PHD COURSE ON

MODEL COMPANIONSHIP RESULTS

FOR SET THEORY

LECTURE 4

TORINO 11/5/2022

By embeddability Lemmas: (1)

Lemma 1: Assume T, S are τ-theories

TFAE:

(i) $T_\forall \supseteq S_\forall = \{\varphi : \varphi \text{ is univ. } S \models \varphi\}$

(ii) $\forall \mathcal{M} \models T \; \exists \mathcal{N} \models S \quad \mathcal{N} \supseteq \mathcal{M}$

Lemma 2 Assume T, S are τ-theories. TFAE

(i) $T_{\forall\vee\exists} \supseteq S_{\forall\vee\exists} = \{\varphi : \varphi \text{ is a bool. comb. of univ-sent. and } S \models \varphi\}$

(ii) $\forall \mathcal{M} \models T \; \exists \mathcal{N} \models S$ s.t. $\mathcal{N} \supseteq \mathcal{M}$ and $\mathcal{N} \equiv_1 \mathcal{M}$

<u>Def</u>: $\mathfrak{M}$ is T-ec if

(i) $\exists\, \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{N} \models T$

(ii) $\forall\, \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{N} \models T \Rightarrow \mathfrak{M} \preceq_1 \mathfrak{N}$

<u>Lemma</u> $\forall\, \mathfrak{N} \models T$ $\exists\, \mathfrak{M} \supseteq \mathfrak{N}$ s.t.
$\mathfrak{M}$ is T-ec.

<u>Lemma</u> $\cancel{\forall}$ TFAE for $\mathfrak{M}$ a τ-structure

(i) $\mathfrak{M}$ is T-ec

(ii) $\mathfrak{M}$ is $T_{\forall\exists}$-ec

(iii) $\mathfrak{M}$ is $T_{\forall}$-ec

Def ~~φ~~ A τ-sentence φ is strongly $T_{\forall\vee\exists}$-cons. (3)
if $\forall R \supseteq T_{\forall\vee\exists}$ $\exists\ \mathfrak{M} \models \varphi + R$

Def Given a τ-theory T

- $KH(T) = \{\varphi : \varphi \text{ is } \Pi_2 \text{ and } \mathfrak{M} \models \varphi \text{ whenever } \mathfrak{M} \text{ is } T\text{-ec}\}$
- $SCH(T) = \{\varphi : \varphi \text{ is } \Pi_2 \text{ and } \varphi \text{ is strongly } T_{\forall\vee\exists}\text{-cons.}\}$

Fact $SCH(T) \subseteq KH(T)$

- $KH(T)_\forall = T_\forall$ $\qquad SCH(T)_{\forall\vee\exists} = T_{\forall\vee\exists}$
- $KH(T)_{\forall\vee\exists} \supseteq T_{\forall\vee\exists}$ $\qquad$ When is $KH(T)_{\forall\vee\exists} = SCH(T)_{\forall\vee\exists} = T_{\forall\vee\exists}$

Model completeness (4)

<u>Def</u>: A τ-theory T is model complete if whenever $\begin{array}{ccc}\mathfrak{M} & \sqsubseteq & \mathfrak{N}\\ \shortparallel & & \shortparallel \\ \vdash & & \vdash \\ T & & T\end{array}$ then $\mathfrak{M} \preceq \mathfrak{N}$.
($\preceq_1$)

<u>Thm.</u> (Robinson's test) TFAE for a τ-theory T:

(i) T is model-complete

(ii) $Mod_T = \{\mathfrak{M} : \mathfrak{M} \text{ is } T\text{-ec}\}$ ($\Rightarrow KH(T) = T$)

(iii) every exist. τ-formula is equiv. to a univ. τ-form

(iv) // X — — — — —

(ii) $\Rightarrow$ (iii): Assume $\phi(\vec{x})$ is Σ_1 for τ (5)

we look for $\psi(\vec{x})$ univ. s.t.

$T \models \forall \vec{x} \left(\phi(\vec{x}) \leftrightarrow \psi(\vec{x}) \right)$.

$\Gamma(\vec{x}) = \{ \psi(\vec{x}) : \psi$ is universal and $T \vdash \phi(\vec{x}) \to \psi(\vec{x})$,

$T, \phi(\vec{x}) \vdash \psi(\vec{x})$

we add constants $c_1 \ldots c_n$ not in τ

and consider

$T \cup \Gamma(\vec{c}) = \{ \psi(\vec{c}) : \psi \in \Gamma(\vec{x}) \}$ which is a $\tau \cup \{c_1 \ldots c_n\}$ theory if $\mathcal{M} \models T + \phi(\vec{x})[x_i / a_i : i \in \{1..n\}]$ $a_1 \dots a_n \in \mathcal{M}$ 6

$(\mathcal{M}, a_1 \dots a_n) \models T + \phi(\vec{c})$

$\Uparrow$

$T + \Gamma(\vec{c})$

Key claim $\boxed{T + \Gamma(\vec{c}) \vdash \phi(\vec{c})}$!

