PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

LECTURE 3

TORINO 10/5/2022

~~Fa~~ <u>Def:</u> φ is ~~$T_{\forall\vee\exists}$~~ strongly $T_{\forall\vee\exists}$-consistent, if for all $R \supseteq T_{\forall\vee\exists}$ consistent

$R_{\forall\vee\exists} + \varphi$ is consistent.

<u>Note</u> if φ is a bool. comb of univ. sent

φ is str. $T_{\forall\vee\exists}$-cons. $\Rightarrow$ $\forall R \supseteq T$

$\varphi + R_{\forall\vee\exists}$ is cons. $\Rightarrow$ $\forall\, \mathfrak{M} \models R$ $\quad \mathfrak{M} \models \varphi \Rightarrow$

$T_{\forall\vee\exists} \models \varphi$

Fact Assume φ is Π_2 and is st. $T_{\forall\vee\exists}$-con.

Then $KH(T) \models \varphi$ e.g. if $\mathfrak{M}$ is T-ec ②

$\mathfrak{M} \models \varphi$.

Pf: Take $\mathfrak{M}$ which is T-ec. Let $R = Th(\mathfrak{M})$

$\cancel{\mathfrak{M} \models \varphi}$ $R_{\forall\vee\exists} \subsetneq (R_{\forall\vee\exists} + \varphi)_{\forall\vee\exists}$ ↯

$R_{\forall\vee\exists} = (R_{\forall\vee\exists} + \varphi)_{\forall\vee\exists}$

(A) We fixed a signature τ and a τ-theory and we proved (2)

<u>Lemma 1</u>: ~~If~~ TFAE

(i) $S_\forall \supseteq T_\forall$

(ii) Any τ-model $\mathcal{M} \models S$ embeds into $\mathcal{N} \models T$

~~Lemma 2~~ <u>Take</u> $\underbrace{(\mathbb{Q}, +, \cdot, 0, 1)}_{\substack{\text{Fields}_0}} \subseteq \underbrace{(\mathbb{C}, +, \cdot, 0, 1)}_{\substack{\text{Fields} \\ ACF_0}}$

$\text{Fields}_\forall = ACF_\forall \quad \tau = \{+, \cdot, 0, 1\}$

(B) <u>Lemma 2</u> Let T, S be τ-theories (3)

TFAE

(i) $T_{\forall\vee\exists} \supseteq S_{\forall\vee\exists}$

(ii) $\forall \mathfrak{M} \models T \quad \exists \mathfrak{N} \supseteq \mathfrak{M} \quad \mathfrak{N} \models \cancel{T}\, S$

and $\mathfrak{N}$ and $\mathfrak{M}$ agree on bool. comb. of univ. sent. $\quad \mathfrak{M} \equiv_1 \mathfrak{N}$

(4)

Def. $\mathfrak{M}$ is T-ec if

(i) $\exists \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{N} \models T$

(ii) $\forall \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{M} \prec_1 \mathfrak{N}$ $\Rightarrow$ $KH(T) = \{\psi : \psi \text{ is } \Pi_2, \mathfrak{M} \models \psi\}$

$\Downarrow$

$\mathfrak{M} \equiv_1 \mathfrak{N}$

If

$KH(T)_{\forall\exists} \overset{?}{\subseteq} T_{\forall\exists}$

$KH(T)_{\forall\exists} \supseteq T_{\forall\exists}$

is model complete

if $\mathfrak{M} \models KH(T) \Rightarrow \mathfrak{M}$ is T-ec

$\Downarrow$

$\exists \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{N} \models T$

$\mathfrak{M} \prec_1 \mathfrak{N} \Rightarrow \mathfrak{M} \equiv_1 \mathfrak{N}$.

The converse inclusion $KH(T)_{\forall\vee\exists} \subseteq T_{\forall\vee\exists}$ fails 5

in general: counterexample Fields and ACF

$$(\mathbb{C},+,\cdot,0,1) \sqsubseteq \underset{T}{\mathfrak{M}} \models \text{Fields}$$

$$\exists x\,(x^2+1=0) \notin \text{Fields}_{\forall\vee\exists}$$

$$KH(T)_\forall = T_\forall$$

$\mathfrak{M} \models T \Rightarrow \exists \mathfrak{N} \sqsupseteq \mathfrak{M}$ $\quad \mathfrak{N}$ is $T_\forall$-ec $\Rightarrow$ $\mathfrak{N} \models KH(T)$

$T_\forall \supseteq \boxed{KH(T)_\forall}$ $\quad$ if $\mathfrak{M}$ is T-ec $\quad \exists \mathfrak{N} \sqsupseteq \mathfrak{M}$ $\quad \mathfrak{N} \models T$ $\quad KH(T)_\forall \subseteq T_\forall$

(6)

(i) $KH(T)_\forall = T_\forall$ for all T

(ii) $KH(T)_{\forall\vee\exists} \supseteq T_{\forall\vee\exists}$ for all T

(iii) $KH(T) \supsetneq T_{\forall\vee\exists}$ for some T (e.g. T = fields in $\{+,\cdot,0,1\}$)

