

ABSOLUTE MODEL COMPANIONSHIP, FORCIBILITY, AND THE CONTINUUM PROBLEM

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ABSTRACT. Absolute model companionship (AMC) is a strict strengthening of model companionship defined as follows: For a theory T , $T_{\exists\forall\forall}$ denotes the logical consequences of T which are boolean combinations of universal sentences. S is the AMC of T if it is model complete and $T_{\exists\forall\forall} = S_{\exists\forall\forall}$.

We use AMC to study the continuum problem and to gauge the expressive power of forcing. We show that (a definable version of) $2^{\aleph_0} = \aleph_2$ is the unique solution to the continuum problem which can be in the AMC of a *partial Morleyization* of the \in -theory $\text{ZFC} + \text{there are class many supercompact cardinals}$. We also show that (assuming large cardinals) forcibility overlaps with the apparently weaker notion of consistency for any mathematical problem ψ expressible as a Π_2 -sentence of a (very large fragment of) third order arithmetic (CH, the Suslin hypothesis, the Whitehead conjecture for free groups are a small sample of such problems ψ).

This paper is divided in two parts: the first part introduces two model theoretic concepts which are then used in the second part to analyze and gauge the complexity of the axiomatization of set theory given by ZFC (eventually enriched with large cardinal axioms). The key theme of the present work is to combine the ideas of Robinson around the notions of model companionship, model completeness, and existentially closed models for a first order theory with those arising in set theory from the analysis of the forcing method. Specifically we will show that a natural strengthening of the notion of model companionship is particularly fit to analyze ZFC: on the one hand it can be used to infer that forcibility overlaps with consistency at least when dealing with a large family of interesting problems of set theory, on the other hand it gives a viable model theoretic tool to tackle the continuum problem and provides an argument to assert that $2^{\aleph_0} = \aleph_2$.

We now briefly outline the model theoretic content and the set theoretic content of the paper trying to avoid technicalities¹.

Model theory.

Absolute model companionship. The first new model theoretic concept of this paper is that of absolute model companionship (AMC), which to our knowledge hasn't been explicitly stated yet. Given a first order theory T in a signature τ , a τ -structure \mathcal{M} is T -existentially closed (T -ec) if and only if it is a substructure of a model of T and it is a Σ_1 -elementary substructure of any τ -superstructure which models T . A standard example is given by Fields (the $\{+, \cdot, 0, 1\}$ -theory of fields), with the Fields-ec models being exactly the algebraically closed fields.

The author acknowledges support from INDAM through GNSAGA and from the project: *PRIN 2017-2019NWTM8R Mathematical Logic: models, sets, computability*. **MSC:** 03C10, 03E57. **Keywords:** *Model Companionship, Generic Absoluteness, Forcing Axioms, Large Cardinals*.

¹The model theory part of this paper is independent from its set theory part and can be read by any one familiar with the basic facts about model companionship and model completeness. The set theory part depends on the model theory part and has two types of arguments: there are basic results whose proofs leverage on classical theorems covered in any master course on the subject (essentially Levy absoluteness, no knowledge of forcing required); there are also advanced results whose proofs require a strong background in forcing axioms and Woodin's work on axiom (*) and take advantage of Asperó and Schindler's recent breakthrough [2].

The way algebraically closed fields sits inside the $\{+, \cdot, 0, 1\}$ -structures which are fields is described by Robinson's notion of model companionship: a τ -theory S is the model companion of a τ -theory T if the elementary class given by the τ -models of S consists exactly of the T -ec models. The $\{+, \cdot, 0, 1\}$ -theory of algebraically closed fields ACF is the model companion of the $\{+, \cdot, 0, 1\}$ -theory Fields ; however for an arbitrary theory T the T -ec models may not form an elementary class (e.g. the existentially closed models for the $\{\cdot, 1\}$ -theory of groups are not an elementary class, hence this axiomatization of groups has no model companion). A very special case occurs when T is its own model companion (e.g. the models of T are exactly the T -ec models), in which case T is *model complete*.

A non-trivial observation is that a τ -structure \mathcal{M} is T -ec if and only if it is T_{\forall} -ec (where T_{\forall} consists of the universal consequences of T in signature τ). Another non-trivial fact is that for a *complete* theory T a T -ec structure \mathcal{M} realizes any Π_2 -sentence which holds true in some model of T_{\forall} . A third non-trivial remark is that if S is the model companion of T , S is axiomatized by its Π_2 -consequences.

Combining these three observations one is led to the speculation that the model companion S of a τ -theory T (if it exists) could be axiomatized by the family of Π_2 -sentences which holds in some model of T_{\forall} . This is an assertion which is slightly too bold and holds true in case T is a *complete*, model companionable theory. However it can fail for non-complete model companionable theories; the standard counterexample being ACF versus Fields : $\forall x \neg(x^2 + 1 = 0)$ is a Π_2 -sentence which is not an axiom of ACF and holds in the field \mathbb{Q} , being therefore consistent with the universal fragment of the theory of fields.

This brings us to introduce the notion of absolute model companionship (AMC): a τ -theory T has an AMC if the class of T -ec models is axiomatized by the Π_2 -sentences which are consistent with the universal and existential fragments of any completion of T (see Def. 1.1, Thm. 1.3). Complete first order theories are model companionable if and only if they admit an AMC, but there are non-complete theories admitting a model companion but not an AMC (e.g. the $\{+, \cdot, 0, 1\}$ -theory of fields).

The AMC-spectrum of a first order theory. The second model theoretic concept we introduce in this paper is that of AMC-spectrum (and model companionship-spectrum) of a mathematical theory T . Model theory has been extremely successful in classifying the complexity of a mathematical theory according to its “structural properties” and has produced a variety of dividing lines to separate the so called “tame” mathematical theories from the others: typically a mathematical theory is considered “wild” or “unclassifiable” if it can code in itself first order arithmetic, hence the \in -theory ZFC is considered unclassifiable. On the other hand several tools have been developed to “classify” mathematical theories, for example stability, simplicity, NIP are structural properties of “tame” mathematical theories.

It is a matter of fact that most mathematical theories admit many different first order axiomatizations in many distinct signatures. A common characteristic of “tameness” properties such as stability, simplicity, NIP is that they are *signature invariant*: more precisely if we take a τ -theory T and we consider its Morleyization T^* in the signature τ^* which adds predicate symbols and axioms granting that all τ -formulae are equivalent to atomic τ^* -formulae (see Notation 1.5), T is stable (NIP, simple) if and only if so is T^* . In contrast Robinson's notion of model companionship is a useful property of a first order theory, but is not signature invariant: for example ACF is the model companion of Fields in signature $\tau = \{+, \cdot, 0, 1\}$, but if we consider their Morleyizations ACF^* and Fields^* in signature τ^* , it no longer holds true that ACF^* is the model companion of Fields^* in signature τ^* . Conversely a τ -theory R may not have a model companion (e.g. the $\sigma = \{\cdot, 1\}$ -theory of groups) but its Morleyization R^* in signature σ^* is its own model companion.

In this paper we consider the classification of the dependence of the existence of a model companion from the signature by itself a useful structural property of a mathematical theory. Specifically we introduce the notion of *partial Morleyzation* of a τ -theory T : given a set A of τ -formulae, we consider the signature τ_A obtained by expanding τ exactly with predicate symbols for the formulae in A and the τ_A -theory $T_{\tau,A}$ with axioms asserting that every formula in A is logically equivalent to the corresponding predicate of $\tau_A \setminus \tau$ (see again² Notation 1.5). The AMC-spectrum of a τ -theory T (Def. 1.2) is given by those sets A of τ -formulae for which $T + T_{\tau,A}$ admits an absolute model companion (mutatis mutandis we can define the model companionship spectrum of T).

A category theoretic perspective on this approach can be enlightening: given a τ -theory T , consider the category \mathcal{C}_T given by the elementary class of its τ -models as objects and the τ -morphisms (e.g. maps which preserve the *atomic* τ -formulae) between them as arrows. By taking A a subset of the τ -formulae we can pass to the category $\mathcal{C}_{T+T_{\tau,A}}$ whose objects are τ_A -models of $T + T_{\tau,A}$ and whose arrows are the τ_A -morphisms (in accordance with Notation 1.5). There is a natural identification of the objects of \mathcal{C}_T and those of $\mathcal{C}_{T+T_{\tau,A}}$, but the arrows of $\mathcal{C}_{T+T_{\tau,A}}$ are now a possibly much narrower subfamily of the arrows of \mathcal{C}_T . This paper (by considering the case of T being set theory) shows that important structural information on a τ -theory T is given (at least from a categorial point of view) by classifying which sets of τ -formulae A produce an elementary class of τ_A -models for which the $T + T_{\tau,A}$ -ec models constitute themselves an elementary class. The model companionship spectrum of T gives exactly this structural information on the arrows of \mathcal{C}_T . Similar considerations apply for the AMC-spectrum of a theory. On the other hand simplicity, stability, NIP provide fundamental structural informations on the class of objects of \mathcal{C}_T , but it is not transparent whether they also convey information on its class of arrows.

Set theory.

What is the right signature for set theory? The \in -signature is certainly sufficient to give by means of ZFC a first order axiomatization of set theory (with eventually other extra hypothesis such as large cardinal axioms), but we can see rightaway that it is not efficient to formalize many basic set theoretic concepts. Consider for example the notion of ordered pair: on the board we write $x = \langle y, z \rangle$ to mean that x is the ordered pair with first component y and second component z . In set theory this concept is formalized by means of Kuratowski's trick stating that $x = \{\{y\}, \{y, z\}\}$. However the "natural" \in -formula formalizing the above is:

$$\exists t \exists u [\forall w (w \in x \leftrightarrow w = t \vee w = u) \wedge \forall v (v \in t \leftrightarrow v = y) \wedge \forall v (v \in u \leftrightarrow v = y \vee v = z)].$$

It is clear that the meaning of this \in -formula is hardly decodable with a rapid glance (unlike $x = \langle y, z \rangle$), moreover just from the point of view of its syntactic complexity it is already Σ_2 . On the other hand we do not regard the notion of ordered pair as a complex or doubtful concept (as is the case for the notion of uncountability, or many of the properties of the continuum such as its correct place in the hierarchy of uncountable cardinals, etc...). Similarly other very basic notions such as: being a function, a binary relation, the domain or the range of a function, etc.. are formalized by rather complicated \in -formulae, both from the point of view of readability for human beings, and from the mere computation of their syntactic complexity according to the Levy hierarchy.

The standard solution adopted by set theorists (e.g. [12, Chapter IV, Def. 3.5]) is to regard as elementary all those properties which can be formalized using \in -formulae all of whose quantifiers are bounded to range over the elements of some set, i.e. the so

²Actually we give in this introduction a simplified version of the Morleyzation procedures we consider; the full details can be found in Notation 1.5.

called Δ_0 -formulae. We adopt this point of view in stating our main set theoretic results, and we maintain it for the remainder of this paper, considering \in_{Δ_0} the correct basic signature where to formalize set theory, where the latter is the signature obtained with the partial Morleyization induced by the Δ_0 -predicates (and slightly more, see Section 1.2 and Notation 3.1.1 for details).

