τ-theories are model companionable if and only if they admit an AMC. Indeed, for a complete theory S, the information conveyed by $S_\forall^\tau$ and by $S^\tau_{\forall\vee\exists}$ is the same. On the other hand we have already seen that there are non-complete τ-theories admitting a model companion but not an AMC, e.g. the $\{+,\cdot,0,1\}$-theory of fields.

Let us briefly analyze more in details what is the obstruction for the $\{+,\cdot,0,1\}$-theory of fields of characteristic 0 to have the $\{+,\cdot,0,1\}$-theory of algebraically closed fields of characteristic 0 as its AMC. The problem is that Diophantine equations can be expressed by atomic formulae of $\{+,\cdot,0,1\}$ and not all of them have rational solutions. On the other hand the existence of solutions for these equations does not contradict the field axioms. Note that the smallest existentially closed $\{+,\cdot,0,1\}$-model $\mathcal{M}$ for the theory of fields is given by the algebraically closed field consisting exactly of the solutions of diophantine equations. Its standard construction builds it as a direct limit of Galois extensions of the rationals each adding new solutions to polynomial equations with coefficients in $\mathbb{N}$ but without rational solutions. By performing this construction we can preserve the field axioms and the characteristic 0 (which holds in $\mathbb{Q}$ as well as in all the Galois extensions of $\mathbb{Q}$ under consideration), but at each stage we may invalidate some universal sentences that are true in $\mathbb{Q}$ (e.g. the universal sentence asserting that a certain diophantine equation does not have rational solutions). In particular the closing-off process performed in signature $\sigma = \{+, \cdot, 0, 1\}$ which brings from the theory of fields T to that of algebraically closed fields cannot be described only by the Π_2-sentences ψ which are strongly T-consistent for σ. Indeed, there are other Π_2-sentences which needs to be realized in an algebraically closed field and which are not strongly T-consistent for σ (e.g. the existential statements asserting the existence of solutions for diophantine equations). On the other hand we will argue that in the case of set theory AMC faithfully describes the closing-off process of models of set theory with respect to basic set-theoretic operations.

5.3. **The (A)MC-spectra of a first order theory.** Model theory has been extremely successful in classifying the complexity of a mathematical theory according to its structural properties, and has produced a variety of properties and criteria to separate the (so called) tame mathematical theories from the (so called) untamed ones (e.g. o-minimality, stability, simplicity, NIP). Typically a mathematical theory is considered hard to classify (and thus untamed) if it can code in itself first order arithmetic. In this respect the $\in$-theory ZFC is untamed.

We have already observed that most mathematical theories admit many different first order axiomatizations in (almost) as many distinct signatures. A common characteristic of tameness properties such as o-minimality, stability, simplicity, and NIP is that they are *signature invariant*. More precisely if we take a τ-theory T and we consider its Morleyization T^* in the signature τ^* (see Notation 5.1), T is o-minimal (stable, simple, NIP) if and only if so is T^*. In contrast, this is not the case for Robinson's notion of model companionship: we have already observed that the theory of algebraically closed fields is the model companion of the theory of commutative (semi-)rings with no zero divisors in signature $\{+, \cdot, 0, 1\}$, but this ceases to be case for the signature $\{+, \cdot, 0, 1, -, ^{-1}\}$. Conversely a σ-theory R may not have a model companion (e.g. the $\sigma = \{\cdot, 1\}$-theory of groups), but its Morleyization R^* in signature σ^* is always its own model companion. We instantiate the model-theoretic criterion formulated in the last part of Section 4 by means of the following:

Definition 5.5. Let T be a τ-theory.

- Its AMC-spectrum ($\mathfrak{spec}_{\mathsf{AMC}}(T, \tau)$) is given by the sets $A \subseteq \mathsf{Form}_\tau \times 2$ such that $T + T_{\tau,A}$ has an AMC (which we denote by $\mathsf{AMC}(T, A)$).
- Its MC-spectrum ($\mathfrak{spec}_{\mathsf{MC}}(T, \tau)$) is given by the sets $A \subseteq \mathsf{Form}_\tau \times 2$ such that $T + T_{\tau,A}$ has a model companion (which we denote by $\mathsf{MC}(T, A)$).

Clearly $\mathfrak{spec}_{\mathsf{MC}}(T, \tau)$ is a superset of $\mathfrak{spec}_{\mathsf{AMC}}(T, \tau)$ and the two can be distinct. Observe that $C = \mathsf{Form}_\tau \times \{0\}$ is always in the AMC-spectrum of a theory T, as $T + T_{\tau,C}$ admits quantifier elimination and therefore it is model complete and its own AMC in signature $\tau_C = \tau^*$. Moreover, $\emptyset$ is in the (A)MC-spectrum of T if and only if T has an (absolute) model companion. For $\sigma = \{+, \cdot, 0, 1\}$ we have that:

- $\emptyset$ is in $\mathfrak{spec}_{\mathsf{MC}}(T, \sigma) \setminus \mathfrak{spec}_{\mathsf{AMC}}(T, \sigma)$ for T the σ-theory of commutative (semi)rings with no 0-divisors or fields considered in Section 3.
- For ϕ the σ-formula defining the additive inverse, $\{(\phi, 1)\}$ is in $\mathfrak{spec}_{\mathsf{MC}}(T, \sigma)$ if T is the theory of commutative rings with no 0-divisors or fields, while $\{(\phi, 1)\}$ is not in $\mathfrak{spec}_{\mathsf{MC}}(S, \sigma)$ if S is the σ-theory of commutative semirings with no 0-divisors.

- For ψ the σ-formula defining the multiplicative inverse, $\{(\psi, 1)\}$ is in $\mathfrak{spec}_{\mathsf{MC}}(T, \sigma)$ if T is the theory of fields, while $\{(\psi, 1)\}$ is not in $\mathfrak{spec}_{\mathsf{MC}}(S, \sigma)$ if S is the σ-theory of commutative (semi)rings with no 0-divisors or the theory of fields.
- It can be shown that $\mathfrak{spec}_{\mathsf{AMC}}(T, \sigma) = \emptyset$ for T the σ-theory of commutative (semi)rings with no 0-divisors or fields.

For A in the AMC-spectrum of a τ-theory T, the problem of axiomatizing the existentially closed τ_A-models for $T + T_{\tau,A}$ becomes a consistency problem: that of checking whether a Π_2-sentence is strongly $T + T_{\tau,A}$-consistent for τ_A.

6. The (A)MC-spectra of set theory

From now on we focus our attention on set theory. Our main goal is to study the model companions of set theory and the relevant signatures for which they exist. We will not only provide a new set of arguments in favour of $2^{\aleph_0} = \aleph_2$, but we will also present a new method to detect set theoretic validities. The realist-minded readers can consider the following results as an attempt to produce a complete (first order) axiomatization of set theory. Indeed we will show that the model companions of set theory describe possible theories of larger and larger segments of the universe of sets $(V, \in)$. More precisely we will show that for each infinite cardinal κ there are appropriate signatures τ_κ which make a certain theory of H_{κ^+} the model companion of set theory in signature τ_κ. Although a realist stance towards mathematics is not needed to understand and prove these results, it is within a realist agenda that we can better appreciate their meaning. We can make more explicit this perspective outlining two important principles that will motivate the present model-theoretic approach.

Semantic realism: the universe of all sets $(V, \in)$ is sufficiently well-defined so that it is possible to provide a *unique* axiomatic first order axiomatization of it. Therefore, any set theoretical statement axiomatizable in first order logic receives, in $(V, \in)$, a well defined truth value.