By compactness exists $\theta_1 \dots \theta_n \in \Gamma(\vec{x})$ s.t.

$T + \theta_1(\vec{x}), \dots \theta_n(\vec{x}) \vdash \phi(\vec{x})$ and $\theta_1 \wedge \dots \theta_n$ is one and in $\Gamma(\vec{x})$

$T \vdash \forall \vec{x} \left(\bigwedge_{i=1}^{n} \theta_i(\vec{x}) \to \phi(\vec{x}) \right)$ (a)

If $\mathfrak{M}$ is a $\tau \cup \{c_1 \dots c_n\}$-theory ⑦

and $\mathfrak{M} \models T + \Gamma(\vec{c})$:

then $T \cup \Delta_0^{\tau \cup \{c_1 \dots c_n\}}(\mathfrak{M}) \cup \{\phi(\vec{c})\}$ is consistent

if $\mathfrak{M}$ is a $\tau \cup \{c_1 \dots c_n\}$-theory and

$\mathfrak{M} \models T \cup \Delta_0^{\tau \cup \{c_1 \dots c_n\}}(\mathfrak{M}) \cup \{\phi(\vec{c})\}$

$\underset{T}{\underset{\parallel}{\underline{\mathfrak{M} \restriction \tau}}} \ \underset{\prec_1}{\sqsubseteq} \ \underset{\phi(c_1^{\mathfrak{M}} \dots c_n^{\mathfrak{M}})}{\underset{\parallel}{\underline{\mathfrak{M} \restriction \tau}}} \models T$ letting $a_i = c_i^{\mathfrak{M}}$

Claim If $\mathcal{M}$ is a $\tau \cup \{c_1 .. c_n\}$-model of 8

$T + \Gamma(\vec{c})$ then $T \cup \Delta_0(\mathcal{M}) \cup \{\phi(\vec{c})\}$ is consistent.

Take $\psi(\vec{x}, \vec{y})$ q.f. formula s.t.

$\psi(\vec{c}, \vec{a}) \in \Delta_0(\mathcal{M})$ with $\vec{a} \in M^{<\omega}$

I must find a model of $T + \psi(\vec{c}, \vec{a}) + \phi(\vec{c})$

$\mathcal{M} \models \exists \vec{y}\, \psi(\vec{c}, \vec{y}) + \Gamma(\vec{c}) + T \Rightarrow$

$T + \phi(\underset{\vec{c}}{\vec{x}}) \nvdash \neg \exists \vec{y}\, \psi(\underset{\vec{c}}{\vec{x}}, \vec{y})$

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(4) $T + \phi(\vec{c}) + \exists \vec{y}\, \psi(\vec{c}, \vec{y})$ is coherent

$T + \phi(\vec{c}) + \psi(\vec{c}, \vec{a})$ is coherent.

$T + \phi(\vec{c}) + \Delta_0(\mathcal{M})$ is coherent

$$\Downarrow$$

$T + \phi(\vec{c}) + \Gamma(\vec{c}) + \Delta_0(\mathcal{M})$

Robinson's Test   TFAE for a τ-theory T (10)

(i) T is model complete $\left(\underset{\models T}{\mathcal{M}} \subseteq \underset{\models T}{\mathcal{N}} \Rightarrow \mathcal{M} \prec \mathcal{N} \right)$

(ii) $Mod_T = \{ \mathcal{M} : \mathcal{M} \text{ is } T\text{-ec} \}$

(iii) every exist. formula is T-eq to a univ. formula.

Corollary: (i) $\Rightarrow$ T admits QE $\Rightarrow$ T is model complete

(iii) $\Rightarrow$ T is axiomatized by the Π_2-fragment:

given $\exists \vec{y}\, \varphi(\vec{x}, \vec{y})$ with φ q.f. let $\theta_\varphi(\vec{x}, \vec{z})$ be q.f.

s.t. $T \vdash \underbrace{\forall \vec{x} \left(\exists \vec{y}\, \varphi(\vec{x}, \vec{y}) \leftrightarrow \forall \vec{z}\, \theta_\varphi(\vec{x}, \vec{z}) \right)}_{\Pi_2}$

if $S = T_\forall \cup \{ \forall \vec{x} \, (\varphi(\vec{x}, \vec{y}) \leftrightarrow \underbrace{\forall \vec{z} \, \delta_\varphi(\vec{x}, \vec{z})}_{T_\forall}) : \varphi \text{ is q.f.} \}$

Thm S is model complete and $S \subseteq T$

$$S_\forall \supseteq T_\forall \qquad S \subseteq T$$

$$S_\forall = T_\forall$$

$$S = KH(S) = KH(S_\forall) = KH(T_\forall) = KH(T) =$$

Model companionship

Def Let T, S be τ-theories.