Existentially closed fragments of the set theoretic universe versus AMC. We now give a non-exhaustive list outlining on the one hand that for certain regular cardinals κ H_κ is not that far from being an existentially closed structure for the appropriate signature (e.g. Levy absoluteness, Shoenfield's absoluteness, BMM, BMM^{++}) or from having a model complete theory³ (e.g. Woodin's absoluteness, MM^{+++}). The reader needs not be familiar with these results, as they only serve as motivation for what we aim to do in the sequel.

Below we denote a τ -structure $(M, R^M : R \in \tau)$ by (M, τ^M) , \sqsubseteq denotes the substructure relation, \prec the elementary substructure relation, \prec_1 the Σ_1 -elementary substructure relation (see Notation 2.0.1 for details). Recall the following results:

Levy absoluteness: (see Lemma 3.1 below) whenever κ is a regular uncountable cardinal,

$$(H_\kappa, \in_{\Delta_0}^V) \prec_1 (V, \in_{\Delta_0}^V).$$

Shoenfield's absoluteness: (see [23, Lemma 1.2] for the apparently weaker formulation we give here) whenever G is V -generic for some forcing notion in V ,

$$(H_{\omega_1}, \in_{\Delta_0}^V) \prec_1 (V[G], \in_{\Delta_0}^{V[G]}).$$

Woodin's absoluteness: (see [23, Lemma 3.2] for the weak form of Woodin's result we give here) whenever G is V -generic for some forcing notion in V (and there are class many Woodin cardinals in V),

$$(H_{\omega_1}^V, \in_{\Delta_0}^V) \prec (H_{\omega_1}^{V[G]}, \in_{\Delta_0}^{V[G]}).$$

Bounded Martin's Maximum (BMM): (see [5]) whenever G is V -generic for some stationary set preserving forcing notion in V ,

$$(H_{\omega_2}, \in_{\Delta_0}^V) \prec_1 (V[G], \in_{\Delta_0}^{V[G]}).$$

BMM⁺⁺: (see [25, Def. 10.91]) whenever G is V -generic for some stationary set preserving forcing notion in V ,

$$(H_{\omega_2}, \in_{\Delta_0}^V, \mathbf{NS}_{\omega_1}^V) \prec_1 (V[G], \in_{\Delta_0}^{V[G]}, \mathbf{NS}_{\omega_1}^{V[G]}),$$

where \mathbf{NS}_{ω_1} is a unary predicate symbol interpreted by the non-stationary ideal on ω_1 .

Bounded category forcing axioms, MM^{+++} , $\text{RA}_\omega(\text{SSP})$: (see⁴ [3, 4, 22]) whenever V and $V[G]$ are models of MM^{+++} ($\text{RA}_\omega(\text{SSP})$, $\text{BCFA}(\text{SSP})$) and G is V -generic for some stationary set preserving forcing notion in V ,

$$(H_{\omega_2}^V, \in_{\Delta_0}^V) \prec (H_{\omega_2}^{V[G]}, \in_{\Delta_0}^{V[G]}).$$

We also want to mention:

Absoluteness between transitive models: (see [10, Lemma 14.21]) whenever G is V -generic for some forcing notion in V ,

$$(V, \in_{\Delta_0}^V) \sqsubseteq (V[G], \in_{\Delta_0}^{V[G]}).$$

³Recall that a τ -theory T is model complete if and only if the substructure relation between its models overlaps with the elementary substructure relation. In particular the mentioned results are weak form of "model completeness" for the theory of H_{\aleph_i} for $i = 1, 2$.

⁴We omit a definition of these axioms since this demands a detour from the main focus of the present paper.

This latter property entails that forcing preserves the meaning of the basic concepts of set theory⁵, e.g. those formalized by atomic \in_{Δ_0} -formulae; furthermore Shoenfield’s absoluteness grants that Π_1 -sentences for \in_{Δ_0} do not change truth value in forcing extensions.

Main set theoretic results. The set theoretic results of the present paper will be outlined in details in Section 1 and systematize the above considerations. They can be informally summarized as follows:

- The theories of the various H_λ for λ an uncountable regular cardinal provide the prototypes of model companions for ZFC in some signature $\{\in\}_A$ with A in the model companionship spectrum of⁶ ZFC. More precisely: For any definable cardinal κ , there is a set of formulae A_κ in the AMC-spectrum of ZFC such that the theory of H_{κ^+} is the model companion of set theory for $\{\in\}_{A_\kappa}$ (see the second item of Thm. 1.6 and the first part of Thm. 1.7); note that for each such κ $\{\in\}_{A_\kappa}$ extends \in_{Δ_0} . Furthermore any existentially closed structure for set theory in some signature extending \in_{Δ_0} looks like an H_λ for some regular uncountable λ (see the first item of Thm. 1.6).
- Forcing suffices to produce the interesting models of a very large fragment of the mathematical universe: for statements of second or third order arithmetic formalizable by Π_2 -sentences of \in_{Δ_0} (among which for example the negation of the continuum hypothesis) any interesting consistency result (if at all possible) can already be obtained by forcing (by the first part of Thm. 1.7).
- The above results provide an argument for $2^{\aleph_0} = \aleph_2$ by taking as granted the following assertions for set theory:
 - (1) Large cardinal axioms are true.
 - (2) The “true” mathematical universe realizes any Π_2 -sentences for third order arithmetic which can hold in some model of set theory enriched with large cardinal axioms.

Note that assertion 2 for set theoretic truths is the standard argument used to motivate forcing axioms and is exactly analogous to the creation process of new numbers bringing from the universal $\{+, \cdot, 0, 1\}$ -theory T of semirings without zero divisors (holding for the natural numbers) to the theory of algebraically closed fields (holding for the algebraic numbers): the algebraic numbers are obtained in a $\{+, \cdot, 0, 1\}$ -structure which realizes T and all the “interesting” Π_2 -sentences which can be individually made consistent with T , e.g. for each n

$$\forall a_0, \dots, a_n \exists x \left(\sum_{i=0}^n a_i x^i = 0 \right).$$

On the basis of assertion 1, assertion 2 (as well as $2^{\aleph_0} = \aleph_2$) can be validated as follows:

⁵We ignored for the moment any consideration regarding universally Baire sets. These sets will play an essential role in the proof of Thm. 1.7.

⁶As a side remark we note that Hirschfeld [9] has proved that ZF has an AMC for the \in -signature; however he himself acknowledges that his result is not that informative on the properties of set theory, since the \in -AMC of ZF is given by a theory which is a small perturbation of the theory of dense linear orders (it interprets the \in -relation as an irreflexive transitive relation which defines a dense and strict preorder). Hirschfeld does not argue that ZF has an AMC but he shows that it decides any universal \in -sentence and has a model companion; under these assumptions the model companion of ZF is its AMC. We believe that a signature in the AMC-spectrum of set theory is meaningful if it allows to prove (i) of Thm. 1.6, e.g. if it is able to show that replacement holds in the AMC of set theory according to the signature. This brings almost automatically to consider signatures which are more or less equivalent to \in_{Δ_0} : they must be able to express the concept of function, relation, domain, codomain, etc by means of terms or of quantifier free formulae.

- Model theory (specifically the notion of AMC) gives us the means to formulate elegantly in precise mathematical terms assertion 2 (by Thm. 1.3).
- $2^{\aleph_0} = \aleph_2$ is the unique solution of the continuum problem which falls in the model companion of set theory enriched with large cardinals for at least one $\{\in\}_A$ with A in the AMC-spectrum of set theory (by Theorems 1.7, 1.9, 1.10).
- Furthermore the Π_2 -consequences for H_{\aleph_2} of BMM^{++} , Woodin’s axiom $(*)$, etc as formalized in an appropriate signature for third order arithmetic of the form $\{\in\}_B$ (for a natural set B of \in -formulae) form the AMC of set theory enriched with large cardinal axioms for $\{\in\}_B$; moreover this AMC exactly overlaps with the forcible Π_2 -sentences for H_{\aleph_2} for this signature (see Thm. 1.7, Thm. 5.1) and has among its Π_2 -axioms one which entails that $2^{\aleph_0} = \aleph_2$ is witnessed by a definable well order (for example by the results of [6, 16, 19]).

Our results outline the role of forcing axioms in detecting the right AMC for set theory describing the theory of H_{\aleph_2} . Key to their proof is the breakthrough [2] by Asperó and Schindler that MM^{++} —in combination with Woodin’s sealing’s theorem (see Def. 5.0.2)— implies the strong form of Woodin’s axiom $(*)$ we call here $(*)$ -UB (see Def. 5.1).

1. MAIN RESULTS

We now give a precise and brief list of the main results of this paper (the proofs are deferred to later sections). We introduce hastily all new concepts and terminology needed to phrase them. Our results are of three types:

Basic model theoretic results: we introduce AMC (a strengthening of model companionship) and we analyze its main properties. We also introduce the notion of partial Morleyization and we show how to combine it with AMC in order to get a useful classification tool for first order theories.

Basic set theoretic results: we outline the general properties of theories which can be the model companion of partial Morleyizations of ZFC leveraging on classical set theoretic results.

Advanced set theoretic result: we show that set theory enriched with large cardinal axioms has an AMC with respect to a natural signature for third order arithmetic⁷. Furthermore this AMC is given by the Π_2 -sentences of a very large fragment of third order arithmetic which are forcible. We also analyze why this result provides an argument in favour of $2^{\aleph_0} = \aleph_2$.

1.1. Absolute model companionship, partial Morleyizations, and the AMC-spectrum of a theory. We introduce a strengthening of model companionship which to our knowledge hasn’t been explicitly stated yet. We refer the reader to [21] or [7, Section 3.5], [18, Sections 3.1-3.2] for a detailed account of model companionship⁸

Notation 1.1. *Let τ be a signature and T be a τ -theory.*

- τ_{\forall} is the set of universal sentences for τ . $\tau_{\forall\exists}$ is the set of boolean combinations of sentences in τ_{\forall} .
- T_{\forall} (respectively $T_{\forall\exists}$) is the set of sentences in τ_{\forall} (respectively $\tau_{\forall\exists}$) which are logical consequences of T .

Example 1.1.1. In signature $\{+, \cdot, 0, 1\}$ let ACF be the theory of algebraically closed fields and Fields be the theory of fields; we have that $\text{ACF}_{\forall} = \text{Fields}_{\forall}$ while $\text{ACF}_{\forall\exists} \not\supseteq \text{Fields}_{\forall\exists}$: $\exists x (x^2 + 1 = 0)$ is in the former but not in the latter.

⁷This signature extends \in_{Δ_0} with predicates for the lightface definable universally Baire sets, a constant for ω_1 , and a unary predicate for the nonstationary ideal on ω_1

⁸The reader will find a detailed analysis of the concepts and results we introduce here in Section 2 (which could be read rightaway after section 1.1 if one has no interest in the set-theoretic content of this article).

1.1.1. *Absolute model companionship.* It is well known that if T, S are τ -theories such that $T_{\forall} = S_{\forall}$, then any model of T is a substructure of a model of S and conversely. However the following holds as well (and to our knowledge hasn't been explicitly stated):

Lemma 1.2. *Let τ be a signature and T, S be τ -theories. TFAE:*

- (1) $T_{\forall\exists} \supseteq S_{\forall\exists}$;
- (2) *Every τ -model \mathcal{M} of T is a τ -substructure of a τ -model \mathcal{N} of S such that \mathcal{M} and \mathcal{N} satisfy exactly the same sentences in τ_{\forall} (or equivalently in $\tau_{\forall\exists}$).*

Recall that a theory T is *model complete* if the substructure relation between its models overlaps with the elementary substructure relation (equivalently if T is its own model companion); recall also that T is the *model companion* of S if and only if $T_{\forall} = S_{\forall}$ and T is model complete. We introduce the following strengthening of model companionship:

Definition 1.1. Let τ be a signature and T, S be τ -theories. T is the *absolute model companion* (AMC) of S if the following conditions are met:

- (1) $T_{\forall\exists} = S_{\forall\exists}$;
- (2) T is model complete.

