PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

(i) Take a mathematical theory T (1)
and look at signatures σ s.t.
T is first order axiomatizable in σ.

(ii) Define a scale of preference among
the σ which axiomatize T

(iii) Detect outermost σ s.t. T as axiomatized in
σ is "nice" (what it means for σ to
be nice for T)

(iv) ~~Apply~~ Once the above is well defined
apply it to set theory.

(2)

(a) Good criterion to say σ is nice for T is given by model companionship

(b) if T axiomatizes set theory in a "natural" signature σ and S is the model companion of T according to σ then for some infinite cardinal κ S is a theory of H_{κ^+} ($H_{\kappa^+} \prec_1 V$ in natural signature)

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(c) if $T + \neg CH$ is consistent then for all "natural" σ CH is not in the AMC of T

(d) Same result with $CH \equiv 2^{\aleph_0} = \aleph_2$ replaced by $2^{\aleph_0} > \aleph_2$ (3)

(e) if T extends ZFC + LC then $\exists\, 6$ s.t. $\boxed{2^{\aleph_0} = \aleph_2}$ (a definable version of it) is in the AMC of T according to 6 even if $T \models CH$ S is the AMC of T in sign. 6

(b) There are plenty of τ s.t. any $T \supseteq ZFC$ admits an AMC in signature τ.

$$V \models T \qquad \underbrace{\left(H_{\omega_2}, 6^V\right)}_{\Pi_1^{CH}} \prec_1 \left(V, 6^V\right) \models CH$$

(4)

in (e) S the AMC of $T \supseteq ZFC + LC$ for 6

is given by $\boxed{\{\psi : T \vDash \exists P \; P \Vdash \psi^{H_{\omega_2}}\} \quad \psi \; \Pi_2\text{-sentences}}$

$\parallel \; T = ZFC + LC$

$\{\psi : T + MM^{++} \vDash \psi^{H_{\omega_2}}\}$

$\mathcal{M}_0 \vDash \psi_0$

$\mathcal{M}_1 \vDash \psi_1 \wedge \neg\psi_0$

$\sqsubseteq \mathcal{M}_2 \vDash \psi_0 \wedge \neg\psi_1 \quad \ldots \quad \mathcal{M}_\omega \vDash \psi_1 \wedge \psi_2$

$\sigma = \{\cdot\}$

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$\forall x, y, z \; (x \cdot y) \cdot z = x \cdot (y \cdot z)$

$\exists x \forall y \; (x \cdot y = y \wedge y \cdot x = y)$

$\forall x \exists y \forall z \; \big((x \cdot y) \cdot z = z \wedge (y \cdot x) \cdot z = z \wedge$

$z \cdot (y \cdot x) = z \wedge z \cdot (x \cdot y) = z \big)$

$\sigma = \{\cdot, e\}$ $\qquad \{\cdot, e, {}^{-1}\}$

$\to \forall y \; (y \cdot e = y \wedge e \cdot y = y)$

$\to \forall x \exists y \; (x \cdot y = e \wedge y \cdot x = e)$

$\sigma = \{\cdot, e, ^{-1}\}$ $\boxed{6}$

$\forall x \, (x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e$

$\{R, e\}$ $\quad R$ is ternary relation symbol

$\forall x \cancel{\forall} y \, \exists! z \; R(x, y, z)$

$\forall x y z w t \, [\; \cdots \;]$

$\forall y \, R(e, y, y) \wedge R(y, e, y)$

$\forall x \, \exists y \, [R(x, y, e) \wedge R(y, x, e)]$

$\tau = \{+, \cdot, 0, 1\}$

$\forall x\, y\, (x \cdot y = 0 \to x = 0 \lor y = 0)$ } η_2 tesent for semirings with no 0-divisor

$\forall x\, \exists y\, (x + y = 0)$ ← rings with no 0-divisor

$\forall x\, \exists y\, (x = 0 \lor x \cdot y = 1)$ ← fields

$\forall x_0 \ldots x_n\, \exists y \left(\sum_{i=0}^{n} x_i y^i = 0 \right)$ ← ACF.