Hilbertian completeness: $(V, \in)$ offers a *complete* picture of what exists in set theory. Therefore, $(V, \in)$ contains any set whose existence is not in direct contradiction with the already recognized truths of set theory.

In the context of a realist conception of sets, Hilbertian completeness yields a notion of *maximality* for set theory. Because of the importance of this notion in set-theoretical investigations, we defer to the last section of this paper a thorough discussion of the role model companionship can play in its clarification. Meanwhile, we can rephrase Hilbertian completeness (still in informal terms) in order to outline its connection with the notion of model companionship and the use we make of this idea.

(IEC) **Informal Existential Compleness** If $P(x, Y)$ is a "simple" property formalizable in the $\in$-signature in parameter some set Y of V and $\exists x\, P(x, Y)$ is consistent with the basic principles of set theory, then in V a witness of this existential statement should exist.

Our task now consists in spelling out a precise mathematical definition of which set theoretic properties are to be regarded as "simple" and which sets Y of V we can accept as parameters for the property P (also attention must be paid to single out what we exactly mean by "consistent with the basic principles of set theory" to avoid trivial counterexamples to (IEC)). Towards this end we can take advantage of our previous discussion about signatures and AMC.

(a) We determine that a property of sets P is "simple" by analyzing the peculiarity of set theory, as a mathematical theory. After choosing $A \subseteq \mathsf{Form}_{\{\in\}} \times 2$ and performing a partial Morleyization with respect to A, we can check whether $P(x, y)$ can be formalized by a quantifier free formula in signature $\{\in\}_A$. If this is the case we consider $P(x, y)$ simple (with respect to A). We clearly need criteria for the choice of A. Indeed, if $A = \mathsf{Form}_{\{\in\}} \times 2$, then any $\in$-formalizable set theoretic property becomes simple, since it is expressible by an atomic formula of $\{\in\}_A$. In order to avoid trivialities, we use set theoretic practice and forcing as a guiding tool to select the appropriate sets A and, consequently, the simple set theoretic properties. We regard a property $P(\vec{x})$ as simple if one or more of the following conditions are met:

(i) its $\in$-formalization can be expressed by a Δ_0-formula;

(ii) its meaning is invariant across forcing extensions: i.e. $P(\vec{a})$ holds in V if and only if it holds in some (any) generic extension of V;

(iii) the basic truths about $P(\vec{x})$ are forcing invariant, which roughly amounts to require that whenever $\psi(\vec{x})$ is an $\in$-formula formalizing $P(\vec{x})$ and $\phi(\vec{y})$ is a boolean combination of Δ_0-formulae and ψ, then the truth value of $\forall \vec{y}\, \phi(\vec{y})$ cannot be changed by forcing[11].

Notice that the second and the third conditions are implied by the first. There are properties $P(x)$ which are not expressible using Δ_0-formulae but for which nonetheless the second or third (or both) conditions apply; for example: the second condition applies to the provably Δ_1-properties for ZFC^- and (assuming large cardinals) to the property *"x belongs to a certain universally Baire set B"*, while the third holds for $P(x)$ being *"x is a stationary subset of* ω_1*"* (see Thm. 7.4 and the comments following it).

We can give separate motivations for why each of the above condition is a simplicity criterion: (i) is motivated by set theoretic practice (we expand on this in Section 6.1). (ii) and (iii) are simplicity criteria as they state that we cannot use forcing to change the basic truths regarding properties satisfying them. We consider forcing invariance a legitimate "simplicity" check, since forcing is the unique known efficient method to produce consistency results for existential statements and —by the results we aim to expose in this paper— also the unique method at all, at least for a large class of statements (we expand more on this in Section 6.3).

(b) Rather than focusing on the particularly delicate task of establishing which sets Y do exist in V and which do not (in order to argue whether $\exists x P(x, Y)$ is consistent), we just investigate whether the Π_2-sentence $\forall x \exists y\, P(x, y)$ is consistent with the basic principles of set theory. This task can be rephrased as checking whether the sentence $\forall x \exists y\, P(x, y)$ belongs to $\mathsf{AMC}(T, A)$, for T some $\in$-theory extending ZFC+*large cardinals*, and $A \subseteq \mathsf{Form}_{\{\in\}} \times 2$ such that $P(x, y)$ is formalizable by a quantifier free formula in signature $\{\in\}_A$. Moreover (assuming (IEC)), if $P(x, y)$ satisfies any of the conditions for simplicity listed in (a), we can regard the Π_2-sentence $\forall x \exists y P(x, y)$ as a set-theoretic validity, since it asserts that the (suitable fragment of the) universe of sets in which it holds is closed-off with respect to the "simple" operations and relations described by P.

Now that we have some insights on which mathematical criteria detect "simplicity" and on what we mean by "consistent with the basic principles of set theory", we can give a first general outline of the main results on the model companions of set theory.

(A) The (A)MC-spectrum of set theory can be used to characterize a theory of the various H_{κ^+} as κ ranges over the infinite cardinals. More precisely, for any signature τ that extends $\{\in\}$ and includes all Δ_0-formulae among its atomic formulae, if the τ-theory of sets admits a model companion, this model companion extends

[11]For example let $P(x)$ be the property *x is a stationary subset of* ω_1, then $\exists x\,[P(x) \wedge P(\omega_1 \setminus x)]$ is a ZFC-theorem hence its truth value cannot be changed by forcing. This is true regardless of the fact that whenever G is V-generic for a forcing collapsing ω_1^V to become countable, the witnesses of the truth of $\exists x\,[P(x) \wedge P(\omega_1 \setminus x)]$ in V and that of $\exists x\,[P(x) \wedge P(\omega_1 \setminus x)]$ in $V[G]$ cannot be the same set.

ZFC^-_τ (i.e. all axioms of ZFC with the exception of power-set and with replacement holding for all τ-formulae). Furthermore for any infinite cardinal κ we can cook up at least one signature τ_κ (extending $\{\in\}$ and including all Δ_0-formulae among its atomic formulae) such that the τ_κ-theory of H_{κ^+} is the AMC of the τ_κ-theory of V.

(B) If S is the $\{\in\}$-theory ZFC+*large cardinals*,[12] there is (more than one) B in $\mathfrak{spec}_{\mathsf{AMC}}(S, \{\in\})$ such that all quantifier free formulae of $\{\in\}_B$ express simple properties according to (a). Furthermore these sets B are chosen so that the corresponding $\mathsf{AMC}(S, B)$ describes a theory of H_{ω_2} and is axiomatized by the Π_2-sentences (in the signature $\{\in\}_B$) which hold in the H_{ω_2} of a model of strong forcing axioms. Among these Π_2-sentences there is a definable version of $2^{\aleph_0} = \aleph_2$.

(C) For any theory S as in (B), neither CH nor $2^{\aleph_0} > \aleph_2$ can be in $\mathsf{AMC}(S, A)$ for any A in $\mathfrak{spec}_{\mathsf{AMC}}(S, \{\in\})$.

(D) Finally for any S and B as in (B), the universal $\{\in\}_B$-theory of $S+T_{\{\in\},B}$ is invariant with respect to forcing. Moreover, any $\Pi_2$sentence ψ of $\{\in\}_B$ is in $\mathsf{AMC}(T, B)$ if and only if $\psi^{H_{\omega_2}}$ is forcible.

The resulting picture of V given by the above results (and on the basis of (IEC)) is that of a cumulative hierarchy whose initial segments are closed-off with respect to more and more complex set-theoretic operations (as κ increases), each initial segment being closed-off with respect to set theoretic operations the same way as an algebraically closed field is closed-off with respect to the solutions to its basic equations. When the H_{κ^+} described by these AMCs are closed-off with respect to basic set-theoretical concepts satisfying simplicity criteria (as those presented in (a)), then their theory realizes (IEC) and the Hilbertian completeness encoded by it. The results outlined in (B) show that this is the case for $\kappa = \omega_1$.

