NOTES ON ABSOLUTE MODEL COMPANIONSHIP

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Abstract. These brief notes contain a self contained account of the main results on model companionship and include a detailed treatment of the notion of absolute model companionship.

We present this topic expanding on [5, Sections 3.1-3.2] and [3, Chapter 3.5].
We introduce the following terminology:

Notation 1.

- $\sqsubseteq$ *denotes the substructure relation between structures.*
- $\mathcal{M} \prec_n \mathcal{N}$ *indicates that* $\mathcal{M}$ *is a* Σ_n*-elementary substructure of* $\mathcal{N}$*, we omit the* n *to denote full-elementarity.*
- *Given a first order signature* τ*,* $\tau_\forall$ *denotes the universal* τ*-sentences; likewise we interpret* $\tau_\exists, \tau_{\forall\exists}, \ldots$*.* $\tau_{\forall\vee\exists}$ *denotes the boolean combinations of universal* τ*-sentences; likewise we interpret* $\tau_{\forall\exists\vee\exists\forall}, \ldots$*.*
- *Given a first order theory* T *in signature* τ*,* $T^\tau_\forall$ *denotes the sentences in* $\tau_\forall$ *which are consequences of* T*, likewise we interpret* $T^\tau_\exists, T^\tau_{\forall\exists}, T^\tau_{\forall\vee\exists}, \ldots$*. If the signature of* T *is clear we omit the superscript* τ *and just write* $T_\forall, \ldots$*.*
- *We often denote a* τ*-structure* $(M, R^M : R \in \tau)$ *by* (M, τ^M)*.*
- *We often identify a* τ*-structure* $\mathcal{M} = (M, \tau^M)$ *with its domain* M *and an ordered tuple* $\vec{a} \in M^{<\omega}$ *with its set of elements.*
- *We often write* $\mathcal{M} \models \phi(\vec{a})$ *rather than* $\mathcal{M} \models \phi(\vec{x})[\vec{x}/\vec{a}]$ *when* $\mathcal{M}$ *is* τ*-structure* $\vec{a} \in \mathcal{M}^{<\omega}$*,* ϕ *is a* τ*-formula.*
- *We let the atomic diagram* $\Delta_0(\mathcal{M})$ *of a* τ*-model* $\mathcal{M} = (M, \tau^M)$ *be the family of quantifier free sentences* $\phi(\vec{a})$ *in signature* $\tau \cup M$ *such that* $\mathcal{M} \models \phi(\vec{a})$*.*

1. Byembeddability versus absolute byembeddability

Let us give a proof of the following well known fact, since it will be helpful to outline the subtle difference between model companionship and absolute model companionship.

Lemma 2. *Let* τ *be a signature and* T*,* S *be* τ*-theories. TFAE:*

(1) $T_\forall \supseteq S_\forall$*.*
(2) For any $\mathcal{M}$ *model of* T *there is* $\mathcal{N}$ *model of* S *superstructure of* $\mathcal{M}$*.*

Proof.

1 implies 2: Assume $\mathcal{M}$ models T and is such that no $\mathcal{N}$ model of S is a superstructure of $\mathcal{M}$. Then $S^* = S \cup \Delta_0(\mathcal{M})$ is not consistent (where $\Delta_0(\mathcal{M})$ is the atomic diagram of $\mathcal{M}$), otherwise if $\mathcal{P}$ is a model of S^* the map $a \mapsto a^{\mathcal{P}}$ is an embedding of $\mathcal{M}$ in $\mathcal{P} \restriction \tau$. By compactness find $\psi(\vec{a}) \in \Delta_0(\mathcal{M})$ quantifier-free sentence such that $S + \psi(\vec{a})$ is inconsistent. This gives that

$$S \models \forall \vec{x} \, \neg\psi(\vec{x})$$

since $\vec{a}$ is a string of constant symbols all outside of τ. Therefore $\forall \vec{x} \, \neg\psi(\vec{x}) \in S_\forall \subseteq T_\forall$. Hence

$$\mathcal{M} \models \forall \vec{x} \, \neg\psi(\vec{x}) \wedge \psi(\vec{a}),$$

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a contradiction.

2 implies 1: Left to the reader.

□

Corollary 3 (Resurrection Lemma)**.** *Assume $\mathcal{M} \prec_1 \mathcal{N}$ are τ-structures. Then there is $\mathcal{Q} \sqsupseteq \mathcal{N}$ which is an elementary extension of $\mathcal{M}$.*

Proof. Let T be the elementary diagram $\Delta_\omega(\mathcal{M})$ of $\mathcal{M}$ in the signature $\tau \cup \mathcal{M}$. It is easy to check that any model of T when restricted to the signature τ is an elementay extension of $\mathcal{M}$. Since $\mathcal{M} \prec_1 \mathcal{N}$, the natural extension of $\mathcal{N}$ to a $\tau \cup \mathcal{M}$-structure realizes the Π_1-fragment of T in the signature $\tau \cup \mathcal{M}$. Now apply the previous Lemma. □

The Resurrection Lemma motivates the resurrection axioms introduced by Hamkins and Johnstone in [4], and their iterated versions introduced by the author and Audrito in [2].

The following is a natural question: assume S and T are τ-theories such that $T_\forall = S_\forall$, can we extend a model $\mathcal{M}$ of T to a superstructure $\mathcal{M}$ of S so that $\mathcal{M}$ and $\mathcal{N}$ satisfy exactly the same universal sentences? The answer is no as shown by $\tau = \{\cdot, +, 0, 1\}$, T the τ-theory of fields, S the τ-theory of algebraically closed fields: it is easy to see that $T_\forall = S_\forall$ in view of Lemma 2 but $\mathbb{Q}$ cannot be extended to an algebraically closed field without killing the universal τ-sentence stating the non existence of the square root of -1.

The clarification of this issue is what has brought our attention to $T_{\forall\vee\exists}$.

Note that any sentence in $T_{\forall\vee\exists}$ is either logically equivalent to $\theta \vee \psi$ or equivalent to $\theta \wedge \psi$ with θ universal and ψ existential.

Note also that $T_{\forall\vee\exists}$ may contain more information than $T_\forall \cup T_\exists$ as there could be a universal $\theta \notin T_\forall$ and an existential $\psi \notin T_\exists$ with $\theta \vee \psi \in T_{\forall\vee\exists}$.

Lemma 4. *Let τ be a signature and T, S be τ-theories. TFAE:*

(1) $T_{\forall\vee\exists} \supseteq S_{\forall\vee\exists}$.

(2) For any $\mathcal{M}$ model of T there is $\mathcal{N}$ model of S superstructure of $\mathcal{M}$ realizing exactly the same universal sentences.

(3) For every boolean combination of universal sentences θ, $T + \theta$ is consistent only if so is $S + \theta$.

Proof.

1 implies 2: Assume $\mathcal{M}$ models T and is such that no $\mathcal{N}$ model of S which is a superstructure of $\mathcal{M}$ realizes exactly the same universal sentences.

For any such $\mathcal{N}$ with $\mathcal{M} \sqsubseteq \mathcal{N} \models S$ we get that some universal τ-sentence $\theta_\mathcal{N}$ true in $\mathcal{M}$ fails in $\mathcal{N}$. We claim that the $\tau \cup \mathcal{M}$-theory

$$S^* = \Delta_0(\mathcal{M}) \cup S \cup \{\theta_\mathcal{N} : \mathcal{M} \sqsubseteq \mathcal{N}, \mathcal{N} \models S\}$$

is inconsistent. If not let $\mathcal{P}^*$ be a model of S^*. Then $\mathcal{P} = (\mathcal{P}^* \restriction \tau) \sqsupseteq \mathcal{M}$ is a model of

$$S \cup \{\theta_\mathcal{N} : \mathcal{M} \sqsubseteq \mathcal{N}, \mathcal{N} \models S\}.$$

Hence it models $\theta_\mathcal{P}$ and $\neg\theta_\mathcal{P}$ at the same time.

By compactness we can find a universal sentence $\phi_\mathcal{M}$ given by the conjunction of a finite set

$$\{\theta_{\mathcal{P}_i} : i = 1, \dots, n, \mathcal{M} \sqsubseteq \mathcal{P}_i \models S\}$$

and a quantifier free sentence $\psi_\mathcal{M}(\vec{a})$ of $\Delta_0(\mathcal{M})$ such that

$$S + \psi_\mathcal{M}(\vec{a}) + \phi_\mathcal{M}$$

is inconsistent. Hence

$$S \models \neg\phi_\mathcal{M} \vee \neg\exists\vec{x}\, \psi_\mathcal{M}(\vec{x}).$$

Now observe that:

- $\neg\phi_{\mathcal{M}} \vee \neg\exists\vec{x}\,\psi_{\mathcal{M}}(\vec{x})$ is a boolean combination of universal sentences,
- $\mathcal{M} \models T + \exists\vec{x}\,\psi_{\mathcal{M}}(\vec{x}) \wedge \phi_{\mathcal{M}}$.

