PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

LECTURE 2 - 4/5/2022

RECAP

1. Byembeddability
2. T-ec models
3. Model companionship and Kaiser Hulls -dependence on the signature.

<u>Lemma</u> Let T, S be ~~a~~ τ-theories. TFAE

(i) $T_\forall \supseteq S_\forall$

(ii) $\forall \mathcal{M} \models T \ \exists \mathcal{N} \supseteq \mathcal{M}$, $\mathcal{N} \models S$

<u>Def.</u> Let T be a τ-theory. ~~TFAE~~ $\mathcal{M}$ is T-ec if:

(i) $\exists \mathcal{N} \supseteq \mathcal{M}$ $\mathcal{N} \models T$

(ii) $\forall \mathcal{N} \supseteq \mathcal{M}$ $(\mathcal{N} \models T \Rightarrow \mathcal{M} \preccurlyeq_1 \mathcal{N})$

<u>Example</u>: $\tau = \{+, \cdot, 0, 1\}$ $T =$ rings with no 0-divis.

$\mathcal{M} \models ACF$

Fact $\mathfrak{M}$ is T-ec iff $\mathfrak{M}$ is $T_\forall$-ec (3)

Def. Let T, S be τ-theories
T is the model companion of S if

(i) $T_\forall^\tau = S_\forall^\tau$

(ii) T is model complete i.e

T ~~the~~ axiomatizes the class of τ-structures given by $T_\forall$-ec models

Example ACF is the model companion of Domains in signature $\{+, \cdot, 0, 1\}$

consider $\sigma = \{+, \cdot, 0, 1, -, ^{-1}\}$ ④

char 0

ACF_0 is not the MC of Domains for σ

because $(\mathbb{C}, \sigma) \subseteq (\mathbb{C}[X], \sigma)$ but not Σ_1-sub.

now assume that you take

$\mathcal{M} \equiv_\sigma \mathbb{C}[X]$ and $\underset{T}{\mathcal{M}} \prec_1 \underset{\mathrm{th}(\mathbb{C}[X])}{\overset{\shortparallel}{\mathcal{N}}}$

MC(Domains, σ)

$\mathcal{M} \wedge T \prec_1 \mathcal{N} \wedge T$

<u>Question</u> Does Domains in sign σ admit a MC? <u>Guess</u>: NO!

Lemma Let $\mathfrak{A} \models T$ then $\exists\, \mathfrak{M} \supseteq \mathfrak{A}$ s.t. $\mathfrak{M}_\kappa$ is T-ec.

Pf: Set $\mathfrak{M}_0 = \mathfrak{A}$ and $\{\varphi_i : i < \lambda\} =$ form of τ

build by induction a chain

$\{\mathfrak{M}_\alpha : \alpha < \underbrace{\lambda + |\mathfrak{M}_0|}_{\kappa}\}$ s.t.

1. $\mathfrak{M}_0 = \mathfrak{A}$

2. $\mathfrak{M}_\alpha \sqsubseteq \mathfrak{M}_{\alpha+1}$, $\mathfrak{M}_{\alpha+1} \models \exists x\, \varphi_{En(\alpha)_0}(x, \vec{a}_{En(\alpha)_1}) + T_\forall$ if this is possible

   $\mathfrak{M}_{\alpha+1} = \mathfrak{M}_\alpha$ otherwise

3. $\mathfrak{M}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{M}_\alpha$ for γ-limit. Note that $\mathfrak{M}_\xi \models T$ $\forall \xi < \lambda + |\mathfrak{M}_0|$ $\mathfrak{M}_\kappa$ is $T_\forall$-ec

Fact If $\mathfrak{M}$ is T-ec and $\mathfrak{N} \preceq_2 \mathfrak{M}$ then $\mathfrak{N}$ is also T-ec.

$$\begin{array}{lll} & & \mathfrak{M} \models T \\ & \subseteq \nearrow & \\ \mathfrak{N} \preceq_2 \mathfrak{M} & & \\ \;\;\; \sqsubseteq & & \\ \;\;\;\; \mathfrak{P} \models T & & \end{array}$$

7

$T \qquad\qquad T\text{-ec}$

$\models \qquad\qquad \downarrow$

$\mathcal{M} \qquad \mathcal{M}$

$\mathcal{M} \preceq \mathcal{B}_1 \qquad \subseteq Q \models T_\forall$

$\sqcap \qquad \subseteq$

$P \models T$

$$\Delta_0(P) = \{\varphi(\vec{a}) : \varphi \text{ q.f. } P \models \varphi(\vec{a})\}$$

$\Delta_0(M)$ . . . . M' . . . .