6.1. **What is the right signature for set theory?** The standard axioms of ZFC in the $\in$-signature are clearly sufficient to provide a first order axiomatization of set theory. However a closer inspection reveals that many simple set-theoretic concepts are not formalized by simple $\in$-formulae.

Consider, for example, the notion of ordered pair. While we informally write $x = \langle y, z\rangle$ to mean that *x is the ordered pair with first component y and second component z*, in set theoretic terms this statement hides a non-trivial coding of the concept of ordered pair (for example) by means of Kuratowski's definition: $x = \{\{y\}, \{y, z\}\}$. A proper definition of the concept of ordered pair in the $\in$-signature can then be given by the following $\in$-formula:

$$\exists t \exists u\, [\forall w\, (w \in x \leftrightarrow w = t \vee w = u) \wedge \forall v\, (v \in t \leftrightarrow v = y) \wedge \forall v\, (v \in u \leftrightarrow v = y \vee v = z)].$$

It is clear that the meaning of this $\in$-formula is hardly recognizable with a rapid glance (unlike $x = \langle y, z\rangle$). Moreover, from a purely logical perspective, its Lévy complexity is already Σ_2. This clashes with our understanding that the concept of ordered pair is simple. Indeed, we do not regard the notion of ordered pair as a complex concept, contrary to other more complicated and theoretically loaded ones like that of uncountability, or many of the properties of the continuum (such as its correct place in the hierarchy of uncountable cardinals). In a similar vein other very basic notions such as being a function, a binary relation, or the domain or the range of a function are formalized by rather complicated $\in$-formulae, both from the point of view of their readability and of their Lévy complexity.

The standard solution adopted in set theory textbooks[13] is to regard as basic all those $\in$-formulae in which the quantifiers are bounded to range over the elements of some set, that

[12]It is not essential here to specify what is meant by *large cardinals.* one could replace this assertion with the statement *there are class many Woodin and a supercompact*, or any axiom implying this hypothesis. See Thm. 7.4 below for a precise formulation of such an S.

[13]See for example [22, Chapter IV, Def. 3.5] or [19, Def. 12.9].

is the Δ_0-formulae. In order to make these observations precise we need to be extremely cautious on our notational conventions.

Notation 6.1. For any $A \subseteq \mathsf{Form}_{\{\in\}} \times 2$ we write $\in_A$ rather than $\{\in\}_A$, and we let $T_{\in,A}$ be the $\in_A$-theory

$$T_{\{\in\},A} + \forall x \left[(\forall y\, y \notin x) \leftrightarrow c_{\{\in\}} = x \right],$$

where the theory $T_{\{\in\},A}$ (according to Notation 5.1 for $\{\in\}$ and A) is reinforced by an axiom asserting that the interpretation of the constant symbol $c_{\{\in\}}$ is the empty set. We use the abbreviations $\in_{\Delta_0}$ and T_{Δ_0} to denote what, according to the above conventions, should rather be slight extensions of[14] $\in_{\Delta_0 \times \{0\}}$ and $T_{\in,\Delta_0\times\{0\}}$.

We also let for any $\tau \supseteq \in_{\Delta_0}$, ZFC_τ denote $\mathsf{ZFC} + T_{\Delta_0}$ enriched with the replacement axiom for all τ-formulae, and ZFC^-_τ denote ZFC_τ without the powerset axiom. ZFC_{Δ_0} denotes ZFC_τ for τ being $\in_{\Delta_0}$, accordingly we define $\mathsf{ZFC}^-_{\Delta_0}$.

The reason why set-theoretic practice regards the concepts expressed by quantifier free formulae of $\in_{\Delta_0}$ as "simple" is due to the fact that the truth value of these formulae is invariant among transitive models of large enough fragments of ZF (e.g. [22, Corollary IV.3.6]), and thus also forcing invariant (e.g. [19, Lemma 14.21]). Furthermore $\in_{\Delta_0}$ is a signature which allows to formalize many fundamental set theoretic concepts using formulae whose Lévy complexity is in accordance to our intuitive understanding (e.g. [19, Chapter 13, Lemma 13.10]).

For example consider the following notions and their logical complexity:

- (x is a cardinal) is the Π_1-formula (for $\in_{\Delta_0}$)
$$(x \text{ is an ordinal}) \wedge \forall f \left[(f \text{ is a function} \wedge \operatorname{dom}(f) \in x) \to \operatorname{ran}(f) \neq x \right].$$
- (x *is* $\aleph_1$) is the boolean combination of Σ_1-formulae
$$(x \text{ is a cardinal}) \wedge (\omega \in x) \wedge$$
$$\wedge \exists F \left[(F : \omega \times x \to x) \wedge \forall \alpha \in x \, (F \restriction \omega \times \{\alpha\} \text{ is a surjection on } \alpha) \right].$$
- CH is the Σ_2-sentence
$$\exists f \left[(f \text{ is a function} \wedge \operatorname{dom}(f) \textit{ is } \aleph_1) \wedge \forall r \subseteq \omega \, (r \in \operatorname{ran}(f)) \right].$$
and $\neg\mathsf{CH}$ is the boolean combination of Π_2-sentences[15]
$$\exists x \, (x \textit{ is } \aleph_1) \wedge \forall f \left[(\operatorname{dom}(f) \textit{ is } \aleph_1 \wedge f \text{ is a function}) \to \exists r \subseteq \omega \, (r \notin \operatorname{ran}(f)) \right].$$
- (x *is* $\aleph_2$) is the Σ_2-formula
$$(x \text{ is a cardinal}) \wedge$$
$$\wedge \exists F \exists y \left[(y \textit{ is } \aleph_1) \wedge (y \in x) \wedge (F : y \times x \to x) \wedge \forall \alpha \in x \, (F \restriction y \times \{\alpha\} \text{ is a surjection on } \alpha) \right].$$
- $2^{\aleph_0} > \aleph_2$ is the boolean combination of Π_2-sentences
$$\exists x \, (x \textit{ is } \aleph_2) \wedge \forall f \left[(f \text{ is a function} \wedge \operatorname{dom}(f) \textit{ is } \aleph_2) \to \exists r \, (r \subseteq \omega \wedge r \notin \operatorname{ran}(f)) \right].$$
- $2^{\aleph_0} \leq \aleph_2$ is the Σ_2-sentence
$$\exists f \left[(f \text{ is a function}) \wedge \operatorname{dom}(f) \textit{ is } \aleph_2 \wedge \forall r \, (r \subseteq \omega \to r \in \operatorname{ran}(f)) \right].$$

Let us also introduce the notation we will use to handle the substructure relation over expanded signatures. The following conventions supplement Notation 5.1.

[14] The precise definition of $\in_{\Delta_0}$ and T_{Δ_0} can be found in [34, Notation 3.1.1]. They are obtained by enriching $\in_{\Delta_0 \times \{0\}}$ and $T_{\in,\Delta_0\times\{0\}}$ with a constant symbol for ω and function symbols for the Goedel operations (as defined in [19, Def. 13.6]) and axioms to interpret them correctly.

[15] We let $\neg\mathsf{CH}$ include the Σ_2-sentence $\exists x \, (x \textit{ is } \aleph_1)$, for otherwise its failure could be witnessed by the assertion that there is no uncountable cardinal, a statement which holds true in H_{ω_1}, regardless of whether CH or its negation is true in the corresponding universe of sets.