Therefore we get that $\neg\phi_{\mathcal{M}} \vee \neg\exists\vec{x}\,\psi_{\mathcal{M}}(\vec{x})$ is in $S_{\forall\vee\exists} \setminus T_{\forall\vee\exists}$.

2 implies 3: Left to the reader.

3 implies 1: If $T_{\forall\vee\exists} \not\supseteq S_{\forall\vee\exists}$ there is $\theta \in S_{\forall\vee\exists} \setminus T_{\forall\vee\exists}$. Then $\neg\theta$ is inconsistent with S and consistent with T.

□

Definition 1.1. Let τ be a signature and T, S be τ-theories.

- T and S are *cotheories* if $T_\forall = S_\forall$.
- T and S are *absolute cotheories* if $T_{\forall\vee\exists} = S_{\forall\vee\exists}$.

Remark 5. *Say that a τ-theory T is* Π_1-complete *if $T \vdash \phi$ or $T \vdash \neg\phi$ for any universal τ-sentence ϕ.*

Now consider the $\{+,\cdot,0,1\}$-theories ACF_0 and Fields_0 expanding ACF and Fields with the axioms fixing the characteristic of their models to be 0. Note that ACF_0 is Π_1-complete (it is actually complete) while Fields_0 is not, even if $(\mathsf{ACF}_0)_\forall = (\mathsf{Fields}_0)_\forall$. In particular $T_\forall = S_\forall$ is well possible with T Π_1-complete and S not Π_1-complete. Absolute cotheories rule out this confusing discrepancy. In particular we will use the following trivial fact crucially in the proof of Lemma 22: if S is a complete theory $S_{\forall\vee\exists}$ is Π_1-complete, while $S_\forall$ may not.

2. Existentially closed models

The objective is now to isolate the "generic" models of some universal theory T (i.e. all axioms of T are universal sentences). These are described by the T-existentially closed models.

Definition 2.1. Given a first order signature τ, let T be any consistent τ-theory. A τ-structure $\mathcal{M}$ is T-existentially closed (T-ec) if

(1) $\mathcal{M}$ can be embedded in a model of T.
(2) $\mathcal{M} \prec_{\Sigma_1} \mathcal{N}$ for all $\mathcal{N} \sqsupseteq \mathcal{M}$ which are models of T.

In general T-ec models need not be models[1] of T, but only of their universal fragment. A standard diagonalization argument shows that for any theory T there are T-ec models, see Lemma 9 below or [5, Lemma 3.2.11].

Trivial observations which will come handy in the sequel are the following:

Fact 6. *Assume $\mathcal{M}$ is a T-ec model and $S \supseteq T$ is such that some $\mathcal{N} \sqsupseteq \mathcal{M}$ models S. Then $\mathcal{M}$ is S-ec.*

Fact 7. *Assume $\mathcal{M}$ is T-ec. Let $\forall\vec{x}\exists\vec{y}\psi(\vec{x},\vec{y},\vec{a})$ be a Π_2-sentence with $\psi(\vec{x},\vec{y},\vec{z})$ quantifier free τ-formula and parameters $\vec{a}$ in $\mathcal{M}^{<\omega}$. Assume it holds in some $\mathcal{N} \sqsupseteq \mathcal{M}$ which models $T_\forall$, then it holds in $\mathcal{M}$.*

Proof. Observe that for all $\vec{b} \in \mathcal{M}^{<\omega}$, $\exists\vec{y}\,\psi(\vec{b},\vec{y},\vec{a})$ holds in $\mathcal{N}$, and therefore in $\mathcal{M}$, since $\mathcal{M}$ is T-ec; hence $\mathcal{M} \models \forall\vec{x}\exists\vec{y}\psi(\vec{x},\vec{y},\vec{a})$. □

In view of the byembeddability Lemmas it is not hard to check the following:

Fact 8. *TFAE for a τ-theory T and a τ-structure $\mathcal{M}$:*

(1) $\mathcal{M}$ is T-ec.
(2) $\mathcal{M}$ is $T_{\forall\vee\exists}$-ec.

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[1]For example let T be the theory of commutative rings with no zero divisors which are not fields in the signature $(+,\cdot,0,1)$. Then the T-ec structures are exactly all the algebraically closed fields, and no T-ec model is a model of T.

(3) $\mathcal{M}$ is $T_\forall$-ec.

Proof. The unique non-trivial implication is to argue that a T-ec model $\mathcal{M}$ is $T_\forall$-ec. Given $\mathcal{N} \sqsupseteq \mathcal{M}$ model of $T_\forall$, by Lemma 2 we can build $\mathcal{P} \sqsupseteq \mathcal{N}$ which is a model of T. Hence $\mathcal{M} \prec_1 \mathcal{P}$, which gives that any Σ_1-formula $\psi(\vec{a})$ with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ holds in $\mathcal{P}$ as well, hence it reflects to $\mathcal{M}$. □

We now show that any structure $\mathcal{M}$ which models T can always be extended to a T-ec superstructure.

Lemma 9. *[5, Lemma 3.2.11] Given a first order τ-theory T, any model of $T_\forall$ can be extended to a τ-superstructure which is T-ec.*

Proof. Given a model $\mathcal{M}$ of T, we construct an ascending chain of $T_\forall$-models as follows. Enumerate all quantifier free τ-formulae as $\{\phi_\alpha(y, \vec{x}_\alpha) : \alpha < |\tau|\}$. Let $\mathcal{M}_0 = \mathcal{M}$ have size $\kappa \geq |\tau| + \aleph_0$. Fix also some enumeration

$$\begin{aligned} \pi : &\kappa \to |\tau| \times \kappa^2 \\ &\alpha \mapsto (\pi_0(\alpha), \pi_1(\alpha), \pi_2(\alpha)) \end{aligned}$$

such that $\pi_2(\alpha) \leq \alpha$ for all $\alpha < \kappa$ and for each $\xi < |\tau|$, and $\eta, \beta < \kappa$ there are unboundedly many $\alpha < \kappa$ such that $\pi(\alpha) = (\xi, \eta, \beta)$.

Let now $\mathcal{M}_\eta$ with enumeration $\left\{\vec{m}^\xi_\eta : \xi < \kappa\right\}$ of $\mathcal{M}_\eta^{<\omega}$ be given for all $\eta \leq \beta$. If $\mathcal{M}_\beta$ is T-ec, stop the construction. Else check whether $T_\forall \cup \Delta_0(\mathcal{M}_\beta) \cup \left\{\exists y \phi_{\pi_0(\alpha)}(y, \vec{m}^{\pi_1(\alpha)}_{\pi_2(\alpha)})\right\}$ is a consistent $\tau \cup \mathcal{M}_\beta$-theory; if so let $\mathcal{M}_{\beta+1}$ have size κ and realize this theory, otherwise let $\mathcal{M}_{\beta+1}$ be $\mathcal{M}_\beta$. At limit stages γ, let $\mathcal{M}_\gamma$ be the direct limit of the chain of τ-structures $\{\mathcal{M}_\beta : \beta < \gamma\}$. Then all $\mathcal{M}_\xi$ are models of $T_\forall$, and at some stage $\beta \leq \kappa$ $\mathcal{M}_\beta$ is $T_\forall$-ec (hence also T-ec), since all existential τ-formulae with parameters in some $\mathcal{M}_\eta$ will be considered along the construction, and realized along the way if this is possible, and all $\mathcal{M}_\eta$ are always models of $T_\forall$ (at limit stages the ascending chain of $T_\forall$-models remains a $T_\forall$-model). □

Compare the above construction with the standard consistency proofs of bounded forcing axioms as given for example in [1, Section 2]. In the latter case to preserve $T_\forall$ at limit stages we use iteration theorems[2].

We also show that T-ec models are closed under Σ_1-elementary substructures.

Proposition 10. *Assume a τ-structure $\mathcal{M}$ is T-ec and $\mathcal{N} \prec_1 \mathcal{M}$. Then $\mathcal{N}$ is also T-ec.*

Proof. Assume $\mathcal{N} \sqsubseteq \mathcal{P}$ for some model of $T_\forall$ $\mathcal{P}$. Let $\Delta_0(\mathcal{P})$ be the atomic diagram of $\mathcal{P}$ in the signature $\tau \cup \mathcal{P} \cup \mathcal{M}$ and $\Delta_0(\mathcal{M})$ be the atomic diagram of $\mathcal{M}$ in the same signature[3].