Lemma Assume $\mathfrak{M} \models T$. Then $\exists \mathfrak{N} \supseteq \mathfrak{M}$ 7

$\mathfrak{N}$ is T-ec. ~~and~~ ~~A~~

Proof we enumerate the q.f. $\forall$ formulae and the parameters and at each stage α we try to make appr. formula with appr. parameter true in $\mathfrak{M}_{\alpha+1} \supseteq \mathfrak{M}_{\alpha}$ if possible otherwise we let $\mathfrak{M}_{\alpha+1} = \mathfrak{M}_{\alpha}$ and al limit stages β

we survive as a model of $T_\forall$ because univ. theories are preserved by chains.

$$(\mathbb{N}, +, \cdot, 0, 1)^{\mathfrak{M}_0} \subseteq (\mathbb{Z}, +, \cdot, 0, 1)^{\mathfrak{M}_1} \subseteq (\mathbb{Q}, +, \cdot, 0, 1)^{\mathfrak{M}_2} \quad 8$$

$\not\models$ $\not\models$ $\not\models$

Fields$_\forall$ $\exists x\, (x + 1 = 0)$ $\exists x\, (n \cdot x = 1)$

$n \in \mathbb{N}$

sem $\parallel$

Theory of comm. semirings with no 0-div.

$\exists x\, (p(x) = q(x))$

$\subseteq \ldots\ldots$

$$\boxed{\forall x_0 \ldots x_n \forall z_0 \ldots z_k \exists y \sum_{i=0}^{n} x_i y^i = \sum_{j=0}^{k} z_j y^j} \subseteq (\overline{\mathbb{Q}}, +, \cdot, 0, 1)$$

$\cancel{\forall \vec{x}\, \exists y \sum}$ $\not\models$ ACF if $\cancel{\emptyset}\ \mathfrak{M} \equiv (\mathbb{Q}, +, \cdot, 0, 1)$ (9)

$\dfrac{\mathfrak{M}}{\text{Fields}_\forall}$ then $\mathfrak{M} \equiv_1 (\overline{\mathbb{Q}}, +, \cdot, 0, 1)$

$\cancel{\forall}\ T_{\forall\vee\exists}$ if $\mathfrak{M} \models T + KH(T)_{\forall\vee\exists}$

$\mathfrak{M}$ is a superstructure of a T-ec modell

Bounded forcing axioms

proof ~~goes~~ by starting with $\mathcal{M}_0 = (V, \in) \models ZFC$

and then pick $f_0(x, y, \omega_1)$ Δ_0-formula in parameter ω_1

and check if $\exists P_0 \in V$ s.t. $(V, \in) \models P_0$ is proper

and $(V, \in) \models P_0 \Vdash \exists x\, f_0(x, a_0, \omega_1)$ $a_0 \in H_{\omega_2}^V$ if so

let $\mathcal{M}_1 = V[G_0]$ with G_0 V-generic for P_0.

take $f_1(x, y, \omega_1)$ and check if $V[G_1] \models \exists P_2$ proper

$P_1 \Vdash \exists x\, f_1(x, a_1, \omega_1)$

$a_1 \in V[G_0]$

By it. thm for proper forcings (12)

$\mathbb{Q}_\omega = \lim\{(\mathbb{Q}_n, \dot{P}_n) : n \in \omega\}$ is proper
$\Longleftrightarrow$

$\mathbb{Q}_\lambda =$ proper limit of $\{(\mathbb{Q}_\alpha, \dot{P}_\alpha) : \alpha < \lambda\}$ is proper

in the limit δ refl we get
$H_{\omega_2}^{V[G_\delta]} \prec_1 V[G_\delta][H]$ whenever H is $V[G_\delta]$-generic for a proper forcing.

(1) Question (1): suppose we start from 12
$(V, \in_{\Delta_0}, \omega_1)$ and force with a proper forcing P with G V-generic for P

$$(V, \in_{\Delta_0}, \omega_1) \equiv_1 (V[G], \in_{\Delta_0}, \omega_1)$$

$\forall$ ω_1 is first unc. In general no

for example $(L, \in_{\Delta_0}, \omega_1^L) \not\equiv_1 (L[G], \in_{\Delta_0}, \omega_1^{L[G]})$

If G is V-generic for Cohen forcing

$\omega_1 = \omega_1^L$ $\forall x \, (x \subseteq \omega \rightarrow L(x))$

(ii) if LC exists then $(V, \in_{\Delta_0}, \omega_1^V, NS_{\omega_2}^V) \equiv_1$ (13)

$(V[G], \in_{\Delta_0}, \omega_1^{V[G]}, NS_{\omega_2}^{V[G]})$

for any forcing $P \in V$ and G V-generic

$\in_{\Delta_0}$ includes Relation symbols for Δ_0-formulae and function symbols for Goedel operations.