Notation 6.2. Given τ-structures $\mathcal{M}$, $\mathcal{N}$ we write $\mathcal{M} \sqsubseteq \mathcal{N}$ to denote the substructure relation, while we use $\mathcal{M} \prec_n \mathcal{N}$ to denote the Σ_n-substructure relation. Morevoer, $\mathcal{M} \prec \mathcal{N}$ denotes the elementary substructure relation.

Notation 6.3. Let $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$ be a signature with κ a constant symbol, (M, τ^M) a τ-structure, and B a subset of $\mathsf{Form}_\tau \times 2$. Then, (M, τ_B^M) is the unique extension of (M, τ) defined in accordance with Notations 5.1 which satisfies $T_{\tau,B}$. In particular (M, τ_B^M) is a shorthand for $(M, S^M : S \in \tau_B)$. If (N, τ^N) is a substructure of (M, τ^M) we also write (N, τ_B^M) as a shorthand for $(N, S^M \restriction N : S \in \tau_B)$.

6.2. **Lévy absoluteness and the possible AMCs of set theory.** Recall that for an infinite cardinal λ, H_λ is the initial transitive fragment of V given by those sets whose transitive closure has size less than λ and is by itself a set in V (see [22, Section IV.6]). We have the following results.

Stratification of V: the universe of sets V can be stratified as the union, along the class of infinite cardinals κ, of the sets H_{κ^+}.

Standard theory of H_λ: [22, Thm. IV.6.5] for any uncountable cardinal λ, $(H_\lambda, \in_{\Delta_0}^V)$ is a model of $\mathsf{ZFC}^-_{\Delta_0}$ (recall Notation 6.1).

Strong Lévy absoluteness: [31, Lemma 5.3] For all $A_i \subseteq \mathcal{P}(\kappa)^{n_i}$ for $i = 1, \dots, k$

$$(H_{\kappa^+}^V, \in_{\Delta_0}^V, A_1, \dots, A_k) \prec_1 (V, \in_{\Delta_0}^V, A_1, \dots, A_k).$$

Second order characterization of H_{κ^+}: Whenever $\mathcal{M}$ is an $\in_{\Delta_0} \cup \{\kappa\}$-model of $\mathsf{ZFC}_{\Delta_0}+$ *κ is an infinite cardinal* $(H_{\kappa^+}^{\mathcal{M}}, \in_{\Delta_0}^{\mathcal{M}}, \kappa^{\mathcal{M}})$ is a model of the Π_2-sentence for $\in_{\Delta_0} \cup \{\kappa\}$

$$\forall x \exists f \, (f : \kappa \to x \text{ is a surjection}). \tag{7}$$

Furthermore $H_{\kappa^+}^V$ can be described as the unique set $M \in V$ such that;

- $\kappa \in M$ and M is transitive;
- $(M, \in_{\Delta_0}^V)$ satisfies (7) and $\mathsf{ZFC}^-_{\Delta_0}$;
- M has the (second order property) that for all $a \in M$ and $b \subseteq a$, $b \in M$ as well.

In particular the theory of each H_{κ^+} offers a Σ_1-approximation of the theory of V with respect to a signatures τ_κ which extends $\in_{\Delta_0} \cup \{\kappa\}$ and contains predicates for subsets of $\mathcal{P}(\kappa)^n$ for some $n \in \mathbb{N}$. Consequently, if we want to study the Σ_1-properties of the reals (second order arithmetic), or the powerset of the reals (third order arithmetic) it is sufficient to study it within the theory of some H_{κ^+} for κ a large enough regular cardinal (and for most purposes $H_{\aleph_1}$ suffices for second order arithmetic, and $H_{\aleph_2}$ for third-order arithmetic). Moreover, in view of Lévy absoluteness, H_{κ^+} and V agree on the computation of many simple set theoretic operations and relations: for example the operations and relations formalizable by means of Δ_0-formulae and Δ_0-definable Skolem functions.

The above results entail that whenever T is a τ-theory that extend $\mathsf{ZFC}_{\Delta_0} + \kappa$ *is a cardinal*, τ is a signature that contain $\in_{\Delta_0} \cup \{\kappa\}$ and is contained in

$$\in_{\Delta_0} \cup \{A : A \subseteq \mathcal{P}(\kappa)^n, \, n \in \mathbb{N}\},$$

and (V, τ^V) models T, then

$$(H_{\kappa^+}^V, \tau^V) \prec_1 (V, \tau^V).$$

Hence (H_{κ^+}, τ^V) witnesses that (7) is consistent with the universal fragment of the τ-theory of V. In particular if T has an AMC for τ, say S, then (7) must be in S, since it is a Π_2-sentence which is strongly T-consistent for τ.

We can now collect all these observations and conclude the following.

(1) The structure of V is uniquely determined by the structure of the various H_{κ^+} as κ ranges among the infinite cardinals.

(2) Given an infinite cardinal κ, consider a signature τ extending $\in_{\Delta_0}$ only with predicates for subsets of $\mathcal{P}(\kappa)$ and a constant symbol for κ. Then, if some $T \supseteq \mathsf{ZFC}_\tau$ has an AMC, this AMC looks like a theory of H_{κ^+} and is axiomatized by the Π_2-sentences which are strongly T-consistent for τ.

Now assume T is some $\in$-theory $T \supseteq \mathsf{ZFC}+$*large cardinals* and $A \subseteq \mathsf{Form}_\in \times 2$ is in $\mathfrak{spec}_{\mathsf{AMC}}(T, \{\in\})$. Assume further that:

- for some constant symbol κ of $\in_A$
$$\mathsf{ZFC}^- + \forall X \exists f\,(f : \kappa \to X \text{ is a surjection})$$
is in $\mathsf{AMC}(T, A)$;
- every atomic formula of $\in_A$ satisfies at least one of the simplicity criteria set forth in (a).

Then (on the basis of (IEC)) we should conclude that $\mathsf{AMC}(T, A)$ describes the correct theory of H_{κ^+}, since criterion (b) is satisfied by all axioms of $\mathsf{AMC}(T, A)$.

6.3. **Existentially closed fragments of the set theoretic universe versus AMC.** We conclude this section giving a non-exhaustive list of results that show that, for certain infinite cardinals κ, the corresponding H_{κ^+} already witnesses some important existential closure properties (e.g. Lévy absoluteness, Shoenfield's absoluteness, BMM, BMM^{++}) and its theory has some traits which are peculiar of model complete theories (e.g. Woodin's absoluteness, MM^{+++}).[16] The reader needs not be familiar with these results, as they only serve as motivation to bring our focus on existentially closed models for set theory.[17]

Lévy absoluteness: [31, Lemma 5.3] Whenever λ is a regular uncountable cardinal,
$$(H_\lambda, \in^V_{\Delta_0}) \prec_1 (V, \in^V_{\Delta_0}).$$

Shoenfield's absoluteness: (see [33, Lemma 1.2] for the apparently weaker formulation we give here) Whenever G is V-generic for some forcing notion in V,
$$(H_{\omega_1}, \in^V_{\Delta_0}) \prec_1 (V[G], \in^{V[G]}_{\Delta_0}).$$

Woodin's absoluteness: (see [33, Lemma 3.2] for the weak form of Woodin's result we give here) Whenever G is V-generic for some forcing notion in V (and there are class many Woodin cardinals in V),
$$(H^V_{\omega_1}, \in^V_{\Delta_0}) \prec (H^{V[G]}_{\omega_1}, \in^{V[G]}_{\Delta_0}).$$

Bounded Martin's Maximum (BMM): [5] Whenever G is V-generic for some stationary set preserving forcing notion in V,
$$(H_{\omega_2}, \in^V_{\Delta_0}) \prec_1 (V[G], \in^{V[G]}_{\Delta_0}).$$

BMM^{++}: [39, Def. 10.91] Whenever G is V-generic for some stationary set preserving forcing notion in V,
$$(H_{\omega_2}, \in^V_{\Delta_0}, \mathbf{NS}^V_{\omega_1}) \prec_1 (V[G], \in^{V[G]}_{\Delta_0}, \mathbf{NS}^{V[G]}_{\omega_1}),$$
where $\mathbf{NS}_{\omega_1}$ is a unary predicate symbol interpreted by the non-stationary ideal on ω_1.