Claim 1. *$T_\forall \cup \Delta_0(\mathcal{P}) \cup \Delta_0(\mathcal{M})$ is a consistent $\tau \cup \mathcal{M} \cup \mathcal{P}$-theory.*

Proof. Assume not. Find $\vec{a} \in (\mathcal{P} \setminus \mathcal{N})^{<\omega}$, $\vec{b} \in (\mathcal{M} \setminus \mathcal{N})^{<\omega}$, $\vec{c} \in \mathcal{N}^{<\omega}$ and τ-formulae $\psi_0(\vec{x}, \vec{z})$, $\psi_1(\vec{y}, \vec{z})$ such that:

- $\psi_0(\vec{a}, \vec{c}) \in \Delta_0(\mathcal{P})$,
- $\psi_1(\vec{b}, \vec{c}) \in \Delta_0(\mathcal{M})$,
- $T \cup \left\{\psi_0(\vec{a}, \vec{c}), \psi_1(\vec{b}, \vec{c})\right\}$ is inconsistent.

[2]Assume G is V-generic for a forcing which is a limit of an iteration of length ω of forcings $\{P_n : n < \omega\}$. In general $H_{\omega_2}^{V[G]}$ is not given by the union of $H_{\omega_2}^{V[G \cap P_n]}$, hence a subtler argument is needed to maintain that $H_{\omega_2}^{V[G]}$ preserves $T_\forall$.

[3]We are considering $\mathcal{P} \cup \mathcal{M}$ as the union of the domains of the structure $\mathcal{P}, \mathcal{M}$ amalgamated over $\mathcal{N}$; in particular we add a new constant for each element of $\mathcal{P} \setminus \mathcal{N}$, a new constant for each element of $\mathcal{M} \setminus \mathcal{N}$, a new constant for each element of $\mathcal{N}$.

Then

$$T \vdash \neg\psi_0(\vec{a}, \vec{c}) \vee \neg\psi_1(\vec{b}, \vec{c}).$$

Since the constants appearing in $\vec{a}, \vec{b}, \vec{c}$ are never appearing in sentences of T, we get that

$$T \vdash \forall \vec{z}\, (\forall \vec{x} \neg\psi_0(\vec{x}, \vec{z})) \vee (\forall \vec{y} \neg\psi_1(\vec{y}, \vec{z})).$$

Since $\mathcal{P}$ models $T_\forall$, and

$$\mathcal{P} \models \psi_0(\vec{x}, \vec{z})[\vec{x}/\vec{a}, \vec{z}/\vec{c}],$$

we get that

$$\mathcal{P} \models \forall \vec{y} \neg\psi_1(\vec{y}, \vec{c}).$$

Therefore

$$\mathcal{N} \models \forall \vec{y} \neg\psi_1(\vec{y}, \vec{c})$$

being a substructure of $\mathcal{P}$, and so does $\mathcal{M}$ since $\mathcal{N} \prec_1 \mathcal{M}$. This contradicts $\psi_1(\vec{b}, \vec{c}) \in \Delta_0(\mathcal{M})$. □

If $\bar{\mathcal{Q}}$ is a model realizing $T_\forall \cup \Delta_0(\mathcal{P}) \cup \Delta_0(\mathcal{M})$, and $\mathcal{Q}$ is the τ-structure obtained forgetting the constant symbols not in τ, we get that:

- $\mathcal{P}$ and $\mathcal{M}$ are both substructures of $\mathcal{Q}$ containing $\mathcal{N}$ as a common substructure;
- $\mathcal{N} \prec_1 \mathcal{M} \prec_1 \mathcal{Q}$, since $\mathcal{Q}$ realizes $T_\forall$ and $\mathcal{M}$ is $T_\forall$-ec.

We can now conclude that if a Σ_1-formula $\psi(\vec{c})$ for $\tau \cup \mathcal{N}$ with parameters in $\mathcal{N}$ holds in $\mathcal{P}$, it holds in $\mathcal{Q}$ as well (since $\mathcal{Q} \sqsupseteq \mathcal{P}$), and therefore also in $\mathcal{N}$ (since $\mathcal{N} \prec_1 \mathcal{Q}$). □

3. Kaiser hulls and strong consistency

Definition 3.1. Let T be a τ-theory.

- A τ-sentence ψ is *strongly $T_{\forall\vee\exists}$-consistent* if $\psi + R_{\forall\vee\exists}$ is consistent for all $R \supseteq T_{\forall\vee\exists}$.
- The *Kaiser hull of T* ($\mathsf{KH}(T)$) consists of the Π_2-sentences for τ which hold in all T-ec models.
- The *strong consistency hull of T* ($\mathsf{SCH}(T)$) consists of the Π_2-sentences for τ which are strongly $T_{\forall\vee\exists}$-consistent.

The Kaiser hull of a theory is a well known notion describing an equivalent of model companionship which can be defined also for non-companionable theories (see for example [5, Lemma 3.2.12, Lemma 3.2.13, Thm. 3.2.14]); the strong consistency hull is a slight weakening of the Kaiser hull not considered till now (at least to my knowledge) and which does the same with respect to the notion of absolute model companionship (defined below in Def. 6.1).

Remark 11. *Strong $T_{\forall\vee\exists}$-consistency sits in between $T_{\forall\vee\exists}$-provability and $T_{\forall\vee\exists}$-consistency. It is clear that if ψ is a boolean combination of universal sentences ψ is strongly $T_{\forall\vee\exists}$-consistent if and only if it is in $T_{\forall\vee\exists}$; on the other hand the Fact and Lemma below yield the following:*

- *If T is Π_1-complete, strong $T_{\forall\vee\exists}$-consistency clearly overlaps with $T_{\forall\vee\exists}$-consistency.*
- *If ψ is a Π_2-sentence which is strongly $T_{\forall\vee\exists}$-consistent it holds in a model of* $\mathsf{KH}(T)$ *(by the Lemma below).*
- $\mathsf{KH}(T)$ *can be a proper superset of* $\mathsf{SCH}(T)$ *(by the Fact below it is always a superset); hence there can be Π_2-sentences which are $T_{\forall\vee\exists}$-consistent but are neither T-provable nor strongly $T_{\forall\vee\exists}$-consistent.*

  For example consider the $\{0, 1, \cdot, +\}$-theories ACF_0 *(of algebraically closed fields of characteristic 0) and* Fields_0 *(of fields of characteristic 0). Note that* ACF_0 *is complete while* Fields_0 *is not. Furthermore* ACF_0 *is the Kaiser hull of* Fields_0 *(note that:* ACF_0 *is axiomatized by its Π_2-fragment; any* Fields_0*-ec model is an algebraically closed field; any model of* ACF_0 *is* Fields_0*-ec).*

We get that $\exists x\,(x^2+1=0)$ is a Π_2-sentence (in fact existential) in the Kaiser hull of Fields_0 but not in its strong consistency hull, since it is not consistent with $R_{\forall\vee\exists}$, where R is the $\{0,1,\cdot,+\}$-theory of the rationals.

Fact 12. *For any τ-theory T:*

(i) $\mathsf{KH}(T)_\forall = T_\forall$;
(ii) $\mathsf{SCH}(T)_{\forall\vee\exists} = T_{\forall\vee\exists}$;
(iii) $\mathsf{SCH}(T) \subseteq \mathsf{KH}(T)$;
(iv) $\mathsf{SCH}(T) = \mathsf{KH}(T)$ *if* $T_{\forall\vee\exists} = \mathsf{KH}(T)_{\forall\vee\exists}$, *which is the case if T is complete.*
(v) For any Π_2-sentence ψ such that $T_{\forall\vee\exists}+\psi$ is consistent, there is a model of $\mathsf{KH}(T)+\psi$.

Proof.

