PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

LECTURE 5

TORINO 17/5/2022

<u>Def</u>: Given τ and φ, ψ R-formula

$$Ax^0_\varphi := \forall \vec{x} \left(\varphi(\vec{x}) \leftrightarrow R_\varphi(\vec{x}) \right)$$

$$Ax^1_\psi := \forall \vec{x} \left[\left(\exists! y\, \psi(\vec{x}, y) \wedge f_\psi(\vec{x}) = y \right) \vee \left(\neg \exists! y\, \psi(\vec{x}, y) \wedge f_\psi(\vec{x}) = c_\tau \right) \right]$$

for $A \subseteq Form_\tau \times 2$

$$T_{\tau,A} = \{ Ax^i_\varphi : (\varphi, i) \in A \} \qquad \tau_A = \{ R_\varphi : (\varphi, 0) \in A \} \cup \{ f_\psi : (\psi, 1) \in A \}$$

$\mathrm{Spec}_{MC}(T) = \{A \subseteq \mathrm{Form}_\tau \times 2 : T + T_{\tau,A}$ has an MC$\}$ 2

$\mathrm{Spec}_{AMC}(T) = \{\ \ldots \qquad$ AMC$\}$

$\in_{\Delta_0} = \in \cup \{R_\varphi : \varphi\ \Delta_0\text{-formula}\} \cup \{f_\varphi : \varphi\ \Delta_0\text{-formula}$ which is the graph of a Goedel operation $\cup \{\emptyset, \omega\}\}$

$\boxed{\text{$x = \langle y, z\rangle$ is Σ_2 in arithmetical hier. on $V^{\{\in\}}$ and atomic on $\in_{\Delta_0}$}}$

<u>Goal</u>: study $\mathrm{Spec}_{AMC}(T)$ for $T \supseteq ZFC$

and looking just at A s.t. $\{\in\}_A \supseteq \in_{\Delta_0}$
$A \subseteq \mathrm{Form}_\in \times 2$

$\boxed{Z^{-}_{\Delta_0} :}$

- Ext
- Found

} those are univ sen $\in_{\Delta_0}$

$$\forall x \forall y (\underbrace{x \subseteq y \wedge y \subseteq x \to y = x}_{\Delta_0})$$

$$\forall x (x \neq \emptyset \to \exists y \in x \, \forall z \in x \; z \notin y)^{\Delta_0}$$

- $\forall \vec{x} y (R_\varphi(\vec{x}, y) \longleftrightarrow f_\varphi(\vec{x}) = y)$ for φ the Δ_0-formula of a Goedel operations

$$\forall \vec{x} y \left(R_{\forall z \in y \psi}(\vec{x}, y) \Longleftrightarrow \forall z \, (z \in y \to R_\psi(z, \vec{x}, y) \right) \quad (4)$$

$$\forall \vec{x} \left(R_{\psi \wedge \varphi}(\vec{x}) \Longleftrightarrow R_\psi(\vec{x}) \wedge R_\varphi(\vec{x}) \right)$$

is M_2 for $\in_{\Delta_0}$

$$\forall \vec{x} \left(\neg R_\psi(\vec{x}) \Longleftrightarrow R_{\neg\psi}(\vec{x}) \right)$$

Z_{Δ_0} is $Z^-_{\Delta_0}$ + Powerset + Comprehension for $\in_{\Delta_0}$-formulae

$ZF^-_{\Delta_0}$ is Z_{Δ_0} + Replacement for Δ_0-formulae

ZC_{Δ_0} is Z_{Δ_0} + AC $ZFC^-_{\Delta_0}$ is $ZF^-_{\Delta_0}$ + AC

ZFC_{Δ_0} is $ZFC^-_{\Delta_0}$ + Powerset.

If $\tau \supseteq \epsilon_{\Delta_0}$ We can define accordingly ZF_τ ZFC_τ ZF^-_τ ZC^-_τ ...

$$x \text{ is } \aleph_1 := (\underbrace{x \text{ is an Ordinal}}_{\Delta_0}) \wedge \underbrace{\omega \in x}_{\Delta_0} \wedge$$

$$\wedge \underbrace{\exists F : \omega \times x \to x \text{ s.t. } \forall \alpha \in x \ F \restriction \omega \times \{\alpha\} \to \alpha \text{ is surjective}}_{\Sigma_1 \text{ for } \epsilon_{\Delta_0}}$$

$$\wedge \underbrace{\forall f \, (\mathrm{dom}(f) = \omega \wedge f \text{ is a function} \to \mathrm{ran}(f) \neq x)}_{\Pi_1}$$

x is $\aleph_2$ ... $\Pi_2 \wedge \Sigma_2$

$2^{\aleph_0} = \aleph_1$ is Σ_2 := $\exists F\,(\overbrace{\mathrm{dom}(F) = \aleph_1 \wedge}^{\Pi_2 \wedge \Sigma_2}$

F is a function $\wedge$

$\forall z\,(z \subseteq \omega \to z \in \mathrm{ran}(F)))$

$2^{\aleph_0} > \aleph_1$ is $\Pi_2 \wedge \Sigma_2$:= $\exists x\,(x \text{ is } \aleph_1) \wedge \neg\, 2^{\aleph_0} = \aleph_1$