$(\mathbb{Z}, \cancel{+}, \cdot, 0, 1) \sqsubseteq (\mathbb{Q}, +, \cdot, 0, 1)$ 8

we add $\{-, {}^{-1}\}$

$\forall x \left[\left(\exists! y \; x+y=0 \cancel{\wedge} y=-x \right) \vee \neg \exists! y \; x+y=0 \wedge y=0 \right]$

$\forall x \left[\left(\exists! y \; x \cdot y = 1 \rightarrow y = x^{-1} \vee \neg \exists! y \; x \cdot y = 1 \wedge y = 0 \right) \right]$

$(\mathbb{Z}, \cancel{+}, \cdot, 0, 1, -, {}^{-1}) \not\sqsubseteq (\mathbb{Q}, +, \cdot, 0, 1, -, {}^{-1})$

$\cancel{\mathbb{Z}}$ $\quad \cancel{+} 2^{-1} = 0 \quad\quad 2^{-1} = \frac{1}{2}$

First idea: select the signature according to the class of morphisms you are interested between models of T. 9

$\{+, \cdot, 0, 1\}$ $\{+, \cdot, 0, 1, {}^{-1}\}$

Third idea: Complexity of axiom

Rings: is Π_2-axiomatized in $\{+, \cdot, 0, 1\}$ and Π_1-axiom in $\{+, \cdot, 0, 1, -\}$

Fields is // in $\{+, \cdot, 0, 1, -\}$ and Π_1-axiom in $\{+, \cdot, 0, 1, -, {}^{-1}\}$

Def: Let T be a τ-theory. A τ-structure $\mathcal{M}$ is T-ec if

(i) $\exists\, \mathcal{N} \supseteq \mathcal{M}$ $\quad \mathcal{N} \models T$

(ii) if $\mathcal{N} \supseteq \mathcal{M}$ $\quad \mathcal{N} \models T$ then $\quad \mathcal{M} \preceq_1 \mathcal{N}$

Example in $\tau = \{+, \cdot, 0, 1\}$

~~A~~ $\mathcal{M} \models ACF \implies \mathcal{M}$ is ~~T-ec~~ Fields-ec

Notation Given a τ-theory T

$$T_{\forall}^{\tau} = \{\psi : T \models \psi \text{ and } \psi \text{ is universal sentences}\}$$

$$T_{\forall\exists}^{\tau} = \{\psi : T \models \psi \text{ and } \psi \text{ is a boolean comb. of univ. sentences}\}$$

~~$\mathrm{Fields}_{\forall}^{\tau} \subsetneq \mathrm{Fiel}$~~

Fact Assume T, S are τ-theories (12)

TFAE:

(i) $T_\forall^\tau \supseteq S_\forall^\tau$

(ii) for any $\mathfrak{M} \models T$ $\exists \mathfrak{N} \models S$ $\mathfrak{N} \supseteq \mathfrak{M}$

Pf: (ii) $\to$ (i): take $\varphi \in S_\forall^\tau$ take $\mathfrak{M} \models T$

by (ii) $\exists \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{N} \models S$ hence $\mathfrak{N} \models \varphi \Rightarrow \mathfrak{M} \models \varphi$ $\square$

(i) $\to$ (ii): Assume not let $\mathfrak{M} \models T$ s.t.

$\forall \mathfrak{N}$ $\mathfrak{N} \supseteq \mathfrak{M} \Rightarrow \mathfrak{N} \not\models S$

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$S \cup \Delta_0(\mathfrak{M})$ is inconsistent

$\shortparallel$

$\{\psi(a_1 \ldots a_n) : a_1 \ldots a_n \in \mathfrak{M} \quad \mathfrak{M} \models \psi(x_1 \ldots x_n)[{}^{x_i}/_{a_i} \; i = 1 \ldots n]\}$