Bounded category forcing axioms, MM^{+++}, $\mathsf{RA}_\omega(\mathsf{SSP})$: [3, 4, 32] Whenever V and $V[G]$ are models of MM^{+++} ($\mathsf{RA}_\omega(\mathsf{SSP})$, $\mathsf{BCFA}(\mathsf{SSP})$) and G is V-generic for some stationary set preserving forcing notion in V,
$$(H^V_{\omega_2}, \in^V_{\Delta_0}) \prec (H^{V[G]}_{\omega_2}, \in^{V[G]}_{\Delta_0}).$$

[16] Recall that a τ-theory T is model complete if and only if the substructure relation between its models overlaps with the elementary substructure relation. In particular the mentioned results are weak forms of model completeness for the theory of $H_{\aleph_i}$ for $i = 1, 2$.

[17] For further examples of existentially closed models of set theory see [30].

7. Main results

We can now present results culminating the discussion of the previous sections. The proofs can be found in [34].

Notation 7.1. For $A \subseteq \mathsf{Form}_\in \times 2$, ZFC_A denotes the $\in_A$-theory $\mathsf{ZFC}_{\in_A}$ defined in Notation 6.1. Accordigly we define ZFC^-_A.

Definition 7.2. Let $T \supseteq \mathsf{ZFC}^-$ be an $\in$-theory. κ is a T-definable cardinal if for some $\in$-formula $\phi_\kappa(x)$, T proves:

- $\exists! x\, \phi_\kappa(x)$ and
$$\forall x\, [\phi_\kappa(x) \to (x \textit{ is a cardinal})].$$
- κ is the constant f_{ϕ_κ} existing in the signature $\in_{\{(\phi_\kappa,1)\}}$.

7.1. **The AMCs of set theory are theories of H_{κ^+}.** The first result shows that the AMC spectrum of set theory isolates a rich set of theories which produce models of ZFC^-, that is, structures which behave like an H_λ for some regular uncountable cardinal λ.

Theorem 7.3. *Let R be an $\in$-theory extending* ZFC.

(i) Assume $A \in \mathfrak{spec}_{\mathsf{MC}}(R, \in)$ and $\in_A \supseteq \in_{\Delta_0}$. Then $\mathsf{MC}(R, A)$ models ZFC^-_A.

(ii) Assume $A \in \mathfrak{spec}_{\mathsf{MC}}(R, \in)$, $\in_A \supseteq \in_{\Delta_0}$, and λ is an R-definable cardinal represented by a constant symbol of $\in_A$ and such that[18]
$$(H^{\mathcal{M}}_{\lambda^+}, \in^{\mathcal{M}}_A) \prec_1 \mathcal{M}$$
whenever $\mathcal{M}$ models $R + T_{\in,A}$. Then $\forall x \exists f\, (f : \lambda \to x \text{ is a surjection})$ is in $\mathsf{MC}(R, A)$.

(iii) Assume λ is an R-definable cardinal. Then there exists $A_\lambda \in \mathfrak{spec}_{\mathsf{AMC}}(R, \in)$ with $\in_{A_\lambda}$ containing $\in_{\Delta_0}$ such that $\mathsf{AMC}(R, A_\lambda)$ is given by the $\in_{A_\lambda}$-theory common to the structures $H^{\mathcal{M}}_{\lambda^+}$ as $\mathcal{M}$ ranges among the $\in_{A_\lambda}$-models of $R + T_{\in,A_\lambda}$.

7.2. **Forcibility versus absolute model companionship.** The following is the major result relating AMC to forcibility and to forcing axioms[19]:

Theorem 7.4. *Let S be the $\in$-theory*
$$\mathsf{ZFC} + \text{there exists class many supercompact cardinals.}$$
Then there is a set $B \in \mathfrak{spec}_{\mathsf{AMC}}(S, \in)$ with $\in_B$ containing $\in_{\Delta_0}$, and such that for any Π_2-sentence ψ for $\in_B$ and any $\in$-theory $R \supseteq S$ the following are equivalent:

(a) $\psi \in \mathsf{AMC}(R, B)$;

(b) $(R + T_{\in,B})^{\in_B}_{\forall\vee\exists} + S + \mathsf{MM}^{++} + T_{\in,B}$ proves[20] *$\psi^{H_{\omega_2}}$;*

(c) R proves that $\psi^{H_{\omega_2}}$ is forcible[21] *by a stationary set preserving forcing;*

(d) R proves that $\psi^{H_{\omega_2}}$ is forcible by some forcing;

(e) For any $R' \supseteq R$, $\psi + (R' + T_{\in,B})^{\in_B}_{\forall\vee\exists}$ is consistent.

Furthermore for any θ which is a boolean combination of Π_1-sentences for $\in_B$ and any $(V, \in)$ model of S, TFAE:

(A) $(V, \in^V_B)$ models θ;

[18] $H^{\mathcal{M}}_{\lambda^+}$ denotes the substructure of $\mathcal{M}$ whose extension is given by the formula defining H_{λ^+} in the model (using the parameter λ).

[19] The reader unaware of what is MM^{++} or a stationary set preserving forcing can skip the second and third items of the theorem.

[20] Here and elsewhere we write ψ^N to denote the relativization of ψ to a definable class (or set) N; see [22, Def. IV.2.1] for details. Recall that for a τ-theory S $S^\tau_{\forall\vee\exists}$ denotes the boolean combinations of universal τ-sentences which follow from S.

[21] Here and in the next item we mean that the $\in$-formula θ which is $T_{\in,B}$-equivalent to ψ is such that $\theta^{H_{\omega_2}}$ is forcible by the appropriate forcing.

(B) $(V, \in_B^V)$ *models that some forcing notion* P *forces* θ*;*
(C) $(V, \in_B^V)$ *models that all forcing notions* P *force* θ*.*

The second part of the theorem shows that forcing cannot change the Π_1-fragment of the theory of V in signature $\in_B \supseteq \in_{\Delta_0}$. Notice also that if $(V, \in)$ is a model of S and R is the $\in_B$-theory of its unique extension to a model of $T_{\in,B}$, we get that a Π_2-sentence ψ for $\in_B$ is consistent with the universal fragment of R if and only if $\psi^{H_{\omega_2}}$ is forcible over V.

We give an accurate definition of $\in_B$ in [34]; here we just mention that B is a recursive set extending $\in_{\Delta_0}$ with a predicate symbol for the non-stationary ideal on ω_1, a constant symbol for ω_1, and predicate symbols for all sets of reals definable by $\in$-formulae without parameters in the Chang model $L(\mathrm{Ord}^\omega)$ (which by an unpublished result of Woodin form an interesting subclass of the universally Baire sets, assuming the large cardinal hypothesis of the Theorem).

We can also drop any reference to AMC and $\in_B$ and prove the following result which relates forcibility to consistency for Π_2-sentences in the signature $\in_{\Delta_0}$.