(i) By definition any model of $\mathsf{KH}(T)$ is a model of $T_\forall$; conversely any model of T can be extended to a T-ec model by Lemma 9. We conclude by Lemma 2.
(ii) Trivial.
(iii) Assume a Π_2-sentence ψ is strongly $T_{\forall\vee\exists}$-consistent. Let $\mathcal{M}$ be a T-ec model. Then $\mathcal{M}$ is $T_{\forall\vee\exists}$-ec. Let R be the τ-theory of $\mathcal{M}$. Since $\mathcal{M}$ is T-ec, any superstructure of $\mathcal{M}$ which models T is also a model of $R_{\forall\vee\exists}$ (by Fact 8). Since ψ is strongly $T_{\forall\vee\exists}$-consistent, $\psi + R_{\forall\vee\exists}$ is consistent. By Lemma 4, ψ holds in some $\mathcal{N} \sqsupseteq \mathcal{M}$ which models $R_{\forall\vee\exists}$. Since $\mathcal{M}$ is T-ec and $R_{\forall\vee\exists} \supseteq T_\forall$, we get that ψ reflects to $\mathcal{M}$ (being a Π_2-sentence which holds in $\mathcal{N}$ which is a Σ_1-superstructure of $\mathcal{M}$).
(iv) Assume a Π_2-sentence ψ is in $\mathsf{KH}(T)$. Let R be any complete extension of T and $\mathcal{M}$ be a model of R. By Lemma 4 there is $\mathcal{N}$ which is a model of $\mathsf{KH}(T)$ and of $R_{\forall\vee\exists}$. In particular $\mathcal{N}$ models $\psi + R_{\forall\vee\exists}$. Since R is arbitrary, ψ is strongly $T_{\forall\vee\exists}$-consistent.
Clearly if T is complete, $T_{\forall\vee\exists}$ is Π_1-complete, and ψ is strongly $T_{\forall\vee\exists}$-consistent if and only if $\psi + T_{\forall\vee\exists}$ is consistent. We conclude also in this case that a Π_2-sentence ψ holds in some T-ec model if and only if it is strongly $T_{\forall\vee\exists}$-consistent.
(v) Note that $\mathsf{KH}(T)$ is axiomatized by its Π_2-fragment and $\mathsf{KH}(T)_{\forall\vee\exists} \supseteq T_{\forall\vee\exists}$. Therefore we can apply Lemma 13 below.

□

Lemma 13. *Let S, T be τ-theories such that $S_{\forall\vee\exists} \subseteq T_{\forall\vee\exists}$ and S is axiomatized by its Π_2-fragment. Then for any Π_2-sentence ψ consistent with T there is a model of $S+\psi$.*

Proof. We prove a stronger conclusion which is the following:

> *Let R be a complete theory extending $T+\psi$. Then there is a model of $R_{\forall\vee\exists}+S+\psi$.*

Let $\{\mathcal{M}_n : n \in \omega\}$ be a sequence of τ-structures such that for all $n \in \omega$:

- $\mathcal{M}_n$ is a τ-substructure of $\mathcal{M}_{n+1}$;
- $\mathcal{M}_n$ models $R_{\forall\vee\exists}$;
- $\mathcal{M}_{2n}$ models R;
- $\mathcal{M}_{2n+1}$ models S.

Such a sequence can be defined letting $\mathcal{M}_0$ be a model of R, $\mathcal{M}_1$ be a model of S which satisfies $R_{\forall\vee\exists}$ (which is possible in view of Lemma 4) and defining $\mathcal{M}_n$ as required for all other n appealing to the fact that $S+R_{\forall\vee\exists}$ and $T+\psi+R_{\forall\vee\exists}$ are absolute cotheories with $R_{\forall\vee\exists}$ being the Π_1-complete fragment shared by both theories. Then $\mathcal{M} = \bigcup_{n\in\omega} \mathcal{M}_n$ is a model of $R_{\forall\vee\exists}+S+\psi$ since it realizes all Π_2-sentences which hold in an infinite set of $\mathcal{M}_n$ (see for example [5, Lemma 3.1.6]) and satisfies exactly the same Π_1-sentences of each of the $\mathcal{M}_n$. □

Note that the above cannot be proved if S, T are just cotheories: performing the above construction under this weaker assumption, we may not be able to define $\mathcal{M}_2$ as required if $\mathcal{M}_1$ does not realize exactly the same universal sentences of $\mathcal{M}_0$.

4. Model completeness

It is possible (depending on the choice of the theory T) that there are models of the Kaiser hull of T which are not T-ec. Robinson has come up with two model theoretic properties (model completeness and model companionship) which describe the case in which the models of the Kaiser hull of T are exactly the class of T-ec models (even in case T is not a complete theory).

Definition 4.1. A τ-theory T is *model complete* if for all τ-models $\mathcal{M}$ and $\mathcal{N}$ of T we have that $\mathcal{M} \sqsubseteq \mathcal{N}$ implies $\mathcal{M} \prec \mathcal{N}$.

Remark that theories admitting quantifier elimination are automatically model complete. On the other hand model complete theories need not be complete[4]. However for theories T which are Π_1-complete, model completeness entails completeness: any two models of a Π_1-complete, model complete T share the same Π_1-theory, therefore if $T_1 \supseteq T$ and $T_2 \supseteq T$ with $\mathcal{M}_i$ a model of T_i, we can suppose (by Lemma 2) that $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$. Since they are both models of T, model completeness entails that $\mathcal{M}_1 \prec \mathcal{M}_2$.

Lemma 14. *[5, Lemma 3.2.7] (Robinson's test) Let T be a τ-theory. The following are equivalent:*

(a) T is model complete.
(b) Any model of T is T-ec.
(c) Each existential *τ-formula $\phi(\vec{x})$ in free variables $\vec{x}$ is T-equivalent to a universal τ-formula $\psi(\vec{x})$ in the same free variables.*
(d) Each τ-formula $\phi(\vec{x})$ in free variables $\vec{x}$ is T-equivalent to a universal τ-formula $\psi(\vec{x})$ in the same free variables.

Remark 15. *(d) (or (c)) shows that being a model complete τ-theory T is expressible by a Δ_0-property in parameters τ, T in any model of* ZFC*, hence it is absolute with respect to forcing. They also show that quantifier elimination implies model completeness. (c) also shows that model complete theories are axiomatized by their Π_2-fragment.*

Proof.

(a) implies (b): Immediate.

(b) implies (c): Fix an existential formula $\phi(\vec{x})$ in free variables $x_1, \ldots, x_n$. If $\phi(\vec{x})$ is not consistent with T it is T-equivalent to the trivial formula $\forall y(y \neq y)$ in free variables $\vec{x}$. Hence we may assume that $T \cup \phi(\vec{x})$ is a consistent theory. Let $\vec{c} = (c_1, \ldots, c_n)$ be a finite set of new constant symbols. Then $T \cup \phi(\vec{c})$ is a consistent $\tau \cup \{c_1, \ldots, c_n\}$-theory.

Let Γ be the set of universal τ-formulae $\theta(\vec{x})$ such that

$$T \vdash \forall \vec{x}\, (\phi(\vec{x}) \to \theta(\vec{x})).$$

Note that Γ is closed under finite conjunctions and disjunctions. Let $\Gamma(\vec{c}) = \{\theta(\vec{c}) : \theta(\vec{x}) \in \Gamma\}$. Note that $T \cup \Gamma(\vec{c})$ is a consistent $\tau \cup \{c_1, \ldots, c_n\}$-theory, since it holds in any $\tau \cup \{c_1, \ldots, c_n\}$-model of $T \cup \phi(\vec{c})$.

It suffices to prove

$$T \cup \Gamma(\vec{c}) \models \phi(\vec{c}); \tag{1}$$

[4]For example the theory of algebraically closed fields is model complete, but algebraically closed fields of different characteristics are elementarily inequivalent.

if this is the case, by compactness, a finite subset $\Gamma_0(\vec{c})$ of $\Gamma(\vec{c})$ is such that

$$T \cup \Gamma_0(\vec{c}) \models \phi(\vec{c});$$

letting $\bar{\theta}(\vec{x}) := \bigwedge \{\psi(\vec{x}) : \psi(\vec{c}) \in \Gamma_0(\vec{c})\}$, the latter gives that

$$T \models \forall \vec{x} \, (\bar{\theta}(\vec{x}) \to \phi(\vec{x}))$$

(since the constants $\vec{c}$ do not appear in T).

$\bar{\theta}(\vec{x}) \in \Gamma$ is a universal formula witnessing (c) for $\phi(\vec{x})$.

So we prove (1):

Proof. Let $\mathcal{M}$ be a $\tau \cup \{c_1, \dots, c_n\}$-model of $T \cup \Gamma(\vec{c})$. We must show that $\mathcal{M}$ models $\phi(\vec{c})$.