$2^{\aleph_0} \leq \aleph_2$ is Σ_2

$2^{\aleph_0} > \aleph_2$ is $\Pi_2 \wedge \Sigma_2$

Thm Let $(M, \in^{M}) \models ZFC$

κ be a cardinal in M and

A

$$\left(H_{\kappa^+}^{M}, \in_{\Delta_0}^{M}, \boxed{A_1 \dots A_k}, f_{\psi_1}^{M} \dots f_{\psi_\ell}^{M}\right) \prec_1 \left(M, \in_{\Delta_0}, A_1 \dots A_k, f_{\psi_1}^{M} \dots f_{\psi_\ell}^{M}\right)$$

if each $A_i \subseteq (P(\kappa))^{M}$ $A_i \in M$

and ψ_i is provably $\Delta_1(ZFC^-)$ and such that

$$ZFC^- \vdash \forall \vec{x}\, \exists! y\, \psi_i(\vec{x}, y)$$

Thm 2 Assume $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$ and $T \supseteq ZFC + \kappa$ is a cardinal 8

$\forall \mathcal{M} \models \cancel{ZFC}\ T \quad \left(H^{\mathcal{M}}_{\kappa^+}, \tau^{\mathcal{M}}\right) \prec_1 \mathcal{M}$

(Note that all signatures of Thm 1 work for assumption of Thm 2)

Then if $MC(T,\tau)$ exists

$\forall x\, \exists F\, (F : \kappa \to x \text{ is surjective}) \in MC(T,\tau)$

and $MC(T,\tau) \supseteq ZFC^{-}_{\tau}$

<u>Thm 3</u> Assume $\tau \geq \epsilon_{\Delta_0}$ and $T \supseteq ZFC_\tau$ [redacted] and $MC(T,\tau)$ exists.

[redacted]

Then $MC(T,\tau) \supseteq ZFC^-_{\tau}$.

[redacted]

<u>Thm 4</u> [redacted]
Assume $T \supseteq ZFC_{\Delta_0} + K$ is a cardinal in signature $\epsilon_{\Delta_0} \cup \{K\}$. Then there is plenty of $A \subseteq Form_{\epsilon_{\Delta_0} \cup \{K\}} \times 2$

s.t. $T \nvdash A \in Spec_{AMC}(T)$ and

$AMC(T, A) \models ZFC^-_{\Delta_0} + \forall x \exists F (F : K \to x$ is a surjection)

(i.e $AMC(T,A)$ is a theory of H_{K^+}).

Thm 5 Assume $T \supseteq ZFC$[redacted]. Then

(i) [redacted] if $T + \neg CH$ is consistent.
Then $CH \notin KH(T)$

(ii) if $T + ($a special version of $2^{\aleph_0} \leq \aleph_2)$ is consistent
Then $\underbrace{2^{\aleph_0} \geq \aleph_2} \notin KH(T)$

(iii) there is (plenty ~~and~~) at least one recursive
$B \subseteq Form_{\in_{\Delta_0}} \times 2$ s.t. for any $T \supseteq ZFC_{\Delta_0} + LC$
we get that $AMC(T, B)$ exists and
it is the theory of $H_{\aleph_2}$ in models of MM^{++}
or strong versions of $(*)$ for example
$AMC(T,B) = \{\psi : \psi$ is Π_2 for [redacted]$(\in_{\Delta_0})_B$ and $T \vdash \exists P \; P \Vdash \psi^{H_{\omega_2}}$

<u>Thm 6</u> ~~for the~~ For the signature

$\tau_J \in_{\Delta_1} \cup \{\omega_1\} \cup \{NS_{\omega_1}\}$ (if one wants also universally Baire sets can be added as predicates)

$\in_B$

$= \in_{\Delta_0} \cup \{R_\varphi : \varphi \text{ is } \Delta_1(ZFC^-)\} \cup \{f_\varphi : \varphi \text{ is } \Delta_1(ZFC^-)$ and $ZFC^- \vdash \forall \vec{x} \exists! y\, \varphi(\vec{x}, y)\}$

TFAE for $(V, \in) \models ZFC + LC$ and

θ a universal sent of τ:

~~(i)~~

(i) $(V, \in) \models \theta$

(ii) $(V, \in) \models \exists P \; P \Vdash \theta$

(iii) $(V, \in) \models \forall P \; P \Vdash \theta$

hence for ψ a Π_2-sentence for τ

$T \vdash \exists P \; P \Vdash \psi^{(H_{\omega_2})} \Rightarrow$

[redacted] ψ is strongly ~~$\forall\vee\exists$ consistent~~

$T + (Ax^i_\psi : (\psi, i) \in B)$ $_{\forall\vee\exists}$-consistent.

<u>Thm</u> There exists (many, but) at least one (12)

(recursive) set $B \subseteq Form_{\{\in\}} \times 2$

$$\{\in\}_B \supseteq \in_{\Delta_1} \cup \{\omega_1\} \cup \{NS_{\omega_1}\} \cup \boxed{\left\{ R_{\psi^{L(Ord^\omega)}} : \psi \ \in\text{-formula},\ ZFC \vdash \forall x\, \psi^{L(Ord^\omega)}(x) \rightarrow x \subseteq \omega \right\}}$$

s.t. TFAE for any $T \supseteq ZFC + LC$ and ψ $\cancel{\in AMC(T,B)}$ Π_2 for $\{\in\}_B$

(i) $\psi \in AMC(T,B)$

(ii) $T \vdash \exists P\ P \Vdash \psi^{H_{\omega_2}}$

(iii) $(T_{\cancel{\forall\vee\exists}} + T_{\in,B})_{\forall\vee\exists} + ZFC + LC + MM^{++} \vdash \psi^{H_{\omega_2}}$

(iv) ψ is strongly $(T + T_{\in,B})_{\forall\vee\exists}$-consistent.