Note that:

- ACF is the model companion of Fields (since ACF is model complete and $\text{ACF}_{\forall} = \text{Fields}_{\forall}$) but not its absolute model companion (as $\exists x(x^2 + 1 = 0) \in \text{ACF}_{\exists\forall\forall} \setminus \text{Fields}_{\exists\forall\forall}$).
- A model complete theory T is the model companion of T_{\forall} and the AMC of $T_{\forall\exists}$.
- The uniqueness of the model companion grants that a theory S admitting a model companion T with $T_{\forall\exists} \neq S_{\forall\exists}$ cannot have an AMC.

The following result is what brought our attention on the notion of AMC:

Theorem 1.3. *Let τ be a signature and T, S be τ -theories. TFAE:*

- (1) T is the AMC of S ;
- (2) T is the model companion of S and is axiomatized by the Π_2 -sentences ψ such that $\psi + R_{\forall\exists}$ is consistent for all τ -theories $R \supseteq S$.

In case S is complete, T is the AMC of S if and only if it is its model companion, and the second item of the above equivalence states that the model companion of S is axiomatized by the Π_2 -sentences consistent with $S_{\forall\exists}$.

The following motivates our terminology for this strengthening of model companionship:

Lemma 1.4. *Assume S, S' are τ -theories such that S' is the AMC of S . Then for all $T \supseteq S$, $S' + T_{\forall}$ is the AMC of T .*

Note that this characterization does not hold for the weaker notion of model companionship: for the signature $\tau = \{+, \cdot, 0, 1\}$ ACF is the model companion of Fields , but for T the theory of the rationals in signature τ , $\text{ACF} + T_{\forall}$ is inconsistent, therefore it cannot be the model companion of T .

1.1.2. *Partial Morleyizations and the AMC-spectrum of a theory.* We now introduce the notation we use to relate AMC to set theory.

Notation 1.5. *Given a signature τ , let $\phi(x_0, \dots, x_n)$ be a τ -formula.*

We let:

- R_{ϕ} be a new $n + 1$ -ary relation symbols,
- f_{ϕ} be a new n -ary function symbols⁹
- c_{τ} be a new constant symbol.

⁹As usual we confuse 0-ary function symbols with constants.

We also let:

$$\mathbf{AX}_\phi^0 := \forall \vec{x} [\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x})],$$

$$\begin{aligned} \mathbf{AX}_\phi^1 := & \forall x_1, \dots, x_n \\ & [(\exists! y \phi(y, x_1, \dots, x_n) \rightarrow \phi(f_\phi(x_1, \dots, x_n), x_1, \dots, x_n)) \wedge \\ & \wedge (\neg \exists! y \phi(y, x_1, \dots, x_n) \rightarrow f_\phi(x_1, \dots, x_n) = c_\tau)] \end{aligned}$$

for $\phi(x_0, \dots, x_n)$ having at least two free variables, and

$$\mathbf{AX}_\phi^1 := [(\exists! y \phi(y)) \rightarrow \phi(f_\phi)] \wedge [(\neg \exists! y \phi(y)) \rightarrow c_\tau = f_\phi].$$

for $\phi(x)$ having exactly one free variable.

Let Form_τ denotes the set of τ -formulae. For $A \subseteq \text{Form}_\tau \times 2$

- τ_A is the signature obtained by adding to τ relation symbols R_ϕ for the $(\phi, 0) \in A$ and function symbols f_ϕ for the $(\phi, 1) \in A$ (together with the special symbol c_τ if at least one $(\phi, 1)$ is in A).
- $T_{\tau, A}$ is the τ_A -theory having as axioms the sentences \mathbf{AX}_ϕ^i for $(\phi, i) \in A$.

Note the following:

- For any τ -theory T , let $A = \text{Form}_\tau \times \{0\}$ and $\tau^* = \tau_A$; then $T^* = T + T_{\tau, A}$ is a τ^* -theory admitting quantifier elimination (the Morleyization of T). Furthermore any τ -structure admits exactly one extension to a τ^* -structure which is a model of $T_{\tau, A}$.
- For any τ -formula ϕ , any τ -structure \mathcal{M} with domain M , and any $a \in M$, there is exactly one extension of \mathcal{M} to a $\tau_{\{\phi\} \times \{1\}}$ -structures which interprets the value of the special constant c_τ as a and models \mathbf{AX}_ϕ^1 .

In the sequel of this paper we are interested to analyze what happens when the Morleyization process is performed on arbitrary subsets of $\text{Form}_\tau \times 2$.

Definition 1.2. The AMC-spectrum of a τ -theory T ($\text{spec}_{\text{AMC}}(T)$) is given by those $A \subseteq \text{Form}_\tau \times 2$ such that $T + T_{\tau, A}$ has an AMC (which we denote by $\text{AMC}(T, A)$).

The MC-spectrum of a τ -theory T ($\text{spec}_{\text{MC}}(T)$) is given by those $A \subseteq \text{Form}_\tau \times 2$ such that $T + T_{\tau, A}$ has a model companion (which we denote by $\text{MC}(T, A)$).

Note that $A = \text{Form}_\tau \times \{0\}$ is always in the model companionship spectrum of a theory T (as $T + T_{\tau, A}$ admits quantifier elimination, hence is model complete and its own AMC in signature τ_A). Note also that \emptyset is in the (A)MC-spectrum of T if and only if T has a model companion (an AMC).

We now show how we can use AMC and model companionship to extract interesting information on the continuum problem and on the relation between forcibility and consistency. Some of the results below holds also if we consider the *model companionship spectrum* of set theory, but the most interesting ones work just in case we focus on the AMC-spectrum¹⁰.

1.2. The AMC-spectrum of set theory. From now on for any $A \subseteq \text{Form}_{\{\in\}} \times 2$ we write \in_A rather than $\{\in\}_A$, and we let $T_{\in, A}$ be the \in_A -theory

$$T_{\{\in\}, A} + \forall x [(\forall y y \notin x) \leftrightarrow c_{\{\in\}} = x],$$

where the theory $T_{\{\in\}, A}$ (according to Notation 1.5 for $\{\in\}$ and A) is reinforced by an axiom asserting that the interpretation of the constant symbol $c_{\{\in\}}$ is the empty set.

We will be interested only in sets $A \subseteq \text{Form}_{\{\in\}} \times 2$ so that \in_A contains a basic signature \in_{Δ_0} where all the basic set theoretic results can be developed (e.g. those exposed in [12,

¹⁰Moreover as of now we are not even able to produce an example of an \in -theory $T \supseteq \text{ZFC}$ which has some $A \subseteq \text{Form} \times 2$ in its model companionship spectrum but not in its AMC-spectrum.

Chapter IV] and/or in [10, Chapter 13]). The specific details on \in_{Δ_0} and the axioms T_{Δ_0} are given in Notation 3.1.1, we anticipate here that:

- \in_{Δ_0} is of the form \in_D for a specific recursive set $D \subseteq \text{Form}_{\{\in\}} \times 2$ so that \in_{Δ_0} includes constant symbols for \emptyset, ω , relation symbols for all Δ_0 -formulae, function symbols for all Goedel operations as defined in [10, Def. 13.6].
- T_{Δ_0} is a family of Π_2 -axioms for \in_{Δ_0} so that $\text{ZF}^- + T_{\Delta_0}$ is equivalent to $\text{ZF}^- + T_{\in, D}$.

ZF^- (ZFC^-) denotes the \in -theory ZF (ZFC) deprived of the powerset axiom.

Definition 1.3. Let $T \supseteq \text{ZFC}^-$ be an \in -theory. κ is a T -definable cardinal if for some \in -formula $\phi_\kappa(x)$, T proves:

- $\exists! x \phi_\kappa(x)$ and

$$\forall x [\phi_\kappa(x) \rightarrow (x \text{ is a cardinal})].$$
- κ is the constant f_{ϕ_κ} existing in the signature $\in_{\{(\phi_\kappa, 1)\}}$.

The first result shows that the AMC spectrum of set theory isolates a rich set of theories which produce models of ZFC^- , e.g. structures which behave like an H_λ for some regular λ .

Theorem 1.6. *Let R be an \in -theory extending ZFC .*

- Assume $A \in \text{spec}_{\text{MC}}(R)$ and $\in_A \supseteq \in_{\Delta_0}$. Then $\text{MC}(R, A)$ models $\text{ZFC}^- + T_{\Delta_0} + T_{\tau, A}$.
- Assume κ is an R -definable cardinal. Then there exists $A_\kappa \in \text{spec}_{\text{AMC}}(R)$ with $\in_A \supseteq \in_{\Delta_0}$ and such that $\text{AMC}(R, A_\kappa)$ is given by the \in_{A_κ} -theory common to the structures¹¹ $H_{\kappa^+}^{\mathcal{M}}$ as \mathcal{M} ranges among the \in_{A_κ} -models of $R + T_{\in, A_\kappa}$.

1.3. Forcibility versus absolute model companionship. The following is the major result of the paper¹²:

Theorem 1.7. *Let S be the \in -theory*

$\text{ZFC} +$ there exists class many supercompact cardinals.

Then there is a set $B \in \text{spec}_{\text{AMC}}(S)$ with $\in_B \supseteq \in_{\Delta_0}$ and such that for any Π_2 -sentence ψ for \in_B and any \in -theory $R \supseteq S$ the following are equivalent:

- $\psi \in \text{AMC}(R, B)$;
- $(R + T_{\in, B})_{\forall\forall\exists} + S + \text{MM}^{++} + T_{\in, B}$ proves¹³ $\psi^{H_{\omega_2}}$;
- R proves that $\psi^{H_{\omega_2}}$ is forcible¹⁴ by a stationary set preserving forcing;
- R proves that $\psi^{H_{\omega_2}}$ is forcible by some forcing;
- For any $R' \supseteq R$, $\psi + (R' + T_{\in, B})_{\forall\forall\exists}$ is consistent.

Furthermore for any θ which is a boolean combination of Π_1 -sentences for \in_B and any (V, \in) model of S , TFAE:

- (V, \in_B^V) models θ ;
- (V, \in_B^V) models that some forcing notion P forces θ ;
- (V, \in_B^V) models that all forcing notions P force θ .

¹¹ $H_{\kappa^+}^{\mathcal{M}}$ denotes the substructure of \mathcal{M} whose extension is given by the formula defining H_{κ^+} in the model (using the parameter κ).

¹²The reader unaware of what is MM^{++} or a stationary set preserving forcing can skip the second and third items of the theorem.

¹³Here and elsewhere we write ψ^N to denote the relativization of ψ to a definable class (or set) N ; see [12, Def. IV.2.1] for details.

¹⁴Here and in the next item we mean that the \in -formula θ which is $T_{\in, B}$ -equivalent to ψ is such that $\theta^{H_{\omega_2}}$ is forcible by the appropriate forcing.