The key step is to prove the following:

Claim 2. *$T \cup \Delta_0(\mathcal{M}) \cup \{\phi(\vec{c})\}$ is consistent (where $\Delta_0(\mathcal{M})$ is the $\tau \cup \{c_1, \dots, c_n\}$-atomic diagram of $\mathcal{M}$ in signature $\tau \cup \{c_1, \dots, c_n\} \cup \mathcal{M}$).*

Assume the Claim holds and let $\mathcal{N}$ realize the above theory. Then

$$\mathcal{M} \sqsubseteq \mathcal{N} \restriction (\tau \cup \{c_1, \dots, c_n\}).$$

Hence

$$\mathcal{M} \restriction \tau \sqsubseteq \mathcal{N} \restriction \tau.$$

By (b)

$$\mathcal{M} \restriction \tau \prec_1 \mathcal{N} \restriction \tau.$$

Now let $b_1, \dots, b_n \in \mathcal{M}$ be the interpretations of $c_1, \dots, c_n$ in the $\tau \cup \{c_1, \dots, c_n\}$-structure $\mathcal{M}$. Then

$$\mathcal{N} \restriction \tau \models \phi(x_1, \dots, x_n)[b_1, \dots, b_n].$$

Since $\phi(\vec{x})$ is Σ_1 for τ and $b_1, \dots, b_n \in \mathcal{M}$, we get that

$$\mathcal{M} \restriction \tau \models \phi(x_1, \dots, x_n)[b_1, \dots, b_n],$$

hence

$$\mathcal{M} \models \phi(c_1, \dots, c_n),$$

and we are done.

So we are left with the proof of the Claim.

Proof. Let $\psi(\vec{x}, \vec{y})$ be a quantifier free τ-formula such that $\psi(\vec{c}, \vec{a}) \in \Delta_0(\mathcal{M})$ for some $\vec{a} \in \mathcal{M}$.

Clearly $\mathcal{M}$ models $\exists \vec{y} \psi(\vec{c}, \vec{y})$.

Then the universal formula $\neg \exists \vec{y} \psi(\vec{c}, \vec{y}) \notin \Gamma(\vec{c})$, since $\mathcal{M}$ models its negation and $\Gamma(\vec{c})$ at the same time.

This gives that

$$T \not\models \forall \vec{x} \, (\phi(\vec{x}) \to \neg \exists \vec{y} \psi(\vec{x}, \vec{y})),$$

i.e.

$$T \cup \{\exists \vec{x} \, [\phi(\vec{x}) \wedge \exists \vec{y} \psi(\vec{x}, \vec{y})]\}$$

is consistent.

Since the argument holds for all $\psi(\vec{c}, \vec{a}) \in \Delta_0(\mathcal{M})$, we conclude that

$$T \cup \{\phi(\vec{c}) \wedge \psi(\vec{c}, \vec{a})\}$$

is consistent for any tuple $a_1, \dots, a_k \in \mathcal{M}$ and quantifier free formula ψ such that $\mathcal{M}$ models $\psi(\vec{c}, \vec{a})$.

This shows that $T \cup \Delta_0(\mathcal{M}) \cup \{\phi(\vec{c})\}$ is consistent. □

(1) is proved. □

(c) implies (d): We prove by induction on n that Π_n-formulae and Σ_n-formulae are T-equivalent to a Π_1-formula.

(c) gives the base case $n = 1$ of the induction for Σ_1-formulae and (trivially) for Π_1-formulae.

Assuming we have proved the implication for all Σ_n formulae for some fixed $n > 0$, we obtain it for Π_{n+1}-formulae $\forall\vec{x}\psi(\vec{x},\vec{y})$ (with $\psi(\vec{x},\vec{y})$ Σ_n) applying the inductive assumptions to $\psi(\vec{x},\vec{y})$; next we observe that a Σ_{n+1}-formula is equivalent to the negation of a Π_{n+1}-formula, which is in turn equivalent to the negation of a universal formula (by what we already argued), which is equivalent to an existential formula, and thus equivalent to a universal formula (by (c)).

(d) implies (a): By (d) every formula is T-equivalent both to a universal formula and to an existential formula (since its negation is T-equivalent to a universal formula).

This gives that $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \sqsubseteq \mathcal{N}$ are models of T, since truth of universal formulae is inherited by substructures, while truth of existential formulae pass to superstructures.

□

We will also need the following:

Fact 16. *Let τ be a signature and T a model complete τ-theory. Let $\sigma \supseteq \tau$ be a signature and $T^* \supseteq T$ a σ-theory such that every σ-formula is T^*-equivalent to a τ-formula. Then T^* is model complete.*

Proof. By the model completeness of T and the assumptions on T^* we get that every σ-formula is equivalent to a Π_1-formula for $\tau \subseteq \sigma$. We conclude by Robinson's test. □

Later on we will show that in most cases model complete theories maximize the family of Π_2-sentences compatible with any Π_1-completion of their universal fragment. This will be part of a broad family of properties for first order theories which require a new concept in order to be properly formulated, that of model companionship.

5. Model companionship

Model completeness comes in pairs with another fundamental concept which generalizes to arbitrary first order theories the relation existing between algebraically closed fields and commutative rings without zero-divisors. As a matter of fact, the case described below occurs when T^* is the theory of algebraically closed fields and T is the theory of commutative rings with no zero divisors.

Definition 5.1. Given two theories T and T^* in the same language τ, T^* is the *model companion* of T if the following conditions holds:

(1) Each model of T can be extended to a model of T^*.
(2) Each model of T^* can be extended to a model of T.
(3) T^* is model complete.

Remark 17. *By Robinson's test and Lemma 2, model companionship is expressible by a Δ_0-property in the relevant parameters in the standard model $(V,\in)$ of* ZFC.

Different theories can have the same model companion, for example the theory of fields and the theory of commutative rings with no zero-divisors which are not fields both have the theory of algebraically closed fields as their model companion.

Theorem 18. *[5, Thm 3.2.14] Let T be a first order theory. If its model companion T^* exists, then*

(1) $T_\forall = T^*_\forall$.
(2) T^* is the theory of the existentially closed models of $T_\forall$.

Proof.

(1) By Lemma 2.
(2) By Robinson's test 14 T^* is the theory realized exactly by the T^*-ec models; by Fact 6 $\mathcal{M}$ is T^*-ec if and only if it is $T^*_\forall$-ec; by (1) $T^*_\forall = T_\forall$.

□

An immediate by-product of the above Theorem is that the model companion of a theory does not necessarily exist, but, if it does, it is unique and is its Kaiser hull.

Theorem 19. *[5, Thm. 3.2.9] Assume T has a model companion T^*. Then T^* is axiomatized by its Π_2-consequences and is the Kaiser hull of $T_\forall$.*

Moreover T^* is the unique model companion of T and is characterized by the property of being the unique model complete theory S such that $S_\forall = T_\forall$.

Proof. For quantifier free formulae $\psi(\vec{x}, \vec{y})$ and $\phi(\vec{x}, \vec{z})$ the assertion

$$\forall \vec{x}\, [\exists \vec{y} \psi(\vec{x}, \vec{y}) \leftrightarrow \forall \vec{z} \phi(\vec{x}, \vec{z})]$$

is a Π_2-sentence.

Let T^{**} be the theory given by the Π_2-consequences of T^*.

Since T^* is model complete, by Robinson's test 14(c), for any Σ_1-formula $\exists \vec{y} \psi(\vec{x}, \vec{y})$ there is a universal formula $\forall \vec{z} \phi(\vec{x}, \vec{z})$ such that

$$\forall \vec{x}\, [\exists \vec{y} \psi(\vec{x}, \vec{y}) \leftrightarrow \forall \vec{z} \phi(\vec{x}, \vec{z})]$$

is in T^{**}.

Again by Robinson's test 14(c) T^{**} is model complete.

Now assume S is a model complete theory such that $S_\forall = T_\forall$. Clearly $T^*_\forall = T_\forall = S_\forall$. By Robinson's test 14(b) and Fact 6, $S_\forall$ holds exactly in the $T_\forall$-ec models, but these are exactly the models of T^*. Hence $T^* = S$.

This shows that any model complete theory is axiomatized by its Π_2-consequences, that the model companion T^* of T is unique, that T^* is also the Kaiser hull of T (being axiomatized by the Π_2-sentences which hold in all T-ec-models), and is characterized by the property of being the unique model complete theory S such that $T_\forall = S_\forall$. □

Thm. 19 provides an equivalent characterization of model companion theories (which is expressible by a Δ_0-property in parameters T and T^*, hence absolute for transitive models of ZFC).

Note also that Robinson's test 14(d) gives an explicit axiomatization of a model complete theory T:

Fact 20. *Assume T is a model complete τ-theory. Let $\psi \mapsto \theta^T_\psi$ be a function assigning to each Σ_1-formula $\psi(\vec{x})$ for τ a Π_1-formula $\theta^T_\psi(\vec{x})$ which is T-equivalent to $\psi(\vec{x})$.*

Then T is axiomatized by $T_\forall$ and the Π_2-sentences

$$\mathsf{AX}^T_\psi \equiv \forall \vec{x} (\psi(\vec{x}) \leftrightarrow \theta^T_\psi(\vec{x}))$$

as $\psi(\vec{x})$ ranges over the Σ_1-formulae for τ.