ψ quantifier free

if $\mathfrak{A} \models S \cup \Delta_0(\mathfrak{M})$

$\mathfrak{A}{\restriction}\tau \supseteq \mathfrak{M}$

$c_i \mapsto a_i$

$\hat{\mathfrak{A}}$ s.t c_i interprets the constant a_i

so $\exists\ \varphi_1 \cdots \varphi_n$ q.f. and $a_1 \cdots a_m$ (14)

s.t. $S + \varphi_1(\vec{a}) \cdots \varphi_n(\vec{a})$ inconsistent

$$\varphi(\vec{a}) = \bigwedge_{i=1}^{n} \varphi_i(\vec{a})$$

$$S \vdash \varphi(\vec{a}) \qquad S \vdash \neg\varphi(\vec{a})$$

$$\Downarrow$$

$$S \vdash \forall \vec{x}\, \neg\varphi(\vec{x})$$

$$\downarrow \text{ univ. sent.}$$

$$S_\forall^\tau \subseteq T_\forall^\tau \Rightarrow T \vdash \forall x \neg\varphi(\vec{x}) \qquad \mathfrak{M} \vDash \varphi(\vec{a}) \wedge \forall x \neg\varphi(\vec{x}) \ \lightning$$

Cor. TFAE (i) $T_\forall^t = S_\forall^t$

(a) every model of T embeds in a model of S and conversely.

Cor. $\mathcal{M}$ is T-ec iff it is $T_\forall$-ec

Pf: Suppose $\mathcal{M}$ is T-ec

(i) $\mathcal{M} \subseteq \mathcal{N} \models T$

(ii) $\mathcal{M} \preceq_1 \mathcal{N}$ for any $\mathcal{N} \models T$

$P \models \exists \vec{x}\, \psi(\vec{x}, \vec{a})$

$$\mathcal{M} \subseteq \underset{\substack{\rotatebox{90}{$\models$} \\ T_\forall}}{\mathcal{N}} \subseteq \underset{\substack{\rotatebox{90}{$\models$} \\ T}}{P} \qquad \mathcal{M} \preceq_1 P$$

$P \models \psi(\vec{b}, \vec{a})$

$\Uparrow$

$\mathcal{N} \models \psi(\vec{b}, \vec{a})$ with $\vec{a} \in \mathcal{M}^{<\omega}$

$(T_\forall)_\forall = T_\forall$

if $\mathfrak{M}$ is $T_\forall$-ec we must show $\mathfrak{M}$ is T-ec 16

(i) $\mathfrak{M} \subseteq \underline{\mathfrak{N}} \subseteq P \not\models T$

$\underline{\mathfrak{N}} \models T_\forall$

(ii) automatic

Def. [Robinson] ~~$\mathfrak{M}$~~ T is model complete if for all $\underset{\underset{T}{\models}}{\mathfrak{N}} \subseteq \underset{\underset{T}{\models}}{\mathfrak{M}}$ we have $\mathfrak{M} \preceq_1 \mathfrak{N}$.

$\prec$

Thm. TFAE

(i) T is model complete

(ii) $C_T = \{\mathfrak{M} : \mathfrak{M} \models T\} = \{\mathfrak{M} : \mathfrak{M} \text{ is } T_\forall\text{-ec}\}$

Def.: T is the model companion of S if (18)

(i) T is model complete

(ii) $T_\forall = S_\forall$

Def.: Let S be a τ-theory. $KH(S)$ is given by $\{\psi : \psi$ is Π_2-snt for T[illegible] and $\mathfrak{M} \models \psi$ for all $\mathfrak{M}$ which are S-ec$\}$

Fact: ACF is the model companion of fields in signature $\tau = \{+, \cdot, 0, 1\}$ ~~in~~ pa

if $\mathfrak{M} \models ACF$ and $\mathfrak{A} \preceq_1 \mathfrak{M}$

Fields

consider $\sigma = \{+, \cdot, 0, 1, -, ^{-1}\}$ 19

then ACF is the MC of Rings with no-0-div. in sign $\tau = \{+, \cdot, 0, 1\}$

$\cancel{\mathbb{C} \sqsubseteq \mathbb{C}[X]}$

ACF is not the MC of Rings with no 0-div.

$(\mathbb{C}, +, \cdot, 0, 1, -, ^{-1}) \sqsubseteq (\mathbb{C}[X], +, \cdot, 0, 1, -, ^{-1})$ in sign σ

$\exists X (X \neq 0 \wedge X^{-1} \simeq 0)$ $\exists X (X \neq 0 \wedge X^{-1} \simeq 0)$

& Given σ, τ signatures which axiomatize (20)
T we can prefer σ to τ if
T axiomatized in σ admits a model companion
while in τ it does not.