The second part of the theorem shows that forcing cannot change the Π_1 -fragment of the theory of V in signature $\in_B \supseteq \in_{\Delta_0}$. Note also that if (V, \in) is a model of S and R is the \in_B -theory of its unique extension to a model of $T_{\in, B}$, we get that a Π_2 -sentence ψ for \in_B is consistent with the universal and existential fragments of R if and only if $\psi^{H_{\omega_2}}$ is forcible over V .

We will give a detailed definition of \in_B at the proper stage; we anticipate here that it is a recursive set extending \in_{Δ_0} with a predicate symbol for the non-stationary ideal on ω_1 , a constant symbol for ω_1 , and predicate symbols for all sets of reals definable by \in -formulae without parameters in the Chang model $L(\text{Ord}^\omega)$ (which by a result of Woodin form an interesting subclass of the universally Baire sets, assuming the large cardinal hypothesis of the Theorem).

We can also drop any reference to AMC and \in_B and prove also the following result relating forcibility to consistency for Π_2 -sentences in the signature \in_{Δ_0} (where most of set theory can be efficiently formalized — see Section 1.4 right below):

Theorem 1.8. *Let S be the theory of Thm. 1.7.*

For any Π_2 -sentence ψ for the signature \in_{Δ_0} and for any \in -theory $R \supseteq S$ the following are equivalent:

- (1) $(R + T_{\Delta_0})_{\forall\forall\exists} + S + \text{MM}^{++} + T_{\Delta_0}$ proves $\psi^{H_{\omega_2}}$;
- (2) R proves that $\psi^{H_{\omega_2}}$ is forcible by a stationary set preserving forcing;
- (3) R proves that $\psi^{H_{\omega_2}}$ is forcible;
- (4) For any consistent \in -theory $R^* \supseteq R$, $\psi + (R^* + T_{\Delta_0})_{\forall\forall\exists}$ is consistent.

1.4. The AMC-spectrum of set theory and the continuum problem. We show that large cardinals place $2^{\aleph_0} = \aleph_2$ in a very special position of the AMC spectrum of set theory.

We formalize CH and $2^{\aleph_0} > \aleph_2$ in signature \in_{Δ_0} as follows:

- $(x \text{ is a cardinal})$ is the Π_1 -formula

$$(x \text{ is an ordinal}) \wedge \forall f [(f \text{ is a function} \wedge \text{dom}(f) \in x) \rightarrow \text{ran}(f) \neq x].$$

- $x := \aleph_1$ is the boolean combination of Σ_1 -formulae

$$(x \text{ is a cardinal}) \wedge (\omega \in x) \wedge$$

$$\wedge \exists F [(F : \omega \times x \rightarrow x) \wedge \forall \alpha \in x (F \upharpoonright \omega \times \{\alpha\} \text{ is a surjection on } \alpha)].$$

- CH is the Σ_2 -sentence

$$\exists f [(f \text{ is a function} \wedge \text{dom}(f) := \aleph_1) \wedge \forall r \subseteq \omega (r \in \text{ran}(f))].$$

and $\neg\text{CH}$ is the boolean combination of Π_2 -sentences¹⁵

$$\exists x (x := \aleph_1) \wedge \forall f [(\text{dom}(f) := \aleph_1 \wedge f \text{ is a function}) \rightarrow \exists r \subseteq \omega (r \notin \text{ran}(f))].$$

- $x := \aleph_2$ is the Σ_2 -formula

$$(x \text{ is a cardinal}) \wedge$$

$$\wedge \exists F \exists y [(y := \aleph_1) \wedge (y \in x) \wedge (F : y \times x \rightarrow x) \wedge \forall \alpha \in x (F \upharpoonright y \times \{\alpha\} \text{ is a surjection on } \alpha)].$$

- $2^{\aleph_0} > \aleph_2$ is the boolean combination of Π_2 -sentences

$$\exists x (x := \aleph_2) \wedge \forall f [(f \text{ is a function} \wedge \text{dom}(f) := \aleph_2) \rightarrow \exists r (r \subseteq \omega \wedge r \notin \text{ran}(f))].$$

- $2^{\aleph_0} \leq \aleph_2$ is the Σ_2 -sentence

$$\exists f [(f \text{ is a function}) \wedge \text{dom}(f) := \aleph_2 \wedge \forall r (r \subseteq \omega \rightarrow r \in \text{ran}(f))].$$

¹⁵We let $\neg\text{CH}$ include the Σ_2 -sentence $\exists x (x := \aleph_1)$, for otherwise its failure could be witnessed by the assertion that there is no uncountable cardinal, a statement which holds true in H_{ω_1} , regardless of whether CH or its negation is true in the corresponding universe of sets.

Theorem 1.9. *Let S be the \in -theory of Thm. 1.7. The following holds:*

- (1) *Let $R \supseteq S$ be an \in -theory. Assume $A \in \mathbf{spec}_{\text{AMC}}(R)$ with $\in_A \supseteq \in_{\Delta_0}$ and $\neg\text{CH} + \text{ZFC} + (R + T_{\in, A})_{\forall\forall\exists}$ is consistent. Then $\text{CH} \notin \text{AMC}(R, A)$.*
- (2) *For the signature \in_B of Thm. 1.7 $\neg\text{CH}$ is in $\text{AMC}(R, B)$ for any \in -theory $R \supseteq S$.*

We can prove exactly the same type of result replacing CH by $2^{\aleph_0} > \aleph_2$. Specifically Moore introduced in [16] a Π_2 -sentence θ_{Moore} for \in_{Δ_0} to show the existence of a definable well order of the reals in type ω_2 in models of the bounded proper forcing axiom¹⁶. We can use θ_{Moore} as follows:

Theorem 1.10. *There is a Π_2 -sentence θ_{Moore} for \in_{Δ_0} such that the following holds:*

- (1) *θ_{Moore} is independent of $S + T_{\Delta_0}$, where S is the \in -theory of Thm. 1.7.*
- (2) *$\text{ZFC}_{\Delta_0}^- + \exists x (x := \aleph_1) + \theta_{\text{Moore}}$ proves that there exists a well-ordering of¹⁷ $\mathcal{P}(\omega)$ in type at most ω_2 .*
- (3) *$\text{ZFC}_{\Delta_0}^- + \exists x (x := \aleph_2) + \theta_{\text{Moore}}$ proves that $2^{\aleph_0} \leq \omega_2$.*
- (4) *For S and \in_B the theory and signature considered in Thm. 1.7, $\exists x (x := \aleph_1), \theta_{\text{Moore}}$ are both in $\text{AMC}(R, B)$ for any \in -theory R extending S .*
- (5) *If R extends S , $A \in \mathbf{spec}_{\text{AMC}}(R)$ is such that $\in_A \supseteq \in_{\Delta_0}$, $\exists x (x := \aleph_2) \in \text{AMC}(R, A)$, and*

$$\theta_{\text{Moore}} + (R + T_{\in, A})_{\forall\forall\exists} + \text{ZFC}$$

is consistent, then $2^{\aleph_0} > \aleph_2$ is not in $\text{AMC}(R, A)$.

The two theorems show an asymmetry between $2^{\aleph_0} = \aleph_2$ and all other solutions of the continuum problem assuming large cardinals: for any R extending $\text{ZFC} + \text{large cardinals}$ there is at least one $B \in \mathbf{spec}_{\text{AMC}}(R)$ with $\neg\text{CH}$ (and a definable version of $2^{\aleph_0} \leq \aleph_2$) in $\text{AMC}(R, B)$, and this occurs even if $R \models \text{CH}$ or $R \models 2^{\aleph_0} > \aleph_2$. On the other hand if CH is independent of R , CH is never in $\text{AMC}(R, A)$ for any $A \in \mathbf{spec}_{\text{AMC}}(R)$ (with $\in_A \supseteq \in_{\Delta_0}$) and similarly if θ_{Moore} is independent of R , $2^{\aleph_0} > \aleph_2$ is never in $\text{AMC}(R, A)$ for any $A \in \mathbf{spec}_{\text{AMC}}(R)$ (with $\in_A \supseteq \in_{\Delta_0}$).

Furthermore the last part of Thm. 1.7 outlines that CH , $2^{\aleph_0} = \aleph_2$, $2^{\aleph_0} > \aleph_2$, θ_{Moore} are all boolean combination of Π_2 -sentences in the signature \in_{Δ_0} which cannot be expressed by boolean combination of Π_1 -sentences for the signature $\in_B \supseteq \in_{\Delta_0}$ in models of S (with S and B as in Thm. 1.7), as their truth value can be changed by forcing.

STRUCTURE OF THE PAPER

The remainder of this paper is organized as follows:

- Section 2 gives an account on the main properties of AMC .
- Section 3 proves Theorem 1.6, part 1 of Thm. 1.9, and Thm. 1.10 (with the exception of part 4).
- Section 4 proves the generic invariance of the theory of V in signature $\in_{\Delta_0} \cup \{\omega_1, \mathbf{NS}_{\omega_1}\}$ (more precisely a stronger version of the second part of Thm. 1.7).
- Section 5 completes the proof of Thm. 1.7 (and of the missing parts of Theorems 1.9, 1.10).
- We conclude the paper with some comments and open questions.

¹⁶We use here the definable well-ordering of the reals in type ω_2 existing in models of bounded forcing axioms isolated by Moore, but Thm. 1.10 could be proved replacing θ_{Moore} with any other coding device which produce the same effects, for example those introduced in [6, 19], or the sentence ψ_{AC} of Woodin as in [13, Section 6].

¹⁷More precisely: there is a $\text{ZFC}_{\Delta_0}^-$ -provably Δ_1 -property $\psi(x, y, z)$ such that in any model \mathcal{M} of the mentioned theory there is a parameter $d \in \mathcal{M}$ such that $\psi(x, y, d)$ defines an injection of $\mathcal{P}(\omega)$ of the model with the class of ordinals of size at most ω_1 of the model.

Any reader familiar enough with model companionship to follow 1.1 can easily grasp the content of Section 2. Section 3 needs Section 2 and familiarity with set theory at the level of [12, Chapters I, III, IV]. (no knowledge of forcing is required). The proofs in Section 4 assume familiarity with Woodin's stationary tower forcing, and (in its second part, cfr. Section 4.3) also with Woodin's \mathbb{P}_{\max} -technology. Section 5 can be fully appreciated only by readers familiar with forcing axioms, Woodin's stationary tower forcing, Woodin's \mathbb{P}_{\max} -technology and take advantage of Asperó and Schidler's proof that MM^{++} implies a strong form of Woodin's axiom (*) [2].

2. EXISTENTIALLY CLOSED STRUCTURES AND ABSOLUTE MODEL COMPANIONSHIP

This section proves the model theoretic results on model companionship stated in 1.1. Absolute model companionship (AMC) isolates those (possibly non-complete) theories T whose model companion is axiomatized by the Π_2 -sentences which are consistent with the universal and existential fragments of any model of T . We also show that AMC is strictly stronger than model companionship and does not imply (nor is implied by) model completion¹⁸.

We introduce the following terminology:

Notation 2.0.1.