Theorem 7.5. *Let* S *be the theory of Thm. 7.4.*

For any Π_2*-sentence* ψ *for the signature* $\in_{\Delta_0}$ *and for any* $\in$*-theory* $R \supseteq S$ *the following are equivalent:*

(1) $(R + T_{\Delta_0})_{\forall\vee\exists}^{\in_{\Delta_0}} + S + \mathsf{MM}^{++} + T_{\Delta_0}$ *proves* $\psi^{H_{\omega_2}}$*;*
(2) R *proves that* $\psi^{H_{\omega_2}}$ *is forcible by a stationary set preserving forcing;*
(3) R *proves that* $\psi^{H_{\omega_2}}$ *is forcible;*
(4) For any consistent $\in$*-theory* $R' \supseteq R$, $\psi + (R' + T_{\Delta_0})_{\forall\vee\exists}^{\in_{\Delta_0}}$ *is consistent.*

7.3. **The AMC-spectrum of set theory and the continuum problem.** The AMC spectrum of set theory places $\aleph_2$ in a very special position among the possible values of the continuum.

Theorem 7.6. *Let* S *be the* $\in$*-theory of Thm. 7.4. The following holds:*

(1) Let $R \supseteq S$ *be an* $\in$*-theory. Assume* $A \in \mathfrak{spec}_{\mathsf{AMC}}(R, \in)$ *is such that* $\in_A$ *contains* $\in_{\Delta_0}$ *and* $\neg\mathsf{CH} + \mathsf{ZFC} + (R + T_{\in,A})_{\forall\vee\exists}^{\in_A}$ *is consistent. Then* $\mathsf{CH} \notin \mathsf{AMC}(R, A)$*.*
(2) For the signature $\in_B$ *of Thm. 7.4* $\neg\mathsf{CH}$ *is in* $\mathsf{AMC}(R, B)$ *for any* $\in$*-theory* $R \supseteq S$*.*

Let us briefly argue why CH cannot be in $\mathsf{AMC}(R, A)$ whenever $R + \neg\mathsf{CH}$ is consistent. We use the following peculiar property of AMC (which can fail for model companionship[22]):

Fact 7.7. *[34, Lemma 2.1.2, Def. 2.2.4, Lemma 2.2.7] Let* T, S *be* τ*-theories with* T *the AMC of* S*. Then for any completion* S' *of* S *there are models* $\mathcal{M} \prec_1 \mathcal{N}$ *such that* $\mathcal{M}$ *models* T *and* $\mathcal{N}$ *models* S'*.*

Now $R + \neg\mathsf{CH}$ is consistent by assumption, hence so is $R + \neg\mathsf{CH} + T_{\in,A}$, since the latter is a conservative extension of the former. Morevoer, since $\in_A \supseteq \in_{\Delta_0}$, we get that $\neg\mathsf{CH}$ is expressible by the boolean combination of a Π_2-sentence of $R + T_{\in,A}$ with the Σ_2-sentence for $\in_{\Delta_0}$ *the first uncountable cardinal exists*. By the above, we can find $\mathcal{M} \prec_1 \mathcal{N}$ with $\mathcal{M}$ a model of $\mathsf{AMC}(R, A)$ and $\mathcal{N}$ a model of $R + T_{\in,A} + \neg\mathsf{CH}$. If $\mathcal{M}$ does not model *there exists an uncountable cardinal* then $\neg\mathsf{CH}$ holds in it; otherwise the Π_2-sentence which is the other conjunct of $\neg\mathsf{CH}$ holds in $\mathcal{N}$ and therefore reflects to $\mathcal{M}$. In either case we conclude that $\neg\mathsf{CH}$ holds in $\mathcal{M}$. Hence $\mathsf{AMC}(R, A)$ cannot prove CH.

Let us also argue why $\neg\mathsf{CH}$ falls in the AMC of $S + T_{\in,B}$ for the set $B \subseteq \mathsf{Form}_\in \times 2$ of (2) above. Towards this aim we appeal to the equivalence between (A) and (C) of Thm. 7.4 (which gives a precise instantiation of simplicity criterion (iii)). First of all notice that

[22] For example no algebraically closed field can be a $\{+, \cdot, 0, 1\}$-substructure of a $\{+, \cdot, 0, 1\}$-structure elementarily equivalent to $\mathbb{Q}$.

an $\in$-theory $R \supseteq S$ is complete if and only if so is $R + T_{\in,B}$. Let R be some $\in$-completion of S and $(V, \in)$ a model of R. We now note that $\in_B$ has a constant κ to denote ω_1, and $\mathsf{AMC}(S, B)$ satisfies the $\in_{\Delta_0} \cup \{\kappa\}$-sentence *"$\kappa$ is the first uncountable cardinal"* Let B be a cba such that $[\![\neg\mathsf{CH}]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$ holds in $(V, \in)$.[23] Let G be any ultrafilter on B. Then

$$V^{\mathsf{B}}/_G \models \forall f (f \text{ is a function with domain } \kappa \to \exists r \subseteq \omega \text{ which is not in its range})$$

(by [19, Lemma 14.14]) and $V^{\mathsf{B}}/_G$ also models the universal $\in_B$-fragment of $R + T_{\in,B}$ (by (A)⇔(C) of Thm. 7.4, and [19, Lemma 14.14]). In particular the Π_2-conjunct of $\neg\mathsf{CH}$ is consistent with the universal fragment of any completion of $S + T_{\in,B}$, hence it belongs to the AMC of $S + T_{\in,B}$, while the Σ_2-conjunct of $\neg\mathsf{CH}$ is in the AMC of $S + T_{\in,B}$ since the axiom Ax^1_ψ for ψ defining the first uncountable cardinal is in $T_{\in,B}$.

In particular we see that forcing becomes a powerful tool to prove that a Π_2-sentence formalizable in $\in_B$ is the AMC of $S + T_{\in,B}$. Indeed, it suffices to prove over S that this sentence is forcible.

We can prove exactly the same type of result replacing CH by $2^{\aleph_0} > \aleph_2$. Specifically Moore introduced in [26] a Π_2-sentence θ_{Moore} for $\in_{\Delta_0}$ to show the existence of a definable well order of the reals in type ω_2 in models of the bounded proper forcing axiom. We can use θ_{Moore} as follows.

Theorem 7.8. *There is a Π_2-sentence θ_{Moore} for $\in_{\Delta_0}$ such that the following holds:*

(1) *θ_{Moore} is independent of $S + T_{\Delta_0}$, where S is the $\in$-theory of Thm. 7.4.*
(2) *$\mathsf{ZFC}^-_{\Delta_0} + \exists x\,(x \text{ is } \aleph_1) + \theta_{\text{Moore}}$ proves that there exists a well-ordering of[24] $\mathcal{P}(\omega)$ in type at most ω_2.*
(3) *$\mathsf{ZFC}^-_{\Delta_0} + \exists x\,(x \text{ is } \aleph_2) + \theta_{\text{Moore}}$ proves that $2^{\aleph_0} \leq \omega_2$.*
(4) *For S and $\in_B$ the theory and signature considered in Thm. 7.4, $\exists x\,(x \text{ is } \aleph_1), \theta_{\text{Moore}}$ are both in $\mathsf{AMC}(R, B)$ for any $\in$-theory R extending S.*
(5) *If R extends S, $A \in \mathfrak{spec}_{\mathsf{AMC}}(R, \in)$ is such that $\in_A$ contains $\in_{\Delta_0}$, $\exists x\,(x \text{ is } \aleph_2) \in \mathsf{AMC}(R, A)$ and*
$$\theta_{\text{Moore}} + (R + T_{\in,A})^{\in_A}_{\forall\vee\exists} + \mathsf{ZFC}$$
is consistent, then $2^{\aleph_0} > \aleph_2$ is not in $\mathsf{AMC}(R, A)$.