(i) T and S are cotheories if $T_\forall = S_\forall$

(ii) T and S are absolute coth. if $T_{\forall\vee\exists} = S_{\forall\vee\exists}$

(a) T is the model companion (MC) of S if $T_\forall = S_\forall$ and T is model complete

(b) T is the absolute MC (AMC) of S if $T_{\forall\vee\exists} = S_{\forall\vee\exists}$ and T is model complete

<u>Thm</u> If S admits an MC its MC is unique.

<u>Pf</u> If MC of S is T we have

$$T = KH(T) = KH(T_\forall) = KH(S_\forall) \qquad \square$$

~~Thm [illegible] AMC~~

<u>Thm</u> S admits an AMC iff $KH(S)$ is model complete and is the $SCH(S)$.

Pf $KH(S)$ is the MC which belongs AMC (14) $\Rightarrow$

$$\underbrace{KH(S)_{\forall\vee\exists} = S_{\forall\vee\exists}} = SCH(S)_{\forall\vee\exists}$$

$$KH(S) \supseteq SCH(S)$$

Pf is Ψ_2 and belongs to $KH(S)$ $\Rightarrow$

for any $R \supseteq S_{\forall\vee\exists}$ if $\mathfrak{M} \models R$ $\exists \mathfrak{N} \supseteq \mathfrak{M}$

$\mathfrak{N} \models KH(S)$ and $\mathfrak{M} \equiv_2 \mathfrak{N}$ $\quad \mathfrak{N} \models R_{\forall\vee\exists} + \Psi$

$$KH(S) \subseteq SCH(S)$$

Fact: if S is complete and S admits a model companion T, ~~the~~ T is the $SCH(S)$ and the AMC of S.

Example: Fields_0 has as MC ACF_0 but not as its AMC because $(\mathrm{Fields}_0)_{\forall\vee\exists} \neq (ACF_0)_{\forall\vee\exists}$

Example: $Th(\mathbb{Z},<)$ has as AMC $Th(\mathbb{Q},<)$

Lemma: Assume T is the AMC of S.

Then for any $R \supseteq S$, $T + R_{\forall\exists}$ is the AMC of R

Counterexample for MC of the Lemma

$R = Th(\mathbb{Q}, +, \cdot, 0, 1)$ $\quad R_\forall + ACF_0$ is mc.

Proof of Lemma: $T + R_{\forall\vee\exists}$ is model complete

$\boxed{T_{\forall\vee\exists} = S_{\forall\vee\exists}}$

$\boxed{(T + R_{\forall\exists})_{\forall\vee\exists}} = R_{\forall\vee\exists}$

take $\mathcal{M}_0 \models R \supseteq S$ then $\exists \mathcal{N}_0 \supseteq \mathcal{M}_0$ which is a model of $T + R_{\forall\vee\exists}$

and conversely if $\mathcal{M}_1 \models T + R_{\forall\exists} \supseteq T \Rightarrow \exists \mathcal{N}_1 \supseteq \mathcal{M}_1$, $\mathcal{N}_1 \models S$ and $\mathcal{N}_1 \succeq_1 \mathcal{M}_1$

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<u>Obs</u>: if T is model complete then T is the AMC of S # for any S s.t

$$S_{\forall\vee\exists} = \underbrace{T_{\forall\vee\exists}}$$

and the MC of S for any S s.t.

$$S_{\forall} = \underbrace{T_{\forall}}$$

the question is whether $T_{\forall}$ conveys really less inf. than $T_{\forall\vee\exists}$ Key point is whether there $S \supseteq T_{\forall}$ and s.t $S + T_{\forall\vee\exists}$ is inc.

(i) The notion of strong $T_{\forall\exists}$-cons identifier for Π_2-sentences cons. with realizability in T-ec models

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(ii) we argued that ~~univ~~ $T_{\forall}^{\tau}$ detects basic properties of the simple concepts of T according to τ.

(iii) T-ec models for τ detect a "closed off" as possible structures which maintain the basic truths of T according to τ i.e $T_{\forall}^{\tau}$.