- \sqsubseteq denotes the substructure relation between structures.
- $\mathcal{M} \prec_n \mathcal{N}$ indicates that \mathcal{M} is a Σ_n -elementary substructure of \mathcal{N} , we omit the n to denote full-elementarity.
- Given a first order signature τ , τ_{\forall} denotes the universal τ -sentences; likewise we interpret $\tau_{\exists}, \tau_{\forall\exists}, \dots$. $\tau_{\forall\forall\exists}$ denotes the boolean combinations of universal τ -sentences; likewise we interpret $\tau_{\forall\exists\forall\exists\forall}, \dots$.
- Given a first order theory T , T_{\forall} denotes the sentences in τ_{\forall} which are consequences of T , likewise we interpret $T_{\exists}, T_{\forall\exists}, T_{\forall\forall\exists}, \dots$.
- We often denote a τ -structure $(M, R^M : R \in \tau)$ by (M, τ^M) .
- We often identify a τ -structure $\mathcal{M} = (M, \tau^M)$ with its domain M and an ordered tuple $\vec{a} \in M^{<\omega}$ with its set of elements.
- We often write $\mathcal{M} \models \phi(\vec{a})$ rather than $\mathcal{M} \models \phi(\vec{x})[\vec{x}/\vec{a}]$ when \mathcal{M} is τ -structure $\vec{a} \in \mathcal{M}^{<\omega}$, ϕ is a τ -formula.
- We let the atomic diagram $\Delta_0(\mathcal{M})$ of a τ -model $\mathcal{M} = (M, \tau^M)$ be the family of quantifier free sentences $\phi(\vec{a})$ in signature $\tau \cup M$ such that $\mathcal{M} \models \phi(\vec{a})$.

2.1. Byembeddability versus absolute byembeddability. Let us give a proof of the following well known fact, since it will be helpful to outline the subtle difference between model companionship and absolute model companionship.

Lemma 2.1.1. *Let τ be a signature and T, S be τ -theories. TFAE:*

- (1) $T_{\forall} \supseteq S_{\forall}$.
- (2) For any \mathcal{M} model of T there is \mathcal{N} model of S superstructure of \mathcal{M} .

Proof.

1 implies 2: Assume \mathcal{M} models T and is such that no \mathcal{N} model of S is a superstructure of \mathcal{M} . Then $S \cup \Delta_0(\mathcal{M})$ is not consistent (where $\Delta_0(\mathcal{M})$ is the atomic diagram of \mathcal{M}). By compactness find $\psi(\vec{a}) \in \Delta_0(\mathcal{M})$ quantifier-free sentence such that $S + \psi(\vec{a})$ is inconsistent. This gives that

$$S \models \forall \vec{x} \neg \psi(\vec{x})$$

¹⁸A detailed account of model companionship, Kaiser Hulls, existentially closed structures in line with our treatment of this topic can be found in the notes [21].

since \vec{a} is a string of constant symbols all outside of τ . Therefore $\forall \vec{x} \neg \psi(\vec{x}) \in S_{\forall} \subseteq T_{\forall}$. Hence

$$\mathcal{M} \models \forall \vec{x} \neg \psi(\vec{x}) \wedge \psi(\vec{a}),$$

a contradiction.

2 implies 1: Left to the reader. □

The following is a natural question: assume S and T are τ -theories such that $T_{\forall} = S_{\forall}$, can we extend a model \mathcal{M} of T to a superstructure \mathcal{N} of S so that \mathcal{M} and \mathcal{N} satisfy exactly the same universal sentences? The answer is no as shown by $\tau = \{\cdot, +, 0, 1\}$, T the τ -theory of fields, S the τ -theory of algebraically closed fields: it is easy to see that $T_{\forall} = S_{\forall}$ in view of Lemma ?? but \mathcal{Q} cannot be extended to an algebraically closed field without killing the universal sentence stating the non existence of the square root of -1 .

The clarification of this issue is what has brought our attention to $T_{\forall\exists}$.

Note that any sentence in $T_{\forall\exists}$ is either logically equivalent to $\theta \vee \psi$ or equivalent to $\theta \wedge \psi$ with θ universal and ψ existential.

Note also that $T_{\forall\exists}$ may contain more information than $T_{\forall} \cup T_{\exists}$ as there could be a universal $\theta \notin T_{\forall}$ and an existential $\psi \notin T_{\exists}$ with $\theta \vee \psi \in T_{\forall\exists}$.

Lemma 2.1.2. *Let τ be a signature and T, S be τ -theories. TFAE:*

- (1) $T_{\forall\exists} \supseteq S_{\forall\exists}$.
- (2) For any \mathcal{M} model of T there is \mathcal{N} model of S superstructure of \mathcal{M} realizing exactly the same universal sentences.
- (3) For every boolean combination of universal sentences θ , $T + \theta$ is consistent only if so is $S + \theta$.

Proof.

1 implies 2: Assume \mathcal{M} models T and is such that no \mathcal{N} model of S which is a superstructure of \mathcal{M} realizes exactly the same universal sentences.

For any such \mathcal{N} with $\mathcal{M} \sqsubseteq \mathcal{N} \models S$ we get that some universal τ -sentence $\theta_{\mathcal{N}}$ true in \mathcal{M} fails in \mathcal{N} . We claim that the $\tau \cup \mathcal{M}$ -theory

$$S^* = \Delta_0(\mathcal{M}) \cup S \cup \{\theta_{\mathcal{N}} : \mathcal{M} \sqsubseteq \mathcal{N}, \mathcal{N} \models S\}$$

is inconsistent. If not let \mathcal{P}^* be a model of S^* . Then $\mathcal{P} = (\mathcal{P}^* \upharpoonright \tau) \sqsupseteq \mathcal{M}$ is a model of

$$S \cup \{\theta_{\mathcal{N}} : \mathcal{M} \sqsubseteq \mathcal{N}, \mathcal{N} \models S\}.$$

Hence it models $\theta_{\mathcal{P}}$ and $\neg \theta_{\mathcal{P}}$ at the same time.

By compactness we can find a universal sentence $\phi_{\mathcal{M}}$ given by the conjunction of a finite set

$$\{\theta_{\mathcal{P}_i} : i = 1, \dots, n, \mathcal{M} \sqsubseteq \mathcal{P}_i \models S\}$$

and a quantifier free sentence $\psi_{\mathcal{M}}(\vec{a})$ of $\Delta_0(\mathcal{M})$ such that

$$S + \psi_{\mathcal{M}}(\vec{a}) + \phi_{\mathcal{M}}$$

is inconsistent. Hence

$$S \models \neg \phi_{\mathcal{M}} \vee \neg \exists \vec{x} \psi_{\mathcal{M}}(\vec{x}).$$

Now observe that:

- $\neg \phi_{\mathcal{M}} \vee \neg \exists \vec{x} \psi_{\mathcal{M}}(\vec{x})$ is a boolean combination of universal sentences,
- $\mathcal{M} \models T + \exists \vec{x} \psi_{\mathcal{M}}(\vec{x}) \wedge \phi_{\mathcal{M}}$.

Therefore we get that $\neg \phi_{\mathcal{M}} \vee \neg \exists \vec{x} \psi_{\mathcal{M}}(\vec{x})$ is in $S_{\forall\exists} \setminus T_{\forall\exists}$.

2 implies 3: Left to the reader.

3 implies 1: If $T_{\forall\exists} \not\supseteq S_{\forall\exists}$ there is $\theta \in S_{\forall\exists} \setminus T_{\forall\exists}$. Then $\neg \theta$ is inconsistent with S and consistent with T .

□

Definition 2.1.3. Let τ be a signature and T, S be τ -theories.

- T and S are *cotheories* if $T_{\forall} = S_{\forall}$.
- T and S are *absolute cotheories* if $T_{\forall\exists} = S_{\forall\exists}$.

Remark 2.1.4. Say that a τ -theory T is Π_1 -complete if $T \vdash \phi$ or $T \vdash \neg\phi$ for any universal τ -sentence ϕ .

Now consider the $\{+, \cdot, 0, 1\}$ -theories ACF_0 and Fields_0 expanding ACF and Fields with the axioms fixing the characteristic of their models to be 0. Note that ACF_0 is Π_1 -complete (it is actually complete) while Fields_0 is not, even if $(\text{ACF}_0)_{\forall} = (\text{Fields}_0)_{\forall}$. In particular $T_{\forall} = S_{\forall}$ is well possible with T Π_1 -complete and S not Π_1 -complete. Absolute cotheories rule out this confusing discrepancy. In particular we will use the following trivial fact crucially in the proof of Lemma 2.3.4: if S is a complete theory $S_{\forall\exists}$ is Π_1 -complete, while S_{\forall} may not.

2.2. Existentially closed models, Kaiser hulls, strong consistency.

Definition 2.2.1. Given a τ -theory T , \mathcal{M} is T -existentially closed (T -ec) if:

- There is some $\mathcal{N} \supseteq \mathcal{M}$ which models T .
- $\mathcal{M} \prec_1 \mathcal{N}$ for all superstructures $\mathcal{N} \supseteq \mathcal{M}$ which model T .

Remark 2.2.2. Among the many nice properties of T -ec structures, note that Π_2 -sentences (with parameters in \mathcal{M}) which hold in some T -model \mathcal{N} superstructure of \mathcal{M} reflect to \mathcal{M} .

In view of the above Lemmas it is not hard to check the following:

Fact 2.2.3. *TFAE for a τ -theory T and a τ -structure \mathcal{M} :*

- (1) \mathcal{M} is T -ec.
- (2) \mathcal{M} is $T_{\forall\exists}$ -ec.
- (3) \mathcal{M} is T_{\forall} -ec.

Definition 2.2.4. Let T be a τ -theory.

- A τ -sentence ψ is *strongly $T_{\forall\exists}$ -consistent* if $\psi + R_{\forall\exists}$ is consistent for all $R \supseteq T$.
- The *Kaiser hull of T* ($\text{KH}(T)$) consists of the Π_2 -sentences for τ which hold in all T -ec models.
- The *strong consistency hull of T* ($\text{SCH}(T)$) consists of the Π_2 -sentences for τ which are strongly $T_{\forall\exists}$ -consistent.

The Kaiser hull of a theory is a well known notion describing an equivalent of model companionship which can be defined also for non-companionable theories (see for example [18, Lemma 3.2.12, Lemma 3.2.13, Thm. 3.2.14]); the strong consistency hull is a slight weakening of the Kaiser hull not considered till now (at least to my knowledge) and which does the same with respect to the notion of absolute model companionship (defined below in Def. 2.3.1).

Fact 2.2.5. *For any τ -theory T :*

- (i) $\text{KH}(T)_{\forall} = T_{\forall}$;
- (ii) $\text{SCH}(T)_{\forall\exists} = T_{\forall\exists}$;
- (iii) $\text{SCH}(T) \subseteq \text{KH}(T)$;
- (iv) $\text{SCH}(T) = \text{KH}(T)$ if $T_{\forall\exists} = \text{KH}(T)_{\forall\exists}$, which is the case if T is complete.
- (v) For any Π_2 -sentence ψ such that $T_{\forall\exists} + \psi$ is consistent, there is a model of $\text{KH}(T) + \psi$.

Proof.