The two theorems single out $2^{\aleph_0} = \aleph_2$ among all possible solutions of the continuum problem.[25] For any $\in$-theory R extending ZFC+*large cardinals* there is at least one $B \in \mathfrak{spec}_{\mathsf{AMC}}(R, \in)$ with $\neg\mathsf{CH}$ (and a definable version of $2^{\aleph_0} \leq \aleph_2$) in $\mathsf{AMC}(R, B)$, and this occurs even if $R \models \mathsf{CH}$ or $R \models 2^{\aleph_0} > \aleph_2$. On the other hand for any $\in$-theory R extending ZFC, if CH is independent of R, then CH is never in $\mathsf{AMC}(R, A)$ for any $A \in \mathfrak{spec}_{\mathsf{AMC}}(R, \in)$ (if $\in_A$ contains $\in_{\Delta_0}$) and similarly if θ_{Moore} is independent of R, $2^{\aleph_0} > \aleph_2$ is never in $\mathsf{AMC}(R, A)$ for any $A \in \mathfrak{spec}_{\mathsf{AMC}}(R, \in)$ (if $\in_A$ contains $\in_{\Delta_0}$). Furthermore the last part of Thm. 7.4 shows that CH, $2^{\aleph_0} = \aleph_2$, $2^{\aleph_0} > \aleph_2$, θ_{Moore} are all boolean combination of Π_2-sentences in the signature $\in_{\Delta_0}$ which cannot be expressed by boolean combination of Π_1-sentences for the signature $\in_B \supseteq \in_{\Delta_0}$ in models of S (with S and B as in Thm. 7.4), as their truth value can be changed by forcing.

[23] For example the boolean completion of the partial order originally devised by Cohen. See for details [19, Thm. 14.32]

[24] More precisely: there is a a $\mathsf{ZFC}^-_{\Delta_0}$-provably Δ_1-property $\psi(x, y, z)$ such that in any model $\mathcal{M}$ of the mentioned theory there is a parameter $d \in \mathcal{M}$ such that $\psi(x, y, d)$ defines an injection of $\mathcal{P}(\omega)$ of the model with the class of ordinals of size at most ω_1 of the model.

[25] These results bring to light the role of forcing axioms in deciding the value of the continuum; an overview of the proofs of $2^{\aleph_0} = \aleph_2$ given by forcing axioms is given for example in [27].

8. On maximality and justification

In this last section we focus on two philosophical aspects of the present approach. First we address the following question: which notion of maximality is displayed by the model companions of ZFC+*large cardinals*? Notice that the role of large cardinals and strong forcing axioms in providing model completeness for the theory of $H_{\aleph_2}$ suggests a justification of the formers in terms of the clarification they provide of the latter. This observation suggests a second question: is there a form of justification of set-theoretical axioms that emerges from the present approach? In the remainder of this section we address these two questions.

8.1. **Maximality.** Maximality principles play an important role in the study of the axiomatic extensions of ZFC. Maximality conveys the idea that the universe of sets is as full as it could be, while maximality principles are (first order) statements that try to capture this idea axiomatically.[26]

There is no unique way to realize maximality in set theory, and its various forms depend on what in fact is maximized. Indeed, the process of maximization can be applied to objects, possibilities, or even domains.

- With respect to objects we find two standard forms of maximization: height maximality (expressing maximality for the lengths of the series of ordinals) and width maximality (expressing the maximality of the power-set operation). The former is successfully exemplified by large cardinals axioms [21], while the latter represents a cluster of notions often connected to the justification of generic absoluteness principles [6] or forms of horizontal reflection like the Inner Model Hypothesis [1].
- For what concerns the maximization of possibilities, we find modal principles of forcing like the so-called Maximality Principle [14] (expressing the idea that everything that is possibly necessary is true) or resurrection axioms [3,4,16] (expressing the idea that if something is true, then it is necessarily possible).
- Finally, the maximization of domains is a strategy inspired by Hilbert's (second order) axiom of completeness that found a (first order) axiomatic realization in McGee's Completeness Principle [25]. This principle, like Hilbert's axiom, is sufficient to prove categoricity for the class of pure sets (in a context of a set theory with Urelemente).

Now, *what form of maximality is displayed by the notion of model companionship?* Can it be compared to any of the above forms of set-theoretical maximality?

At a very general level it is hard to say whether the maximality provided by models companions concerns syntax, semantics, or ontology. Indeed, not only model companions are model complete theories (a semantic property), but they also maximize the Π_2-sentences consistent with their universal fragment (a syntactic property). Furthermore, a model companion can be considered as a maximization of the ontology of its models, in view of their closure properties. Given this multifaceted character of model companionship, let us compare it with all the three forms of maximality outlined above.

We can start by considering the maximization of objects (or, more appropriately, of witnesses of existential sentences). For what concerns height maximality, this notion is successfully realized by large cardinal axioms. While the model companionship results for set theory do not seem to enforce this form of maximality, they can nonetheless be used to justify it, as we will show in the next section. On the other hand, for what concerns width maximality, notice that bounded forcing axiom (and similar principles meant to express forms of horizontal maximality) can be reformulated in terms of generic absoluteness properties for H_{κ^+}. The key observation, then, consists in noticing that

[26]See [18] for a complete and informative survey on the topic of maximality principles in set theory.

these generic absoluteness results are weakenings of the notion of model companionship, since they (only) describe the relationship of the $H_{\kappa^+}^V$ with the generic multiverse (i.e. the collection of all $H_{\kappa^+}^{V[G]}$ as $V[G]$ ranges over the forcing extensions of V). However, the model-theoretic properties of model companions cannot be reduced to generic absoluteness results. Indeed, the invariance of the theory of $H_{\kappa^+}^V$ (as the model companion of the theory of V expressed in a signature which satisfies the simplicity criteria outlined in (a)) does not only hold with respect to forcing extensions of V, but also with respect to *any* model-theoretic extension whatsoever. In this sense, model companionship results for signatures satisfying the criteria of (a) yield a form of absoluteness for H_{κ^+} that is stronger than that provided by generic absoluteness.

For what concerns the modal principles of forcing, we can argue as for width maximality. As a matter of fact, the Maximality Principle and the resurrection axioms as exposed in [3, 4, 14, 16] are, by all means, principles of generic absoluteness for the theory of $H_{\aleph_2}$ (at least in the presence of mild forms of bounded forcing axioms). From this perspective it is sufficient to notice that (by Thm. 7.4) the $\in_B$-theory of $H_{\aleph_2}$ is the model companion of set theory. This allows to recover the generic absoluteness properties and the modal principles of forcing of [3, 4, 14, 16] leveraging on Robinson's notion of infinite forcing and its relation with model companionship (see [17]).

Finally, what is the connection between the maximization given by model companionship results and that provided by Hilbert's axioms of completeness? One of the presuppositions of our whole approach is a form of completeness that we named Hilbertian. The reason for this choice is more conceptual than formal. Indeed, we can see the notion of model companion as a formal tool implementing the idea that the universe $(V, \in)$ contains *all* sets. This interpretation is made explicit via the principle of Informal Existential Completeness (IEC): the maximality of $(V, \in)$ is thus realized by the ability to contain witnesses for all existential sentences (expressed in the appropriate language). Consequently (IEC) can be seen as a (first order) realization of Hilbert's idea of completeness that avoids reference to domains, but that instead maximizes set theory syntactically with respect to Σ_1-properties. Another point of convergence between model companions and Hilbert's notion of completeness is their final goal: *unicity.* Because of its second order character, Hilbert's Axiom of Completeness yields categoricity for geometry, consequently making its theory complete. Although in a weaker form, also an absolute model companion T^* (of a theory T) is able to fix univocally (via maximization) the Π_2-sentences that are consistent with the Π_1-fragment of T.