Proof. First of all

$$T^* = \left\{\mathsf{AX}^T_\psi : \psi \text{ a } \tau\text{-formula}\right\}$$

is a model complete theory, since T^* satisfies Robinson's test 14(d). Let $S = T^* + T_\forall$. Note that S is also model complete (by Robinson's test 14(d)). Moreover $S \subseteq T$ (since $\mathsf{AX}^T_\psi \in T$ for all Σ_1-formulae ψ), and $S_\forall \supseteq T_\forall$ (since $T_\forall$ is certainly among the universal consequences of S). We conclude that $S_\forall = T_\forall$. Therefore S is the model companion of T. $S = T$ by uniqueness of the model companion. □

6. Absolute model companionship

Definition 6.1. A τ-theory T is the *absolute model companion* (AMC) of a τ-theory S if T and S are absolute cotheories and T is model complete.

Remark 21. *Again Robinson's Test amd Lemma 4 grant that AMC is expressible by a Δ_0-formula in the relevant parameters in $\in$-models of* ZFC.

The following characterization of absolute model companionship has brought our attention to this notion.

Lemma 22. *Assume T, T' are τ-theories and T' is model complete. TFAE:*

(i) T' is the absolute model companion of T.
(ii) T' is axiomatized by the strong consistency hull of T.

Proof.

(i) implies (ii): First of all we note that any model complete theory S is axiomatized by its strong consistency hull in view of Robinson's test (d) and Fact 8.

We also note that for absolute cotheories T, T', their strong consistency hull overlap (in view of Lemma 4).

Putting everything together we obtain the desired implication.

(ii) implies (i): Note that for θ a boolean combination of universal τ-sentences, we have that θ is in the strong consistency hull of some τ-theory S if and only if $\theta \in S_{\forall\vee\exists}$. Combined with (ii), this gives that $T'_{\forall\vee\exists} = T_{\forall\vee\exists}$.

□

Finally the following Lemma motivates our terminology for AMC:

Lemma 23. *Assume T, T' are τ-structures such that T' is the AMC of T. Then any S extending T has as AMC $T' + S_\forall$.*

Note that this fails for the standard notion of model companionship: ACF is the model companion of Fields in signature $\tau = \{0, 1, \cdot, +\}$, but if S is the theory of the rationals in signature τ, $S_\forall +$ ACF is inconsistent, hence it cannot be the model companion of S.

Proof. Assume $S \supseteq T$ is consistent. If $\mathcal{M} \models S$, $\mathcal{M}$ has a superstructure which models $T' + S_{\forall\vee\exists}$, since T and T' are absolute cotheories. This gives that $S' = T' + S_\forall$ is consistent. Since T' is model complete, so is S' by Robinson's test (cfr. Remark 15 and Lemma 14(c)). Now observe that S' and S satisfy item 2 of Lemma 4 (since $S' \supseteq T'$ and $S \supseteq T$ with T and T' absolute cotheories), yielding easily that $S_{\forall\vee\exists} = S'_{\forall\vee\exists}$. Therefore S' is the AMC of S. □

Remark 24. *Absolute model companionship is strictly stronger than model companionship: if T is model complete, T is the model companion of $T_\forall$ and the absolute model companion of $T_{\forall\vee\exists}$; the two notions do not coincide whenever $T_\forall$ is strictly weaker than $T_{\forall\vee\exists}$.*

If T' is the model companion of T, $T'_{\forall\vee\exists} \supseteq T_{\forall\vee\exists}$: assume $\mathcal{M} \models T'$, then there is a superstructure $\mathcal{N}$ of $\mathcal{M}$ which models T (since T' is the model companion of T). Now $\mathcal{M} \prec_1 \mathcal{N}$, since $\mathcal{M}$ is T-ec. Hence $\mathcal{N}$ has the same Π_1-theory of $\mathcal{M}$. The inclusion can be strict as shown by the counterexample given by Fields *versus* ACF.

Note that these theories also show that if T is the model companion of S then some R extending S may not have $T + R_\forall$ as its model companion: the theory R of the rationals is such that ACF $+ R_\forall$ *is inconsistent, hence the latter cannot be the model companion of R*

Recall that T is the model completion of S if it is its model companion and admits quantifier elimination (see [3, Prop. 3.5.19]). Absolute model companionship does not imply model completion (see [6, Fact 2.3.7]).

7. Partial Morleyizations and the AMC-spectrum of a theory

We now introduce the notation we use to relate AMC to set theory.

Notation 25. *Given a signature τ, let $\phi(x_0, \ldots, x_n)$ be a τ-formula.*

We let:

- *R_ϕ be a new $n+1$-ary relation symbols,*
- *f_ϕ be a new n-ary function symbols*[5]
- *c_τ be a new constant symbol.*

We also let:

$$\mathsf{AX}^0_\phi := \forall \vec{x}[\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x})],$$

$$\begin{aligned}\mathsf{AX}^1_\phi :=& \forall x_1, \ldots, x_n \\ & [(\exists! y \phi(y, x_1, \ldots, x_n) \rightarrow \phi(f_\phi(x_1, \ldots, x_n), x_1, \ldots, x_n)) \wedge \\ & \wedge (\neg \exists! y \phi(y, x_1, \ldots, x_n) \rightarrow f_\phi(x_1, \ldots, x_n) = c_\tau)]\end{aligned}$$

for $\phi(x_0, \ldots, x_n)$ having at least two free variables, and

$$\mathsf{AX}^1_\phi := [(\exists! y \phi(y)) \rightarrow \phi(f_\phi)] \wedge [(\neg \exists! y \phi(y)) \rightarrow c_\tau = f_\phi] .$$

for $\phi(x)$ having exactly one free variable.

Let Form_τ denotes the set of τ-formulae. For $A \subseteq \mathsf{Form}_\tau \times 2$

- *τ_A is the signature obtained by adding to τ relation symbols R_ϕ for the $(\phi, 0) \in A$ and function symbols f_ϕ for the $(\phi, 1) \in A$ (together with the special symbol c_τ if at least one $(\phi, 1)$ is in A).*
- *$T_{\tau,A}$ is the τ_A-theory having as axioms the sentences AX^i_ϕ for $(\phi, i) \in A$.*

Note the following:

- For any τ-theory T, let $A = \mathsf{Form}_\tau \times \{0\}$ and $\tau^* = \tau_A$; then $T^* = T + T_{\tau,A}$ is a τ^*-theory admitting quantifier elimination (the Morleyization of T, see Prop. 26 below). Furthermore any τ-structure admits exactly one extension to a τ^*-structure which is a model of $T_{\tau,A}$.
- For any τ-formula ϕ, any τ-structure $\mathcal{M}$ with domain M, and any $a \in M$, there is exactly one extension of $\mathcal{M}$ to a $\tau_{\{\phi\}\times\{1\}}$-structures which interprets the value of the special constant c_τ as a and models AX^1_ϕ.

In the sequel of this paper we are interested to analyze what happens when the Morleyization process is performed on arbitrary subsets of $\mathsf{Form}_\tau \times 2$.

Definition 7.1. The AMC-spectrum of a τ-theory T ($\mathfrak{spec}_{\mathsf{AMC}}(T)$) is given by those $A \subseteq \mathsf{Form}_\tau \times 2$ such that $T + T_{\tau,A}$ has an AMC (which we denote by $\mathsf{AMC}(T, A)$).

The MC-spectrum of a τ-theory T ($\mathfrak{spec}_{\mathsf{MC}}(T)$) is given by those $A \subseteq \mathsf{Form}_\tau \times 2$ such that $T + T_{\tau,A}$ has a model companion (which we denote by $\mathsf{MC}(T, A)$).

Note that $A = \mathsf{Form}_\tau \times \{0\}$ is always in the model companionship spectrum of a theory T (as $T + T_{\tau,A}$ admits quantifier elimination, hence is model complete and its own AMC in signature τ_A). Note also that $\emptyset$ is in the (A)MC-spectrum of T if and only if T has a model companion (an AMC).

Proposition 26. *Given a signature τ consider the signature τ^* which adds an n-ary predicate symbol R_ϕ for any τ-formula $\phi(x_1, \ldots, x_n)$ with displayed free variables.*

Let T_τ be the following τ^*-theory:

- *$\forall \vec{x}\, (\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x}))$ for all quantifier free τ-formulae $\phi(\vec{x})$,*
- *$\forall \vec{x}\, [R_{\phi\wedge\psi}(\vec{x}) \leftrightarrow (R_\psi(\vec{x}) \wedge R_\phi(\vec{x}))]$ for all τ-formulae $\phi(\vec{x}), \psi(\vec{x})$,*

[5]As usual we confuse 0-ary function symbols with constants.