What we do is the following (19)

(i) vary τ and find criteria to detect which τ are expressing "simple concepts of the theory"

(ii) Assume we can argue that τ expresses with q.f. formulae simple concepts of T. T in signature τ admits an (A)MC iff it does the (A)MC of T is as "closed off as possible" w. r. to the operations given by τ and $T_\tau^\forall$.

Def: Given ~~$A \subseteq \mathrm{Form}_\tau$ &~~ $\varphi(\vec{x})$ a τ-formula

$$Ax^0_\varphi := \forall \vec{x}\,\big(\varphi(\vec{x}) \longleftrightarrow R_\varphi(\vec{x})\big)$$ in signature $\tau \cup \{R_\varphi\}$

Given $\psi(\vec{x}, y)$

$$Ax^1_\psi := \forall \vec{x}\Big[\big(\exists!\, y\, \psi(\vec{x}, y) \wedge f_\psi(\vec{x}) = y\big) \vee \big(\neg \exists!\, y\, \psi(\vec{x}, y) \wedge f_\psi(\vec{x}) = c_\tau\big)\Big]$$

in signature $\tau \cup \{f_\psi, c_\tau\}$

Given $A \subseteq Form_\tau \times 2$

$$\tau_A = \tau \cup \{R_\phi : (\phi, 0) \in A\} \cup \{f_\varphi : (\varphi, 1) \in A\} \boxed{\cup \{c_\tau\}}$$

$\to$ only $\exists (\varphi, 2) \in A$

~~It is clear~~ that

$$T_{\tau, A} = \{Ax^i_\varphi : (\varphi, i) \in A\}$$

clear that if $\cancel{\mathfrak{M} \vdash}$ $\mathfrak{M}$ is a τ-structure and $d \in \mathfrak{M}$ there is uniquen expansion of $\mathfrak{M}$ to a τ_A-structure which interprets c_τ as d and models $T_{\tau, A}$

So if T is a mathematical theory axiomatized in signature τ the

$\{\tau_A : A \subseteq Form_\tau \times 2\}$ ~~de~~ and $T + T_{\tau,A}$ describes all possible axiomatizations of T in signatures extending τ.

T groups $\quad \tau = \{\cdot, e, {}^{-1}\}$

$\sigma = R$ for R ternary

$\tau_A = \cancel{\tau \cup} \sigma$

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<u>Def</u>: $\mathrm{spec}_{MC}(T,\tau) = \{ A \subseteq \mathrm{Form}_\tau \times 2 : T + T_{\tau,A}$ admits ~~an~~ ~~AMC~~ AMC $\}$

$\mathrm{spec}_{AMC}(T,\tau) = \ldots\ldots$

Goal of remainder of the course:

study $\mathrm{spec}_{(A)MC}(T, \in_{\Delta_0})$ for $T \supseteq ZFC$

to be specified

(add

Thm (Hirschorn) ZF in signature $\in$ has an AMC which (modulo slight perturbations) is the theory of dense linear orders w. end points

$$\underbrace{\langle x, y \rangle = z}$$

$$z = \{\{x\}, \{x, y\}\}$$

$\exists t$

Def.: $\in_{\Delta_0}$ is enriching $\in$ w..

R_φ for φ Δ_0

f_ψ for ψ the Δ_0-formula for a rudimentary function

$\emptyset$ for the emptyset and ω for natural numbers

Clear that $\in_{\Delta_0} = \{\in\}_A$ for $A \subseteq \mathrm{Form}_{\{\in\}} \times 2$

and from now $\in_{\Delta_0}$ is the basic signature over which we look fo other signatures

~~ZFC_{Δ_0}~~ is Z_{Δ_0} is $\forall x \neg(x \in \phi)$

$\forall x (x \in w \to x = \phi \lor \exists y \in w \; y+1 = x)$

Ext. (Universal $\in_{\Delta_0}$-sent.)

Found

$$\forall \vec{x} \left(R_\varphi(\vec{x}) \land R_\psi(\vec{x}) \iff R_{\varphi \land \psi}(\vec{x}) \right)$$

$$\forall \vec{x}, z \left(R_{\forall y \in z \varphi}(\vec{x}, z) \iff \forall y \left(y \in z \to R_\varphi(y, \vec{x}, z) \right) \right)$$

$$\forall \vec{x} \left(\neg R_\varphi(\vec{x}) \iff R_{\neg\varphi}(\vec{x}) \right)$$

$$\forall \vec{x} y \left(f_\psi(\vec{x}) = y \iff R_\psi(\vec{x}, y) \right)$$ for ψ the Δ_0-formula of 1 rud. function