In conclusion, it seems that the form of maximality provided by (absolute) model companionship does not precisely fit with any of the standard forms of set-theoretical maximality; although there are formal similarities with generic absoluteness results and conceptual similarities with Hilbert's idea of completeness. For this reason we can see the notion of model companionship as a formalization of the somewhat vague ideas of width maximality and Hilbertian completeness. Our opinion is that the model companions of set theory yield a new form of maximality, that we may call *algebraic maximality.* The inspiration clearly comes from algebraically closed fields. We consider maximization as a two steps process: first we determine which are the basic, simple, concepts that our theory can express and then, once we have extended our language with symbols for these concepts, we close our theory with respect to the existential formulae of this new language. In the case of fields, the basic concepts are those expressed by Diophantine equations, while the algebraic maximality of ACF is what guarantees that all these equations have solutions. In the case of set theory, the choice of basic concepts is what determines the relevant signatures for set theory; furthermore the basic concepts should certainly include the Δ_0-properties. Once a signature τ is fixed, the algebraic maximality of the absolute model companions of set theory guarantees the existential closure of the H_{κ^+} and the maximization of the Π_2-sentences (for signature τ) realized in these structures.

8.2. **Justification.** Let us now turn to the topic of justification. When it comes to justifying new set-theoretical principles, the proposed reasons normally fall into two main categories: *intrinsic* or *extrinsic*. These forms of justification have been developed within the so-called Gödel's program[27]: a step by step extension of ZFC, aimed to coherently complete our picture of the universe of sets. In this sense, the present approach is clearly in the wake of Gödel's program. Indeed, the axiomatization of the model companions of set theory should, eventually, provide a description of the theory of V as the stratification of the theories of the various H_{κ^+}.

The role of large cardinals and of strong forcing axioms in the individuation of the model companions of set theory (in terms of the theories of $H_{\aleph_1}$ and $H_{\aleph_2}$) therefore suggests a justification of the appropriateness of the formers in terms of the nice model-theoretic properties of the latters. In order to understand this process of justification, let us briefly revise what we mean by intrinsic and extrinsic reasons.

- A form of justification is considered intrinsic when it is based on intrinsic features of the concept of set or, derivatively (given the foundational role of set theory), on notions that are key to the whole edifice of mathematics and logic. In this sense, a new axiom is justified when it captures an essential aspect of the concept of set or when it formalizes notions that are fundamental for mathematics and logic. This form of justification is utterly conceptual, since it rests on the theoretical priority of the notion of set over its formalization. Consequently, an extension of ZFC is well justified, when it faithfully represents the correct concept of set (e.g. the iterative conception [9] or the quasi-combinatorial one [8]).
- A form of justification is considered extrinsic when we are forced to accept it by the abundance of its desirable consequences; even in the absence of intrinsic reasons. In this case an axiom is justified even if it does not seem *prima facie* to capture any relevant aspect of the concept of set. This second form of justification is clearly meant to overcome the limits of intrinsic reasons and to account for an experimental methodology in the context of the foundations of mathematics. Extrinsic justifications are pragmatic in nature, since they rely on the fruitfulness of an axiom in solving new problems, shortening proofs of already known results, and unifying substantial bodies of theory. The form of justification put forward by extrinsic reasons is akin to an inference to the best explanation within mathematics. Therefore, an extrinsically justified axiom is judged by its consequence and not by its meaning.

Intrinsic and extrinsic reasons have been extensively studied in the philosophy of set theory [23, 24] and they have been widely applied, in recent debates, for the justifications of competing programs [2, 20]. These two forms of justification have also been criticized for their opacity in offering clear criteria of application and for the lack of demarcation between intrinsic and extrinsic reasons [7]. Moreover, as it happened in the debate on the analytic-synthetic distinction, there is also no shortage of contributions that reject the problem of justification at its very base. Following a naturalistic account of mathematics, authors like Hamkins (or before him Cohen [11]) propose to dismiss the problem of justification, together with the issue of independence, by defending the liberty of mathematicinas to study different universes of set theory and by declaring the study of the variation of truth values among the models of ZFC to be all that mathematically matters for the study of independence [15].

[27]Presented by Gödel in his seminal paper [13].

Now, *in which sense the nice model-theoretic properties of model companions are able to justify large cardinals and strong forcing axioms*? Is the role of these new set-theoretical principle in the construction of a model companion able to provide an intrinsic or an extrinsic justification for them?

A first complication that we face in addressing these questions is that the present approach does not only deal with axiomatic extensions of ZFC, but also with linguistic ones: the introduction of new signatures for set theory. For this reason the justification of large cardinals and strong forcing axioms based on model companionship results for set theory contain elements of both intrinsic and extrinsic arguments. Indeed, on the one hand strong forcing axioms provide a maximization of the Π_2-statements realized over $H_{\aleph_2}$ and, thus, have tremendously abundant consequences on third order arithmetic (and on this ground they can be extrinsically justified). On the other hand, because of the possibility to provide (generic) absoluteness results, large cardinal axioms determine the "simple" concepts that need to be included in the new signatures for set theory. Therefore, large cardinals and strong forcing axioms help us understanding what are the basic concepts on which third order arithmetic should be based (and for this reason they can be intrinsically justified).

Another aspect of the present approach that places the justification of new axioms somewhat outside the standard practice, is the focus on models instead of sets (like the intrinsic reasons) or problems (like the extrinsic ones). As a matter of fact, large cardinals and strong forcing axioms play an important role in the construction of model companions of set theory and, therefore, in detemining the stability of the theories of second and third order arithmetic. Derivatively, these principles can be intrinsically or extrinsically justified, that is: because of the nice model-theoretic properties of model companions for set theory assuming large cardinals (model companions whose theory is determined by strong forcing axioms) we are able to clarify the notion of set and to have fruitful consequences.The difficulty in choosing whether the nice model-theoretic consequences of an axiom account for its intrinsic or extrinsic justification is not new. Similar troubles can be found in any attempts to justify generic absoluteness principles in terms of this standard dichotomy.[28] What the present approach makes evident is the role model-theoretic properties can play in determining the success of new axioms. In our opinion, what needs to be praised of these model companionship properties of set theory are not only their intrinsic or extrinsic virtues, but mostly the clarification they provide of the model theory of set theory. Because of the limits of the standard intrinsic-extrinsic dichotomy to capture the form of justification of this model-theoretic approach to set theoretic validities, we may propose a completely new name for this novel form of justification and call it *meta-theoretical*.

There is another sense in which these model companionship results, the justification they offer to new axioms, and the solution they yield for the Continuum problem are utterly Hilbertian. This is the autonomy they provide for mathematics with respect to its foundations. As a matter of fact, the use of the notion of model companionship to fix the theory of second and third order arithmetic (and to enforce algebraic maximality) does not introduce any element that is foreign to mathematical practice. The sought solution to the Continuum problem is a solution obtained with formal tools and that is justified on a purely mathematical ground. Moreover, the possibility to put on a par the notions of provability, forceability, and consistency provides a solution to the Continuum problem that encompasses all mathematical means at disposal to prove independence. In this sense these results realize in full the possibility of an autonomous foundation for mathematics, as the one sought by Hilbert [12].

[28]See [29] for a discussion of a conceptual justification of forcing axioms in terms of the notion of arbitrary set.

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