- (i) By definition any model of $\text{KH}(T)$ is a model of $T_{\forall\exists}$; conversely any model of T can be extended to a T -ec model (see for example [18, Lemma 3.2.11]). We conclude by Lemma 2.1.1.
- (ii) Trivial.
- (iii) Assume a Π_2 -sentence ψ is strongly $T_{\forall\exists}$ -consistent. Let \mathcal{M} be a T -ec model. Then \mathcal{M} is $T_{\forall\exists}$ -ec. Let R be the τ -theory of \mathcal{M} . Since \mathcal{M} is T -ec, any superstructure of \mathcal{M} which models T is also a model of $R_{\forall\exists}$ (by Fact 2.2.3). Since ψ is strongly $T_{\forall\exists}$ -consistent, $\psi + R_{\forall\exists}$ is consistent. By Lemma 2.1.2, ψ holds in some $\mathcal{N} \sqsupseteq \mathcal{M}$ which models $R_{\forall\exists}$. Since \mathcal{M} is T -ec and $R_{\forall\exists} \supseteq T_{\forall}$, we get that ψ reflects to \mathcal{M} (being a Π_2 -sentence which holds in \mathcal{N} which is a Σ_1 -superstructure of \mathcal{M}).
- (iv) Assume a Π_2 -sentence ψ is in $\text{KH}(T)$. Let R be any complete extension of T and \mathcal{M} be a model of R . By Lemma 2.1.2 there is \mathcal{N} which is a model of $\text{KH}(T)$ and of $R_{\forall\exists}$. In particular \mathcal{N} models $\psi + R_{\forall\exists}$. Since R is arbitrary, ψ is strongly $T_{\forall\exists}$ -consistent.
Clearly if T is complete, $T_{\forall\exists}$ is Π_1 -complete, and ψ is strongly $T_{\forall\exists}$ -consistent if and only if $\psi + T_{\forall\exists}$ is consistent. We conclude also in this case that a Π_2 -sentence ψ holds in some T -ec model if and only if it is strongly $T_{\forall\exists}$ -consistent.
- (v) Note that $\text{KH}(T)$ is axiomatized by its Π_2 -fragment and $\text{KH}(T)_{\forall\exists} \supseteq T_{\forall\exists}$. Therefore we can apply Lemma 2.2.7 below.

□

Remark 2.2.6. There can be τ -theories T whose Kaiser hull strictly contains its strong consistency hull.

Consider the $\{0, 1, \cdot, +\}$ -theories ACF_0 (of algebraically closed fields of characteristic 0) and Fields_0 (of fields of characteristic 0). Note that ACF_0 is complete while Fields_0 is not. Furthermore ACF_0 is the Kaiser hull of Fields_0 (note that: ACF_0 is axiomatized by its Π_2 -fragment; any Fields_0 -ec model is an algebraically closed field; any model of ACF_0 is Fields_0 -ec).

We get that $\exists x(x^2 + 1 = 0)$ is a Π_2 -sentence (in fact existential) in the Kaiser hull of Fields_0 but not in its strong consistency hull, since it is not consistent with $R_{\forall\exists}$, where R is the $\{0, 1, \cdot, +\}$ -theory of the rationals.

Lemma 2.2.7. *Let S, T be τ -theories such that $S_{\forall\exists} \subseteq T_{\forall\exists}$ and S is axiomatized by its Π_2 -fragment. Then for any Π_2 -sentence ψ consistent with T there is a model of $S + \psi$.*

Proof. We prove a stronger conclusion which is the following:

Let R be a complete theory extending $T + \psi$. Then there is a model of $R_{\forall\exists} + S + \psi$.

Let $\{\mathcal{M}_n : n \in \omega\}$ be a sequence of τ -structures such that for all $n \in \omega$:

- \mathcal{M}_n is a τ -substructure of \mathcal{M}_{n+1} ;
- \mathcal{M}_n models $R_{\forall\exists}$;
- \mathcal{M}_{2n} models R ;
- \mathcal{M}_{2n+1} models S .

Such a sequence can be defined letting \mathcal{M}_0 be a model of R , \mathcal{M}_1 be a model of S which satisfies $R_{\forall\exists}$ (which is possible in view of Lemma 2.1.2) and defining \mathcal{M}_n as required for all other n appealing to the fact that $S + R_{\forall\exists}$ and $T + \psi + R_{\forall\exists}$ are absolute cotheories with $R_{\forall\exists}$ being the Π_1 -complete fragment shared by both theories. Then $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$ is a model of $R_{\forall\exists} + S + \psi$ since it realizes all Π_2 -sentences which hold in an infinite set of \mathcal{M}_n (see for example [18, Lemma 3.1.6]) and satisfies exactly the same Π_1 -sentences of each of the \mathcal{M}_n . □

Note that the above cannot be proved if S, T are just cotheories: performing the above construction under this weaker assumption, we may not be able to define \mathcal{M}_2 as required if \mathcal{M}_1 does not realize exactly the same universal sentences of \mathcal{M}_0 .

2.3. Absolute model companionship.

Definition 2.3.1. A τ -theory T is:

- *model complete* if $\mathcal{M} \models T$ if and only if it is T_{\forall} -ec;
- the *model companion* of a τ -theory S if T and S are cotheories and T is model complete;
- the *absolute model companion* (AMC) of a τ -theory S if T and S are absolute cotheories and T is model complete.

Our definition of model completeness and model companionship takes advantage of [7, Prop. 3.5.15].

We will use repeatedly Robinson's test providing different equivalent characterizations of model completeness:

Lemma 2.3.2. [18, Lemma 3.2.7] (*Robinson's test*) *Let T be a τ -theory. The following are equivalent:*

- (a) T is model complete.
- (b) Whenever $\mathcal{M} \sqsubseteq \mathcal{N}$ are models of T , $\mathcal{M} \prec \mathcal{N}$.
- (c) Each existential τ -formula $\phi(\vec{x})$ in free variables \vec{x} is T -equivalent to a universal τ -formula $\psi(\vec{x})$ in the same free variables.
- (d) Each τ -formula $\phi(\vec{x})$ in free variables \vec{x} is T -equivalent to a universal τ -formula $\psi(\vec{x})$ in the same free variables.

Remark 2.3.3. (d) (or (c)) shows that being a model complete τ -theory T is expressible by a Δ_0 -property in parameters τ, T in any model of ZFC, hence it is absolute with respect to forcing. They also show that quantifier elimination implies model completeness. (c) also shows that model complete theories are axiomatized by their Π_2 -fragment.

The following characterization of absolute model companionship has brought our attention to this notion.

Lemma 2.3.4. *Assume T, T' are τ -theories and T' is model complete. TFAE:*

- (i) T' is the absolute model companion of T .
- (ii) T' is axiomatized by the strong consistency hull of T .

Proof.

(i) implies (ii): First of all we note that any model complete theory S is axiomatized by its strong consistency hull in view of Robinson's test (d) and Fact 2.2.3.

We also note that for absolute cotheories T, T' , their strong consistency hull overlap (in view of Lemma 2.1.2).

Putting everything together we obtain the desired implication.

(ii) implies (i): Note that for θ a boolean combination of universal τ -sentences, we have that θ is in the strong consistency hull of some τ -theory S if and only if $\theta \in S_{\forall\exists}$. Combined with (ii), this gives that $T'_{\forall\exists} = T_{\forall\exists}$.

□

Finally the following Lemma motivates our terminology for AMC:

Lemma 2.3.5. *Assume T, T' are τ -structures such that T' is the AMC of T . Then any S extending T has as AMC $T' + S_{\forall}$.*

Note that this fails for the standard notion of model companionship: ACF is the model companion of Fields in signature $\tau = \{0, 1, \cdot, +\}$, but if S is the theory of the rationals in signature τ , $S_{\forall} + \text{ACF}$ is inconsistent, hence it cannot be the model companion of S .

Proof. Assume $S \supseteq T$ is consistent. If $\mathcal{M} \models S$, \mathcal{M} has a superstructure which models $T' + S_{\forall\exists}$, since T and T' are absolute cotheories. This gives that $S' = T' + S_{\forall}$ is consistent. Since T' is model complete, so is S' by Robinson's test (cfr. Remark 2.3.3 and Lemma 2.3.2(c)). Now observe that S' and S satisfy item 2 of Lemma 2.1.2 (since $S' \supseteq T'$ and $S \supseteq T$ with T and T' absolute cotheories), yielding easily that $S_{\forall\exists} = S'_{\forall\exists}$. Therefore S' is the AMC of S . \square

Remark 2.3.6. Absolute model companionship is strictly stronger than model companionship: if T is model complete, T is the model companion of T_{\forall} and the absolute model companion of $T_{\forall\exists}$; the two notions do not coincide whenever T_{\forall} is strictly weaker than $T_{\forall\exists}$.

If T' is the model companion of T , $T'_{\forall\exists} \supseteq T_{\forall\exists}$: assume $\mathcal{M} \models T'$, then there is a superstructure \mathcal{N} of \mathcal{M} which models T (since T' is the model companion of T). Now $\mathcal{M} \prec_1 \mathcal{N}$, since \mathcal{M} is T -ec. Hence \mathcal{N} has the same Π_1 -theory of \mathcal{M} . The inclusion can be strict as shown by the counterexample given by Fields versus ACF .

Recall that T is the model completion of S if it is its model companion and admits quantifier elimination (See [7, Prop. 3.5.19]).

Absolute model companionship does not imply model completion:

Fact 2.3.7. *Let \in_B and S be the signature and theory appearing in Theorem 1.7. Then $\text{AMC}(S, B)$ does not admit quantifier elimination, hence $S + T_{\in, B}$ does not have the amalgamation property and has no model completion.*

Proof. We show that $S + T_{\in, B}$ does not have the amalgamation property. This suffices to prove the Fact by [7, Prop. 3.5.19]. Given (V, \in) model of $S + \text{MM}^{++}$, let G be V -generic for Namba forcing at \aleph_2 and H be V -generic for $\text{Coll}(\omega_1, \omega_2)$. Note that (by Thm. 4.2.1, since all predicate symbols of \in_B are either universally Baire sets or Δ_0 -definable formulae or the non-stationary ideal on ω_1)

$$(V, \in_B^V) \sqsubseteq (V[G], \in_B^{V[G]}), (V[H], \in_B^{V[H]}).$$

If W is an \in_B -amalgamation of $V[G]$ and $V[H]$ over V , and W models $S + T_{\in, B}$, we get that in W ω_2^V has cofinality ω_1^V and ω at the same time (both properties are expressible by Δ_0 -formulae in parameters $\omega, \omega_1^V, \omega_2^V, f, g$ stating that $f: \omega \rightarrow \omega_2^V$ is cofinal and $g: \omega_1 \rightarrow \omega_2^V$ is cofinal), hence ω_1^V is countable in W . This is impossible since $(H_{\omega_2}^V, \in_B^V) \prec_1 (W, \in_A^W)$ and ω_1^V is uncountable in $(H_{\omega_2}^V, \in_B^V)$. \square

2.4. Preservation of the substructure relation and of Σ_1 -elementarity by expansions via definable Skolem functions. These technical result will be needed for the enhanced version of Levy absoluteness given by Lemma 3.1. This is crucial for the proofs of Thm. 1.6(i) and in Sections 4 and 5 to define the signature \in_B of Thm. 1.7. Recall Notation 1.5.

Fact 2.4.1. *Assume $\mathcal{M} \sqsubseteq \mathcal{N}$ are τ -structures and $\phi(\vec{x}, y)$ is a τ -formula such that both structures satisfy*

$$\forall \vec{x}, y [\phi(\vec{x}, y) \leftrightarrow \forall \vec{u} \psi_{\phi}(\vec{x}, y, \vec{u}) \leftrightarrow \exists \vec{z} \theta_{\phi}(\vec{x}, y, \vec{z})]$$

with $\psi_{\phi}, \theta_{\phi}$ quantifier free τ -formulae. Then the unique expansions of \mathcal{M}, \mathcal{N} to τ_A -models of $T_{\tau, A}$ for $A = \{\langle \phi, 0 \rangle\}$ are still τ_A -substructures.