- $\forall\vec{x}\,[R_{\neg\phi}(\vec{x}) \leftrightarrow \neg R_\phi(\vec{x})]$ *for all* τ*-formulae* $\phi(\vec{x})$,
- $\forall\vec{x}\,[\exists y R_\phi(y,\vec{x}) \leftrightarrow R_{\exists y\phi}(\vec{x})]$ *for all* τ*-formulae* $\phi(y,\vec{x})$.

Then any τ*-structure* $\mathcal{N}$ *admits a unique extension to a* τ^**-structure* $\mathcal{N}^*$ *which models* T_τ*. Moreover every* τ^**-formula is* T_τ*-equivalent to an atomic* τ^**-formula. In particular for any* τ*-model* $\mathcal{N}$*, the algebras of its* τ*-definable subsets and of the* τ^**-definable subsets of* $\mathcal{N}^*$ *are the same.*

Therefore for any consistent τ*-theory* T*,* $T \cup T_\tau$ *is consistent and admits quantifier elimination, hence is model complete.*

Proof. By an easy induction one can prove that any τ-formula $\phi(\vec{x})$ is T_τ-equivalent to the atomic τ^*-formula $R_\phi(\vec{x})$.

Another simple inductive argument brings that any τ^*-formula $\phi(\vec{x})$ is T_τ-equivalent to the τ-formula obtained by replacing all symbols $R_\psi(\vec{x})$ occurring in ϕ by the τ-formula $\psi(\vec{x})$. Combining these observations together we get that any τ^*-formula is equivalent to an atomic τ^*-formula.

T_τ forces the $\mathcal{M}^*$-interpretation of any relation symbol $R_\phi(\vec{x})$ in $\tau^* \setminus \tau$ to be the $\mathcal{M}$-interpretation of the τ-formula $\phi(\vec{x})$ to which it is T_τ-equivalent. □

Observe that the expansion of the language from τ to τ^* behaves well with respect to several model theoretic notions of tameness distinct from model completeness: for example T is a *stable* τ-theory if and only if so is the τ^*-theory $T \cup T_\tau$, the same holds for NIP-theories, or for o-minimal theories, or for κ-categorical theories.

The passage from τ-structures to τ^*-structures which model T_τ can have effects on the embeddability relation; for example assume $\mathcal{M} \sqsubseteq \mathcal{N}$ is a non-elementary embedding of τ-structures; then $\mathcal{M}^* \not\sqsubseteq \mathcal{N}^*$: if the non-atomic τ-formula $\phi(\vec{a})$ in parameter $\vec{a} \in \mathcal{M}^{<\omega}$ holds in $\mathcal{M}$ and does not hold in $\mathcal{N}$, the atomic τ^*-formula $R_\phi(\vec{a})$ holds in $\mathcal{M}^*$ and does not hold in $\mathcal{N}^*$.

However if T is a model complete τ-theory, then for $\mathcal{M} \sqsubseteq \mathcal{N}$ τ-models of T, we get that $\mathcal{M} \prec \mathcal{N}$; this entails that $\mathcal{M}^* \sqsubseteq \mathcal{N}^*$, which (by the quantifier elimination of $T \cup T_\tau$) gives that $\mathcal{M}^* \prec \mathcal{N}^*$. In particular for a model complete τ-theory T and $\mathcal{M}, \mathcal{N}$ τ-models of T, $\mathcal{M} \sqsubseteq \mathcal{N}$ if and only if $\mathcal{M}^* \sqsubseteq \mathcal{N}^*$.

Let us now investigate the case of model companionship. If T is the model companion of S with $S \neq T$ in the signature τ, $T \cup T_\tau$ and $S \cup T_\tau$ are both model complete theories in the signature τ^*. But $T \cup T_\tau$ cannot be the model companion of $S \cup T_\tau$, by uniqueness of the model companion, since each of these theories is the model companion of itself and they are distinct. Moreover if T and S are also complete, no τ^*-model of $S \cup T_\tau$ can embed into a τ^*-model of $T \cup T_\tau$: since T is the model companion of S and $S \neq T$, $T_\forall = S_\forall$ and there is some Π_2-sentence ψ $\forall x \exists y \phi(x,y)$ with ϕ-quantifer free in $T \setminus S$. Therefore $\forall x\, R_{\exists y\phi}(x) \in (T \cup T_\tau)_\forall \setminus (S \cup T_\tau)_\forall$; we conclude by Lemma 2, since $T \cup T_\tau$ and $S \cup T_\tau$ are complete, hence the above sentence separates $(T \cup T_\tau)_\forall$ from $(S \cup T_\tau)_\forall$.

8. Preservation of the substructure relation and of Σ_1-elementarity by expansions via definable Skolem functions

Fact 27. *Assume* $\mathcal{M} \sqsubseteq \mathcal{N}$ *are* τ*-structures and* $\phi(\vec{x}, y)$ *is a* τ*-formula such that both structures satisfy*

$$\forall\vec{x}, y\,[\phi(\vec{x}, y) \leftrightarrow \forall\vec{u}\,\psi_\phi(\vec{x}, y, \vec{u}) \leftrightarrow \exists\vec{z}\,\theta_\phi(\vec{x}, y, \vec{z})]$$

with ψ_ϕ, θ_ϕ *quantifier free* τ*-formulae. Then the unique expansions of* $\mathcal{M}, \mathcal{N}$ *to* τ_A*-models of* $T_{\tau,A}$ *for* $A = \{\langle\phi, 0\rangle\}$ *are still* τ_A*-substructures.*

Assume further that $\mathcal{M} \sqsubseteq \mathcal{N}$ *both satisfy*

$$\forall\vec{x}\exists! y\phi(\vec{x}, y).$$

Then the unique expansions of $\mathcal{M},\mathcal{N}$ to τ_B-models of $T_{\tau,B}$ for $B=\{\phi\}\times 2$ are still τ_B-substructures.

Proof. The first point is a basic argument left to the reader. The second point mimicks the argument from the first part of the proof of the next Lemma (and is also elementary). □

Lemma 28. *Let τ be a first order signature. Assume $\mathcal{M}\prec_1\mathcal{N}$ are τ-structures. Let $\phi(x_0,\dots,x_n)$ be a Σ_1-formula for τ, $A=\{\phi\}\times 2$, and:*

- *$\mathcal{N}_1$ be some extension of $\mathcal{N}$ to a τ_A-structures which models $T_{\tau,A}$ and interprets c_τ by an element of $\mathcal{M}$;*
- *$\mathcal{M}_1$ be the unique extension of $\mathcal{M}$ to a τ_A-structures which models $T_{\tau,A}$ and interprets c_τ the same way $\mathcal{N}_1$ does.*

Then it still holds that

$$\mathcal{M}_1\prec_1\mathcal{N}_1.$$

Proof. Let $\mathcal{M}^*$, $\mathcal{N}^*$ be the unique extensions of $\mathcal{M}$ and $\mathcal{N}$ to $\tau\cup\{R_\phi,c_\tau\}$-structures which interpret c_τ as $\mathcal{M}_1,\mathcal{N}_1$ do and interpret R_ϕ according to Ax^0_ϕ. Clearly

$$\mathcal{M}^*\prec_1\mathcal{N}^*.$$

Now we show that $\mathcal{M}_1\sqsubseteq\mathcal{N}_1$, e.g. we must show that $f_\phi^{\mathcal{M}_1}=f_\phi^{\mathcal{N}_1}\restriction\mathcal{M}$.

We can suppose that $\phi(x_0,\dots,x_n)$ is of the form $\exists\vec{z}\psi_\phi(x_0,\dots,x_n,\vec{z})$ with ψ_ϕ quantifier free. Note that $\exists! y\,\phi(y,x_1,\dots,x_n)$ is logically equivalent to the boolean combination of Π_1-formulae for τ

(2)
$$\exists y\exists\vec{z}\,\psi_\phi(y,x_1,\dots,x_n,\vec{z}))\wedge\forall u,v\forall\vec{z}\,\left[(\psi_\phi(u,x_1,\dots,x_n,\vec{z})\wedge\psi_\phi(v,x_1,\dots,x_n,\vec{z}))\to u=v\right].$$

Since $\mathcal{M}\prec_1\mathcal{N}$, we get that for any $\vec{b}\in\mathcal{M}^n$

$$\mathcal{M}\models\exists! y\,\phi(y,\vec{b})$$

if and only if

$$\mathcal{N}\models\exists! y\,\phi(y,\vec{b}).$$

Therefore for all $\vec{b}\in\mathcal{M}^n$ and $a\in\mathcal{M}$ we get that

$$\mathcal{M}_1\models f_\phi(\vec{b})=a$$

if and only if

$$\mathcal{N}_1\models f_\phi(\vec{b})=a;$$

hence $\mathcal{M}_1\sqsubseteq\mathcal{N}_1$.