<u>Claim</u> $T_\forall \cup \Delta_0(P) \cup \Delta_0(\mathcal{M})$ is consistent

Assume not $\vec{a} \in (P \setminus M)^{<\omega}$ and $\vec{b} \in (M \setminus N)^{<\omega}$

and $\vec{c} \in N^{<\omega}$ and $\varphi_0(\vec{x},\vec{y})$ $\varphi_1(\vec{z},\vec{y})$ q.f.

s.t. $\frac{P}{\Pi}$ $\frac{M}{\Pi}$

$T_\forall + \psi_0(\vec{a},\vec{c}) + \psi_1(\vec{b},\vec{c})$ is inconsistent

$T_\forall \vdash \neg\psi_0(\vec{a},\vec{c}) \vee \neg\psi_1(\vec{b},\vec{c})$

$T_\forall \vdash \forall \vec{z}\,(\underbrace{\forall x \neg\psi_0(\vec{x},\vec{z})}_{\Pi} \vee \underbrace{\forall \vec{y} \neg\psi_1(\vec{y},\vec{z})})$

$\cancel{\Pi}$

$P \models \psi_0(\vec{a},\vec{c})$ $\cancel{\Pi}$ $\Rightarrow$ $P \models \forall \vec{y} \neg\psi_1(y,\vec{c}) \Rightarrow M \models \forall \vec{y} \neg\psi_1(\vec{y},$

$\mathfrak{N} \leq_1 \mathfrak{M} \models \forall \vec{y}\, \neg \varphi_2(\vec{y}, \vec{c})$

$\varphi_2(\vec{b}, \vec{c})$

$\perp$

$\mathfrak{M}$ Tec

↯

$\mathfrak{M} \leq_1$

$\mathfrak{N} \leq_1 \mathfrak{M} \prec$

$\leq p \subseteq Q \models T_\forall$

$\perp$

$T_\forall$

1. $\forall \mathfrak{A} \models T \quad \exists \mathfrak{M} \supseteq \mathfrak{A} \quad \mathfrak{M}$ is T-ec. 10

2. if $\mathfrak{M}$ is T-ec and $\mathfrak{A} \leq_2 \mathfrak{M}$

   $\mathfrak{A}$ is also T-ec

4. $\forall \vec{x} \exists \vec{y} \, \psi(\vec{x}, \vec{y}, \vec{a}) \quad \psi(\vec{x}, \vec{y}, \vec{z})$ q.f.

s.t. some $\mathfrak{A} \supseteq \mathfrak{M} \quad \mathfrak{M} \models T_\forall + \forall \vec{x} \exists \vec{y} \, \psi(\vec{x}, \vec{y},$

Then $\mathfrak{M} \models \forall \vec{x} \exists \vec{y} \, \psi(\vec{x}, \vec{y}, \vec{a})$

take $\vec{b} \in \mathfrak{M}^{<\omega}$ then $\mathfrak{A} \models \exists \vec{y} \, \psi(\vec{b}, \vec{y}, \vec{a})$

$\Rightarrow \mathfrak{M} \models \exists \vec{y} \, \psi(\vec{b}, \vec{y}, \vec{a}) \quad \Rightarrow \mathfrak{M} \models \forall \vec{x} \exists \vec{y} \, \psi(\vec{x}, \vec{y}, \vec{a})$

<u>Def</u> Let T be a τ-theory. We say that ψ τ-sentence is <u>strongly</u> $T_{\forall\vee\exists}$-consistent if $\forall\, R \supseteq T_{\forall\vee\exists}$ ~~R comp~~ R consistent $\psi + \boxed{R_{\forall\vee\exists}}$ is consistent

---

$$R^{\tau}_{\forall\vee\exists} = \{\psi : \psi \text{ is bool. comb. of univ. sent. and } R \models \psi\}$$

$$\cup\!|$$

$$R^{\tau}_{\forall}$$

Lemma Let T, S be τ-theories

TFAE:

(i) $\boxed{T_{\forall\vee\exists} \supseteq S_{\forall\vee\exists}}$

(ii) $\forall\, \mathfrak{M} \models T \;\; \exists\, \mathfrak{N} \supseteq \mathfrak{M} \;\; \mathfrak{N} \models S$

and $\mathfrak{M}$ and $\mathfrak{N}$ agree on the universal sentences

Lemma Let T, S be τ-theories TFAE

(i) $T_\forall \supseteq S_\forall$ ...