Assume further that $\mathcal{M} \sqsubseteq \mathcal{N}$ both satisfy

$$\forall \vec{x} \exists! y \phi(\vec{x}, y).$$

Then the unique expansions of \mathcal{M}, \mathcal{N} to τ_B -models of $T_{\tau, B}$ for $B = \{\phi\} \times 2$ are still τ_B -substructures.

Proof. The first point is a basic argument left to the reader. The second point mimicks the argument from the first part of the proof of the next Lemma (and is also elementary). \square

Lemma 2.4.2. *Let τ be a first order signature. Assume $\mathcal{M} \prec_1 \mathcal{N}$ are τ -structures. Let $\phi(x_0, \dots, x_n)$ be a Σ_1 -formula for τ , $A = \{\phi\} \times 2$, and:*

- \mathcal{N}_1 be some extension of \mathcal{N} to a τ_A -structures which models $T_{\tau, A}$ and interprets c_τ by an element of \mathcal{M} ;
- \mathcal{M}_1 be the unique extension of \mathcal{M} to a τ_A -structures which models $T_{\tau, A}$ and interprets c_τ the same way \mathcal{N}_1 does.

Then it still holds that

$$\mathcal{M}_1 \prec_1 \mathcal{N}_1.$$

Proof. Let $\mathcal{M}^*, \mathcal{N}^*$ be the unique extensions of \mathcal{M} and \mathcal{N} to $\tau \cup \{R_\phi, c_\tau\}$ -structures which interpret c_τ as $\mathcal{M}_1, \mathcal{N}_1$ do and interpret R_ϕ according to Ax_ϕ^0 . Clearly

$$\mathcal{M}^* \prec_1 \mathcal{N}^*.$$

Now we show that $\mathcal{M}_1 \sqsubseteq \mathcal{N}_1$, e.g. we must show that $f_\phi^{\mathcal{M}_1} = f_\phi^{\mathcal{N}_1} \upharpoonright \mathcal{M}$.

We can suppose that $\phi(x_0, \dots, x_n)$ is of the form $\exists \vec{z} \psi_\phi(x_0, \dots, x_n, \vec{z})$ with ψ_ϕ quantifier free. Note that $\exists! y \phi(y, x_1, \dots, x_n)$ is logically equivalent to the boolean combination of Π_1 -formulae for τ

$$(1) \quad \exists y \exists \vec{z} \psi_\phi(y, x_1, \dots, x_n, \vec{z}) \wedge \forall u, v \forall \vec{z} [(\psi_\phi(u, x_1, \dots, x_n, \vec{z}) \wedge \psi_\phi(v, x_1, \dots, x_n, \vec{z})) \rightarrow u = v].$$

Since $\mathcal{M} \prec_1 \mathcal{N}$, we get that for any $\vec{b} \in \mathcal{M}^n$

$$\mathcal{M} \models \exists! y \phi(y, \vec{b})$$

if and only if

$$\mathcal{N} \models \exists! y \phi(y, \vec{b}).$$

Therefore for all $\vec{b} \in \mathcal{M}^n$ and $a \in \mathcal{M}$ we get that

$$\mathcal{M}_1 \models f_\phi(\vec{b}) = a$$

if and only if

$$\mathcal{N}_1 \models f_\phi(\vec{b}) = a;$$

hence $\mathcal{M}_1 \sqsubseteq \mathcal{N}_1$.

Now we want to show that $\mathcal{M}_1 \prec_1 \mathcal{N}_1$. The key point is to analyze the complexity of the formula $y = t(x_1, \dots, x_n)$ for t a τ_A -term. We can prove the following:

Claim 1. *For any τ_A -term $t(x_1, \dots, x_n)$ in displayed variables, there are a Π_2 -formula $\theta_t(x_1, \dots, x_n)$ and a Σ_2 -formula $\psi_t(x_1, \dots, x_n)$ for $\tau \cup \{c_\tau\}$ such that*

$$(2) \quad T_{\tau, A} \models \forall x_1, \dots, x_n, y [\psi_t(y, x_1, \dots, x_n) \leftrightarrow t(x_1, \dots, x_n) = y \leftrightarrow \theta_t(y, x_1, \dots, x_n)].$$

Assume the Claim holds, and notice that any existential τ_A -formula $\psi(x_1, \dots, x_n)$ is of the form

$$\exists x_{n+1}, \dots, x_m \theta(t_1(x_1, \dots, x_m), \dots, t_k(x_1, \dots, x_m))$$

with θ a quantifier free τ -formula and t_1, \dots, t_k τ_A -terms; by the Claim ψ is $T_{\tau, A}$ -equivalent to the Π_2 -formula for $\tau \cup \{c_\tau\}$

$$\forall y_1, \dots, y_k \left[\left(\exists x_{n+1}, \dots, x_m \bigwedge_{i=1}^k \psi_{t_i}(y_i, x_1, \dots, x_m) \right) \rightarrow \theta(y_1, \dots, y_k) \right].$$

This gives that for $b_1, \dots, b_n \in \mathcal{M}$ such that

$$\mathcal{N}_1 \models \exists x_{n+1}, \dots, x_m \theta(t_1(b_1, \dots, b_n, x_{n+1}, \dots, x_m), \dots, t_k(b_1, \dots, b_n, x_{n+1}, \dots, x_m)),$$

we get that

$$\mathcal{N}^* \models \forall y_1, \dots, y_k \left[(\exists x_{n+1}, \dots, x_m \bigwedge_{i=1}^k \psi_{t_i}(y_i, b_1, \dots, b_n, x_{n+1}, \dots, x_m)) \rightarrow \theta(y_1, \dots, y_k) \right];$$

therefore (since $\mathcal{M}^* \prec_1 \mathcal{N}^*$, and the above is a Π_2 -formula for $\tau \cup \{c_\tau\}$ in parameters b_1, \dots, b_n)

$$\mathcal{M}^* \models \forall y_1, \dots, y_k \left[(\exists x_{n+1}, \dots, x_m \bigwedge_{i=1}^k \psi_{t_i}(y_i, b_1, \dots, b_n, x_{n+1}, \dots, x_m)) \rightarrow \theta(y_1, \dots, y_k) \right].$$

Now observe that for all $i = 1, \dots, k$ by 2

$$\mathcal{M}_1 \models \forall x_{n+1}, \dots, x_m [\psi_{t_i}(y_i, b_1, \dots, b_n, x_{n+1}, \dots, x_m) \leftrightarrow t_i(b_1, \dots, b_n, x_{n+1}, \dots, x_m) = y_i].$$

Therefore

$$\mathcal{M}_1 \models \exists x_{n+1}, \dots, x_m \theta(t_1(b_1, \dots, b_n, x_{n+1}, \dots, x_m), \dots, t_k(b_1, \dots, b_n, x_{n+1}, \dots, x_m)).$$

We are done.

We are left with the proof of the Claim:

Proof. We proceed by induction on the depth of the τ_A -term t . If t is a term of depth 0, then t is a constant or a variable and there is almost nothing to prove (i.e. the unique term of depth 0 in $\tau_A \setminus \tau$ is c_τ ; we can let ψ_t and θ_t be the formula $y = t$).

Now assume the Claim holds for all terms of depth n . Let

$$t = f(t_1(x_1, \dots, x_n), \dots, t_k(x_1, \dots, x_n))$$

be a term of depth $n + 1$ with f a function symbol of τ_A . By inductive assumption there are $\theta_{t_j}(y_j, x_1, \dots, x_n)$ and $\psi_{t_j}(y_j, x_1, \dots, x_n)$ for $j = 1, \dots, k$ which are respectively Π_2 for $\tau \cup \{c_\tau\}$ and Σ_2 for $\tau \cup \{c_\tau\}$ such that:

$$(3) \quad T_{\tau, A} \models \forall x_1, \dots, x_n, y [\psi_{t_j}(y, x_1, \dots, x_n) \leftrightarrow t_j(x_1, \dots, x_n) = y \leftrightarrow \theta_{t_j}(y, x_1, \dots, x_n)].$$

This gives that $y = f(t_1(x_1, \dots, x_n), \dots, t_k(x_1, \dots, x_n))$ is $T_{\tau, A}$ -equivalent to the Σ_2 -formula for τ_A

$$(4) \quad \psi_t^*(x_1, \dots, x_n) := \exists y_1, \dots, y_k \left[\bigwedge_{j=1}^k \psi_{t_j}(y_j, x_1, \dots, x_n) \wedge y = f(y_1, \dots, y_k) \right]$$

and to the Π_2 -formula for τ_A

$$(5) \quad \theta_t^*(x_1, \dots, x_n) := \forall y_1, \dots, y_k \left[\bigwedge_{j=1}^k \psi_{t_j}(y_j, x_1, \dots, x_n) \rightarrow y = f(y_1, \dots, y_k) \right].$$

If f is a function symbol of $\tau \cup \{c_\tau\}$ we let ψ_t be ψ_t^* and θ_t be θ_t^* . These are $\tau \cup \{c_\tau\}$ -formulae, since $y = f(y_1, \dots, y_k)$ is already an atomic τ -formula, and all the other symbols occurring in 4 and 5 are also in $\tau \cup \{c_\tau\}$, and we easily get that 2 holds for $\psi_t := \psi_t^*$, $\theta_t := \theta_t^*$.

Else f is f_ϕ (therefore $k = n$) and we are considering the atomic τ_A -formula

$$y = f_\phi(y_1, \dots, y_n).$$

Now observe that $\exists! y \phi(y, x_1, \dots, x_n)$ is a boolean combination of Π_1 -formulae for τ by 1.

Therefore

$$y = f_\phi(y_1, \dots, y_n)$$

is $T_{\tau,A}$ -equivalent to the boolean combination of Π_1 -formulae for $\tau \cup \{c_\tau\}$

$$(6) \quad [\exists! z \phi(z, y_1, \dots, y_n) \wedge z = y] \vee [\neg \exists! z \phi(z, y_1, \dots, y_n) \wedge y = c_\tau].$$

Also in this case we are done: replacing in 4 and 5 the τ_A -formula $f_\phi(x_1, \dots, x_n) = y$ with the $\tau \cup \{c_\tau\}$ -formula 6 does not change the complexity of the $\tau \cup \{c_\tau\}$ -formulae so obtained. We let ψ_t and θ_t be the $\tau \cup \{c_\tau\}$ -formulae obtained from ψ_t^* and θ_t^* by this substitution. A minimal variant of the argument given above shows that ψ_t and θ_t are $T_{\in,A}$ -equivalent to $y = t(x_1, \dots, x_n)$. □

The Lemma is proved. □

2.5. Summing up.

- We see model completeness, model companionship, AMC as tameness properties of elementary classes \mathcal{E} defined by a theory T rather than of the theory T itself: these model-theoretic notions outline certain regularity patterns for the substructure relation on models of \mathcal{E} , patterns which may be unfolded only when passing to a signature distinct from the one in which \mathcal{E} is first axiomatized (much the same way as it occurs for Birkhoff's characterization of algebraic varieties in terms of universal theories).
- We will see in the next sections that set theory together with large cardinal axioms has (until now unexpected) tameness properties when formalized in certain natural signatures (already implicitly considered in most of the prominent set theoretic results of the last decades). These tameness properties couple perfectly with well known (or at least published) generic absoluteness results. The notion of AMC-spectrum gives an additional model theoretic criterium for selecting these “natural” signatures out of the continuum many signatures which produce definable extensions of ZFC.