Now we want to show that $\mathcal{M}_1\prec_1\mathcal{N}_1$. The key point is to analyze the complexity of the formula $y=t(x_1,\dots,x_n)$ for t a τ_A-term. We can prove the following:

Claim 3. *For any τ_A-term $t(x_1,\dots,x_n)$ in displayed variables, there are a Π_2-formula $\theta_t(x_1,\dots,x_n)$ and a Σ_2-formula $\psi_t(x_1,\dots,x_n)$ for $\tau\cup\{c_\tau\}$ such that*

$$T_{\tau,A}\models\forall x_1,\dots,x_n,y\,\left[\psi_t(y,x_1,\dots,x_n)\leftrightarrow t(x_1,\dots,x_n)=y\leftrightarrow\theta_t(y,x_1,\dots,x_n)\right]. \tag{3}$$

Assume the Claim holds, and notice that any existential τ_A-formula $\psi(x_1,\dots,x_n)$ is of the form

$$\exists x_{n+1},\dots,x_m\,\theta(t_1(x_1,\dots,x_m),\dots,t_k(x_1,\dots,x_m))$$

with θ a quantifer free τ-formula and $t_1,\dots,t_k$ τ_A-terms; by the Claim ψ is $T_{\tau,A}$-equivalent to the Π_2-formula for $\tau\cup\{c_\tau\}$

$$\forall y_1,\dots,y_k\left[(\exists x_{n+1},\dots,x_m\bigwedge_{i=1}^{k}\psi_{t_i}(y_i,x_1,\dots,x_m))\to\theta(y_1,\dots,y_k)\right].$$

This gives that for $b_1,\dots,b_n \in \mathcal{M}$ such that

$$\mathcal{N}_1 \models \exists x_{n+1},\dots,x_m\, \theta(t_1(b_1,\dots,b_n,x_{n+1},\dots,x_m),\dots,t_k(b_1,\dots,b_n,x_{n+1},\dots,x_m)),$$

we get that

$$\mathcal{N}^* \models \forall y_1,\dots,y_k \left[(\exists x_{n+1},\dots,x_m \bigwedge_{i=1}^{k} \psi_{t_i}(y_i,b_1,\dots,b_n,x_{n+1},\dots,x_m)) \to \theta(y_1,\dots,y_k)\right];$$

therefore (since $\mathcal{M}^* \prec_1 \mathcal{N}^*$, and the above is a Π_2-formula for $\tau \cup \{c_\tau\}$ in parameters $b_1,\dots,b_n$)

$$\mathcal{M}^* \models \forall y_1,\dots,y_k \left[(\exists x_{n+1},\dots,x_m \bigwedge_{i=1}^{k} \psi_{t_i}(y_i,b_1,\dots,b_n,x_{n+1},\dots,x_m)) \to \theta(y_1,\dots,y_k)\right].$$

Now observe that for all $i = 1,\dots,k$ by 3

$$\mathcal{M}_1 \models \forall x_{n+1},\dots,x_m\, [\psi_{t_i}(y_i,b_1,\dots,b_n,x_{n+1},\dots,x_m) \leftrightarrow t_i(b_1,\dots,b_n,x_{n+1},\dots,x_m) = y_i].$$

Therefore

$$\mathcal{M}_1 \models \exists x_{n+1},\dots,x_m\, \theta(t_1(b_1,\dots,b_n,x_{n+1},\dots,x_m),\dots,t_k(b_1,\dots,b_n,x_{n+1},\dots,x_m)).$$

We are done.

We are left with the proof of the Claim:

Proof. We proceed by induction on the depth of the τ_A-term t. If t is a term of depth 0, then t is a constant or a variable and there is almost nothing to prove (i.e. the unique term of depth 0 in $\tau_A \setminus \tau$ is c_τ; we can let ψ_t and θ_t be the formula $y = t$).

Now assume the Claim holds for all terms of depth n. Let

$$t = f(t_1(x_1,\dots,x_n),\dots,t_k(x_1,\dots,x_n))$$

be a term of depth $n+1$ with f a function symbol of τ_A. By inductive assumption there are $\theta_{t_j}(y_j,x_1,\dots,x_n)$ and $\psi_{t_j}(y_j,x_1,\dots,x_n)$ for $j = 1,\dots,k$ which are respectively Π_2 for $\tau \cup \{c_\tau\}$ and Σ_2 for $\tau \cup \{c_\tau\}$ such that:

$$(4)\quad T_{\tau,A} \models \forall x_1,\dots,x_n,y \left[\psi_{t_j}(y,x_1,\dots,x_n) \leftrightarrow t_j(x_1,\dots,x_n) = y \leftrightarrow \theta_{t_j}(y,x_1,\dots,x_n)\right].$$

This gives that $y = f(t_1(x_1,\dots,x_n),\dots,t_k(x_1,\dots,x_n))$ is $T_{\tau,A}$-equivalent to the Σ_2-formula for τ_A

$$(5)\qquad \psi_t^*(x_1,\dots,x_n) := \exists y_1,\dots,y_k \left[\bigwedge_{j=1}^{k} \psi_j(y_j,x_1,\dots,x_n) \wedge y = f(y_1,\dots,y_k)\right]$$

and to the Π_2-formula for τ_A

$$(6)\qquad \theta_t^*(x_1,\dots,x_n) := \forall y_1,\dots,y_k \left[\bigwedge_{j=1}^{k} \psi_j(y_j,x_1,\dots,x_n) \to y = f(y_1,\dots,y_k)\right].$$

If f is a function symbol of $\tau \cup \{c_\tau\}$ we let ψ_t be ψ_t^* and θ_t be θ_t^*. These are $\tau \cup \{c_\tau\}$-formulae, since $y = f(y_1,\dots,y_k)$ is already an atomic τ-formula, and all the other symbols occuring in 5 and 6 are also in $\tau \cup \{c_\tau\}$, and we easily get that 3 holds for $\psi_t := \psi_t^*, \theta_t := \theta_t^*$.

Else f is f_ϕ (therefore $k = n$) and we are considering the atomic τ_A-formula

$$y = f_\phi(y_1,\dots,y_n).$$

Now observe that $\exists! y\, \phi(y,x_1,\dots,x_n)$ is a boolean combination of Π_1-formulae for τ by 2.

Therefore

$$y = f_\phi(y_1,\dots,y_n)$$

is $T_{\tau,A}$-equivalent to the boolean combination of Π_1-formulae for $\tau \cup \{c_\tau\}$

$$[\exists! z\, \phi(z, y_1, \ldots, y_n) \wedge z = y)] \vee [\neg\exists! z\, \phi(z, y_1, \ldots, y_n) \wedge y = c_\tau]\,. \tag{7}$$

Also in this case we are done: replacing in 5 and 6 the τ_A-formula $f_\phi(x_1, \ldots, x_n) = y$ with the $\tau \cup \{c_\tau\}$-formula 7 does not change the complexity of the $\tau \cup \{c_\tau\}$-formulae so obtained. We let ψ_t and θ_t be the $\tau \cup \{c_\tau\}$-formulae obtained from ψ_t^* and θ_t^* by this substitution. A minimal variant of the argument given above shows that ψ_t and θ_t are $T_{\in,A}$-equivalent to $y = t(x_1, \ldots, x_n)$.

□

The Lemma is proved. □

9. Summing up

- We see model completeness, model companionship, AMC as tameness properties of elementary classes $\mathcal{E}$ defined by a theory T rather than of the theory T itself: these model-theoretic notions outline certain regularity patterns for the substructure relation on models of $\mathcal{E}$, patterns which may be unfolded only when passing to a signature distinct from the one in which $\mathcal{E}$ is first axiomatized (much the same way as it occurs for Birkhoff's characterization of algebraic varieties in terms of universal theories).
- Set theory together with large cardinal axioms has (until now unexpected) tameness properties when formalized in certain natural signatures (already implicitly considered in most of the prominent set theoretic results of the last decades). These tameness properties couple perfectly with well known (or at least published) generic absoluteness results. The notion of AMC-spectrum gives an additional model theoretic criterium for selecting these "natural" signatures out of the continuum many signatures which produce definable extensions of ZFC.

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