(ii) $\forall\, \mathfrak{M} \models \cancel{T}\, \mathfrak{N} \;\exists\, \mathfrak{N} \models S \;\; \mathfrak{N} \supseteq \mathfrak{M}$

take $\tau = \{+, \cdot, 0, 1\}$ $\qquad S = \text{Fields}$ $\qquad$ 13

$T = ACF$

$$\begin{array}{ccc}
S & & T \\
\rotatebox{90}{$\models$} & & \rotatebox{90}{$\models$} \\
(\mathbb{Q}, +, \cdot, 0, 1) & \subseteq & (\mathbb{C}, +, \cdot, 0, 1) \\
\rotatebox{-90}{$\models$} & & \rotatebox{-90}{$\models$} \\
\forall x \, \neg(x \cdot x + 1 = 0) & & \exists x \, (x \cdot x + 1 = 0)
\end{array}$$

(ii) $\Rightarrow$ (iii) $T_{\forall\vee\exists} \not\supseteq S_{\forall\vee\exists} \Rightarrow$ ~~$\nexists$~~ exists $\theta \in S_{\forall\vee\exists} \setminus T_{\forall\vee\exists}$

14

$S + \neg\theta$ is inconsistent ~~with~~ with S and consistent with T

if $\mathfrak{M} \models T + \neg\theta$ and $\mathfrak{N} \supseteq \mathfrak{M}$ s.t. $\mathfrak{N} \models S$

then ~~$\mathfrak{N} \not\models \theta$~~ $\mathfrak{N} \models \theta$

(i) $\Rightarrow$ (ii): $\neg$(ii) as witnessed by $\mathfrak{M} \models T$

$\forall \mathfrak{N}$ $\underset{\models S}{\mathfrak{N} \supseteq \mathfrak{M}}$ $\exists\, \theta_{\mathfrak{N}}$ universal s.t. $\mathfrak{M} \models \theta_{\mathfrak{N}}$

and $\mathfrak{N} \models \neg\theta_{\mathfrak{N}}$

$S^* = S \cup \Delta_0(\mathfrak{M}) \cup \{\theta_{\mathfrak{N}} : \mathfrak{M} \subseteq \mathfrak{N} \models S\}$ is inconsistent suppose not ie $P^* \models S^*$ $\quad P = P^* \upharpoonright \tau \supseteq \mathfrak{M}$ (15)

$a^P \leftarrow\!\dashv a$

then there is θ_p ~~bx~~ universal s.t.

$\mathfrak{M} \models \theta_p$ and ~~$P \upharpoonright \tau \models$~~ $P \models \neg\theta_p$

which implies that $P^* \models \theta_p \wedge \neg\theta_p$ $\lightning$

~~and~~ S^* inc. $\Rightarrow \exists\ \psi(\vec{a}) \in \Delta_0(\mathfrak{M})$ $\exists\ \theta_{\mathfrak{M}_1} \dots \theta_{\mathfrak{M}_k} \in X$

s.t. $S + \psi(\vec{a}) + \theta_{\mathfrak{M}_1} + \dots + \theta_{\mathfrak{M}_k}$ is inconsistent

$S \vdash \bigwedge_{i=1}^{n} \theta_{\mathfrak{M}_i} \to \neg\psi(\vec{a})$ $\qquad S \vdash \neg\theta \vee \overbrace{\forall x \neg\psi(\vec{x})}$

θ $\qquad$ exist $\qquad$ univ.

$$\mathfrak{M} \models \psi(\vec{a}) \quad \cancel{x}$$

$$\mathfrak{M} \models \exists x\, \psi(\vec{x})$$

$$\mathfrak{M} \models \bigwedge_{i=1}^{m} \vartheta_{m_i}$$

$$\mathfrak{M} \not\models \forall x\, \neg\psi(\vec{x})$$

$$\Downarrow$$

$$\mathfrak{M} \not\models \neg\vartheta$$

$$\mathfrak{M} \not\models \neg\vartheta \vee \forall x\, \neg\psi(\vec{x})$$

$$\mathfrak{M} \models T$$

$S_{\forall\vee\exists} \not\subseteq T_{\forall\vee\exists}$ as witnessed by $\neg\vartheta \vee \forall\vec{x}\, \neg\psi(\vec{x})$ $\blacksquare$

Fact Assume φ is a Π_2-sent. for τ (17)
s.t. φ is strongly $T_{\forall\vee\exists}$-consistent.
Let $\mathfrak{M}$ be a T-ec model.
Then $\mathfrak{M} \models \varphi$.

Pf: Assume $\mathfrak{M}$ is T-ec. Let $R = Th(\mathfrak{M})$

~~Then~~ $\cancel{R_{\forall\vee\exists} \supseteq (T_{\forall\vee\exists}+\varphi)_{\forall\vee\exists} \Rightarrow \exists \mathfrak{N} \supseteq \mathfrak{M} \quad \mathfrak{N} \models T_\forall \quad \mathfrak{N} \models}$

$R_{\forall\vee\exists}+\varphi$ is consistent because φ is strongly T-con.