(14)

<u>Fact</u> ~~iff~~ $\forall$ for any Π_2-sent. φ

φ is strongly $T_{\forall\vee\exists}$ iff $KH(T) \models \varphi$

<u>Question 1</u> when is $KH(T)$ model complete

$$\{\mathcal{M} : \mathcal{M} \models KH(T)\} = \{\mathcal{M} : \mathcal{M} \text{ is } T\text{-ec}\}$$

<u>Question 2</u>: when is $\boxed{KH(T)_{\forall\vee\exists} = T_{\forall\vee\exists}}$

$\supseteq$ holds always

$\subseteq$? depends

Model completeness

Def. ~~∀~~ A τ-theory T is model complete iff $\forall \underset{\models T}{\mathcal{M}} \subseteq \underset{\models T}{\mathcal{N}} \Rightarrow \mathcal{M} \prec \mathcal{N}$.

Cor If T admits Q.e. T is model complete

$n=1$ is (c)

$n=2$ Let $\exists \vec{z}\, \forall \vec{y}\, \varphi(\vec{x}, \vec{y}, \vec{z})$ with φ q.f.

$\forall \vec{y}\, \varphi(\vec{x}, \vec{y}, \vec{z})$ by (c) is T-equivalent to

$\exists \vec{w}\, \theta(\vec{x}, \vec{w}, \vec{z})$ $\quad \exists \vec{z}\, \forall \vec{y}\, \varphi(\vec{x}, \vec{y}, \vec{z}) \equiv_T \exists \vec{z} \exists \vec{w}\, \theta(\vec{x}, \vec{w}, \vec{z})$

q.f.

$\square$

(d) $\Rightarrow$ (a): by (d) every formula is T-equiv to a univ. formula and to an exist. formula.

$T \cup \mathcal{M} \quad T_{\mathcal{N}}$

$\mathcal{M} \subseteq \mathcal{N} \Rightarrow \mathcal{M} \prec \mathcal{N}$

$\square$

Robinson's test

Let T be a τ-theory TFAE:

(a) T is model complete

(b) any model of T is T-ec (or $T_\forall$-ec)

(c) any existential τ-formula $\phi(\vec{x})$ is T-equiv. to a univ. formula $\psi(\vec{x})$

(d) any τ-formula $\phi(\vec{x})$ is T-equiv to a univ. formula $\psi(\vec{x})$.

(c) $\Rightarrow$ (d): by induction on n we show that a Σ_n-formula is equiv. to a univ. formula.

(a) ⇒ (b) if $\mathfrak{M} \models T$ and $\mathfrak{N} \supseteq \mathfrak{M}$, $\mathfrak{N} \models T$

trivial ☒

(b) ⇒ (c): $F(x) = \exists \vec{y}\, \phi(\vec{x}, \vec{y})$ with ϕ q.f.

$\Gamma(\vec{x}) = \{ \underbrace{\forall \vec{z}\, \psi(\vec{x}, \vec{z})}_{\theta_\psi(\vec{x})}$: ψ is q.f. and

$T, \exists \vec{y}\, \phi(\vec{x}, \vec{y}) \vdash \forall \vec{z}\, \psi(\vec{x}, \vec{z}) \}$

$\Gamma(\vec{c}) = \{ \theta_\psi(\vec{c})$; with $c_1 \cdots c_n$ fresh constants for x

We can assume $\exists y \phi(\vec{x}, \vec{y}) + T$ cons. 38

otw. $\exists \vec{y}\, \phi(\vec{x}, \vec{y}) \equiv_T \forall x (x \neq x)$

$\Downarrow$

$T \cup \Gamma(\vec{c})$ is consistent as it holds

in $\mathfrak{M} \models T + \exists \vec{y}\, \phi(\vec{x}, \vec{y})$

if $(\mathfrak{M}, c_1^{\mathfrak{M}} \dots c_n^{\mathfrak{M}}) \models \cancel{\phi(x} \exists \vec{y}\, \phi(\vec{c}, \vec{y})$

$(\mathfrak{M}, c_1^{\mathfrak{M}} \dots c_n^{\mathfrak{M}}) \models \theta_\psi(\vec{c})$ for all $\theta_\psi \in \Gamma(\vec{x})$

$T \cup \Gamma(\vec{c}) \models \exists \vec{y}\, \phi(\vec{c}, \vec{y})$ Key claim (19)

$\mathcal{M} \models T \cup \Gamma(\vec{c})$

$\boxed{\text{second key claim: } T \cup \Delta_0(\mathcal{M}) \cup \{\phi(\vec{c})\} \text{ is consistent}}$

second claim $\Rightarrow$ Key claim.

~~suppose~~ second claim

$\mathcal{M} \models T \cup \Delta_0(\mathcal{M}) \cup \{\exists \phi(\vec{c}, \vec{y})\}$

$T \ni \mathcal{M} \wedge \mathcal{C} \underset{\Sigma_1}{\supseteq} \mathcal{M} \models \mathcal{C}$

$\mathcal{M} \wedge \mathcal{C} \models \exists \vec{y}\, \phi(c_1^{\mathcal{M}} \dots c_n^{\mathcal{M}}, \vec{y})$

$\mathcal{M} \models \exists \vec{y}\, \phi(c_1^{\mathcal{M}} \dots c_n^{\mathcal{M}}, \vec{y})$

$\mathcal{M} \wedge \mathcal{C} \models \exists \cancel{\vec{y}}\, \phi(\vec{z})$