$(R_{\forall\vee\exists}+\varphi)_{\forall\vee\exists} = R_{\forall\vee\exists} \Rightarrow \mathfrak{M} \preceq_1 \mathfrak{N} \models R_{\forall\vee\exists}+\varphi$

$R_{\forall\vee\exists} \supseteq T_\forall \Rightarrow \mathfrak{M} \models \varphi$ ∎

ERROR: undefined
OFFENDING COMMAND: ^.!r`&»

Def Given a τ-theory T

$KH(T) = \{\psi : \psi \text{ is } \Pi_2 \text{ for } \tau,\ \mathcal{M} \models \psi\ \forall\, \mathcal{M}\ T\text{-ec}\}$

$SCH(T) = \{\psi : \psi \text{ is } \Pi_2 \text{ for } \tau,\ \psi \text{ is strongly } T\text{-cons.}_{\forall\vee\exists}\}$

$MC(T) =$ model companion of T (if it exists).

$$SCH(T) \subseteq KH(T) \overset{=}{\subseteq} MC(T) \leftarrow \text{unique}$$

$\uparrow$ true but not yet proved

$=$

Def T is the AMC of S in signature τ if

~~(i)~~ $SCH(S)$ ~~$= AMC(S)$ and (ii)~~ is model complete.

<u>Thm</u> Let T, S be τ-theories
TFAE

(i) T is the AMC of ~~f~~ S i.e.
$SCH(S) = T$ and
T is model
complete

(ii) $T_{\forall\vee\exists} = S_{\forall\vee\exists}$ and T is model complete

<u>Example</u> $Th(\mathbb{Q},<)$ is the AMC

$Th(\mathbb{Z},<)$

<u>First fact</u> $Th(\mathbb{Q},<)$ admits q.e.

which gives $\mathcal{M} \subseteq_{<} \mathcal{N}$ ⟹ $\mathcal{M} \preceq \mathcal{N}$

$$\begin{array}{ccc} \mathcal{M} & \subseteq & \mathcal{N} \\ \shortparallel & < & \shortparallel \\ Th(\mathbb{Q},<) & & Th(\mathbb{Q},<) \end{array}$$

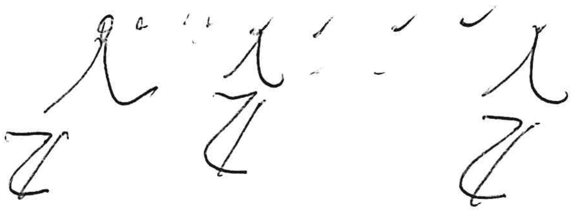
<u>Second fact</u> $Th(\mathbb{Z},<)$ and $Th(\mathbb{Q},<)$ are complete

so if $R \models Th(\mathbb{Z},<)_{\forall\vee\exists}$ we have that in all models of $Th(\mathbb{Z},<)$

$\mathfrak{M} \models Th(\mathbb{Q})$

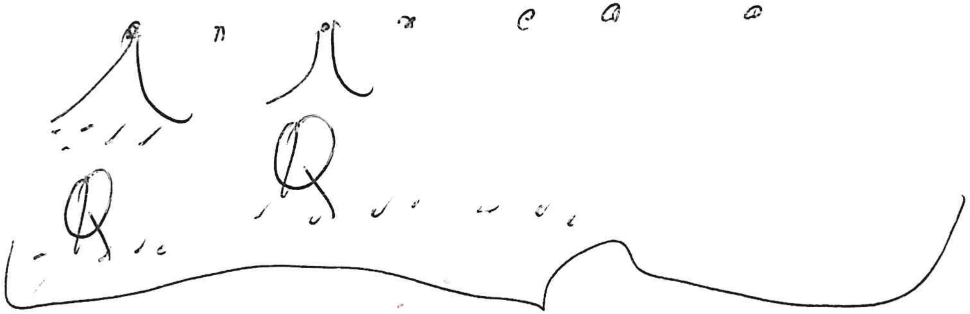
$\mathfrak{M} \models Th(\mathbb{Q},<)$

Π



$\mathfrak{N} \models Th(\mathbb{Z},<)$

$\mathfrak{N} \models Th(\mathbb{Z},<)$



Π

$Th(\mathbb{Q},<)$

ACF is MC of Fields in $\{+,\cdot,0,1\}$

but not AMC of Fields

$(\mathbb{Q},+,\cdot,0,1) \preceq (\mathbb{C},+,\cdot,0,1)$

Lemma Let T be the AMC of S in sign. τ (22)

then for all $R \supseteq \cancel{S}$ $\cancel{S} + R_\forall^\tau$ is the AMC(R, τ)

Counterexample ACF is the MC of ~~F~~ Field in $\{+, \cdot, 0, 1\}$ but $ACF + Th(\mathbb{Q})_\forall$ is not consistent hence it is not the MC of $Th(\mathbb{Q}, +, \cdot, 0, 1) \supseteq S$