3. THE AMC SPECTRUM OF SET THEORY

In this section we prove the basic properties of the AMC spectrum of set theory. Specifically we prove Thm. 1.6, Thm. 1.9, Thm. 1.10 (items 2 of Thm. 1.9 and 4 of Thm. 1.10 are proved conditionally to the proof of Thm. 1.7).

In 3.1 we define precisely the signatures in which we need to formalize set theory in order to prove all our main results.

3.2 formulates in precise mathematical terms the role Levy absoluteness plays in the proofs of the above theorems.

In 3.3 we show that for any $A \subseteq \text{Form}_\in \times 2$ with $\in_A \supseteq \in_{\Delta_0}$ in the model companionship spectrum of set theory and $R \supseteq \text{ZFC}$, $\text{MC}(R, A)$ extends ZFC_A^- , e.g. it is a model of replacement and choice closed under Goedel operations. Typically these axioms characterize the structures of type H_κ for κ a regular cardinal.

In 3.4 we show how to produce sets A such that $\in_A \supseteq \in_{\Delta_0}$ and the AMC of set theory with respect to \in_A exists and is the theory of H_{κ^+} for some infinite cardinal κ .

In 3.5 we show why it is artificial to put CH and $2^{\aleph_0} > \aleph_2$ in some AMC of set theory.

3.3, 3.4, 3.5 can be read independently of one another.

All proofs in this section are elementary (e.g. knowledge of [12, Chapters I, III, IV] suffices to follow the arguments); however —especially in 3.4— the notation is heavy. We haven't been able to simplify it.

3.1. Basic notation and terminology. We introduce the basic signatures and fragments of set theory we will always include in any signature of interest to us.

Notation 3.1.1. We let \in_{Δ_0} be \in_D for $D \subseteq \text{Form}_\in \times 2$ extending the set $\Delta_0 \times \{0\}$ with the pairs $(\phi, 1)$ as ϕ ranges over the following Δ_0 -formulae:

- The Δ_0 -formulae $\phi_\omega(x)$, $\phi_\emptyset(x)$ defining \emptyset and ω in any model of ZF^- , where the latter includes all axioms of ZF with the exception of power-set axiom (also we denote by ω and \emptyset the constants f_{ϕ_\emptyset} , f_{ϕ_ω}).
- The Δ_0 -formulae $\phi_i(\vec{x}, y)$ as G_i ranges over the Goedel operations G_1, \dots, G_{10} as defined in [10, Def. 13.6] and $\phi_i(\vec{x}, y)$ is the Δ_0 -formula defining the graph of G_i in any \in -model of ¹⁹ ZF^- .

We let T_{Δ_0} be given by the axioms:

$$(7) \quad \forall \vec{x} (R_{\forall z \in y \phi}(y, z, \vec{x}) \leftrightarrow \forall z (z \in y \rightarrow R_\phi(y, z, \vec{x})),$$

$$(8) \quad \forall \vec{x} [R_{\phi \wedge \psi}(\vec{x}) \leftrightarrow (R_\phi(\vec{x}) \wedge R_\psi(\vec{x}))],$$

$$(9) \quad \forall \vec{x} [R_{\neg \phi}(\vec{x}) \leftrightarrow \neg R_\phi(\vec{x})]$$

$$(10) \quad \forall x (x \notin \emptyset)$$

$$(11) \quad \omega \text{ is a non-empty ordinal all whose elements are successor ordinals or } \emptyset.$$

$$(12) \quad \forall \vec{x} \exists ! y (y = G_i(\vec{x}))$$

$$(13) \quad \forall \vec{x} \forall y [y = G_i(\vec{x}) \leftrightarrow R_{\phi_i}(\vec{x}, y)]$$

for the Goedel operations G_1, \dots, G_{10} .

We axiomatize suitable fragments of the \in -theory $\text{ZFC} + T_{\Delta_0}$ as follows:

- $Z_{\Delta_0}^-$ stands for the \in_{Δ_0} -theory given by:

(a) the Extensionality Axiom

$$\forall x, y, z [(z \in x \leftrightarrow z \in y) \rightarrow x = y],$$

(b) the Foundation Axiom

$$\forall x [x = \emptyset \vee \exists y \in x \forall z \in x (z \notin y)],$$

(c) T_{Δ_0} .

- Z_{Δ_0} enriches $Z_{\Delta_0}^-$ adding the power-set axiom

$$\forall x \exists y [\forall z (z \subseteq x \leftrightarrow z \in y)].$$

- $\text{ZC}_{\Delta_0}^-$ enriches $Z_{\Delta_0}^-$ adding the axiom of choice AC

$$\forall x \exists f [(f \text{ is a bijection}) \wedge \text{dom}(f) = x \wedge (\text{ran}(f) \text{ is an ordinal})].$$

- $\text{ZF}_{\Delta_0}^-$ enriches $Z_{\Delta_0}^-$ adding the replacement axiom for all \in_{Δ_0} -formulae.
- $\text{ZFC}_{\Delta_0}^-$, ZF_{Δ_0} , ZFC_{Δ_0} are defined as expected.

Remark 3.1.2. We took the pain of giving an explicit axiomatization of $Z_{\Delta_0}^-$ using Extensionality, Foundation, and axioms 7, ..., 13 because this axiomatization is given by Π_2 -sentences of \in_{Δ_0} , hence it is preserved by Σ_1 -substructures. Note that AC is a Π_2 -axiom of \in_{Δ_0} while the power-set axiom and the replacement schema for a quantifier free \in_{Δ_0} -formula are both Π_3 .

A simple inductive argument shows that $\text{ZF}^- + T_{\in, D}$ (where D is the subset of $\text{Form}_\in \times 2$ used in Not. 3.1.1 to define \in_{Δ_0}) is logically equivalent to ZF^- enriched with axioms 7, ..., 13 (with \emptyset taking the place of c_\in and ω being the constant of \in_{Δ_0} associated to the Δ_0 -formula defining it). We skip the details.

We now introduce the terminology to handle set theory formalized in signatures richer than \in_{Δ_0} .

¹⁹In models of ZF^- the Goedel operations G_1, \dots, G_{10} as listed and defined in [10, Def. 13.6] and their compositions have as graph the extension of a Δ_0 -formula (by [10, Lemma 13.7]).

Notation 3.1.3. Let $\tau \supseteq \in_{\Delta_0}$. For a τ -formula $\phi(\vec{x}, \vec{y}, \vec{z})$:

- The *Replacement Axiom* for ϕ ($\text{Rep}(\phi)$) states:

$\forall \vec{z} \forall X [(\forall x \in X \exists! y \phi(x, y, \vec{z})) \rightarrow \exists F (F \text{ is a function} \wedge \text{dom}(F) = X \wedge \forall x \in X \phi(x, F(x), \vec{z}))]$;

- Rep_τ holds if $\text{Rep}(\phi)$ holds for all τ -formulae ϕ .
- ZFC_τ^- is $\text{ZFC}_\tau^- + \text{Rep}_\tau$.
- Accordingly we define ZFC_τ , ZFC_τ^- , ZF_τ , ZFC_τ, \dots
- We write ZFC_{Δ_0} rather than ZFC_τ when $\tau = \in_{\Delta_0}$, etc.
- If $A \subseteq \text{Form}_\in \times 2$ is such that $\in_{\Delta_0} \subseteq \in_A$, we write ZFC_A^- rather than $\text{ZFC}^- + T_{\in, A}, \dots$

Clearly (the suitable fragment of) $\text{ZFC} + T_{\in, A}$ is logically equivalent to (the suitable fragment of) ZFC_A .

3.1.1. *Further notational conventions.* Let us introduce notation we will use to handle the substructure relation over expanded signatures. The following supplements Notations 1.5, 2.0.1.

Notation 3.1.4. Given some signature $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$ and a τ -structure (M, τ^M) and some $B \subseteq \text{Form}_\tau \times 2$, (M, τ_B^M) is the unique extension of (M, τ) defined in accordance with Notations 1.5, 2.0.1 which satisfies $T_{\tau, B}$. In particular (M, τ_B^M) is a shorthand for $(M, S^M : S \in \tau_B)$. If (N, τ^N) is a substructure of (M, τ^M) we also write (N, τ_B^M) as a shorthand for $(N, S^M \upharpoonright N : S \in \tau_B)$.

Remark 3.1.5. Note that in principle if (N, τ^N) is a substructure of (M, τ^M) , (N, τ_B^M) and (N, τ_B^N) could be different structures and (N, τ_B^N) may not be a substructure of (M, τ_B^M) :

- (N, τ_B^M) is obtained by restricting to N the interpretation of the new symbols of τ_B in M according to how M realizes $T_{\tau, B}$;
- (N, τ_B^N) is obtained by interpreting the new symbols of τ_B in N according to $T_{\tau, B}$ as realized in N .

We are spending a great deal of attention to isolate those set theoretic concepts which grant that (N, τ_B^M) and (N, τ_B^N) are the same (at least when $(N, \tau^N) \prec_1 (M, \tau^M)$).

Notation 3.1.6. Let S be an \in -theory. An \in -formula $\phi(\vec{x})$ is $\Delta_1(S)$ if there are quantifier free \in_{Δ_0} -formulae $\psi_\phi(\vec{x}, \vec{y})$ and $\theta_\phi(\vec{x}, \vec{z})$ such that $S + T_{\Delta_0}$ proves

$$\forall \vec{x} [\phi(\vec{x}) \leftrightarrow \forall \vec{y} \psi_\phi(\vec{x}, \vec{y}) \leftrightarrow \exists \vec{z} \theta_\phi(\vec{x}, \vec{z})].$$

Let (M, E) be an \in -structure. For $a \in M$,

$$\text{Ext}_\in^M(a) = \{b \in M : M \models b \in a\}.$$

$N \subseteq M$ is a *transitive subset* of M if $\text{Ext}_\in^M(a) = \text{Ext}_\in^N(a)$ for all $a \in N$.

There are many basic set theoretic properties which are established in standard textbooks (such as [10, 12]) only for transitive models of fragments of ZFC and can be established for arbitrary models of these fragments:

Fact 3.1.7. Let (M, E) be an \in -structure with $N \subseteq M$ a transitive subset. Then:

- For all Δ_0 -formula ϕ and $\vec{a} \in N^{<\omega}$, $N \models \phi(\vec{a})$ if and only if $M \models \phi(\vec{a})$.
- Furthermore assume (M, E) and (N, E) are both models of ZFC^- and let

$$\phi_1(\vec{x}_1), \dots, \phi_k(\vec{x}_k), \psi_1(\vec{x}_1, y), \dots, \psi_n(\vec{x}_n, y)$$

be \in -formulae which are $\Delta_1(\text{ZFC}^-)$ and such that

$$\text{ZFC}^- \models \forall \vec{x} \exists! y \psi_i(\vec{x}_i, y)$$

for $i = 1, \dots, n$. Then

$$\mathcal{M} \supseteq \mathcal{N},$$

where \mathcal{M}, \mathcal{N} are the unique extensions of respectively (M, E) , (N, E) to models for the signature

$$\in_{\Delta_0} \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$$

of

$$T_{\Delta_0} + \text{Ax}_{\phi_j}^0 + \text{Ax}_{\psi_l}^1 \text{ for } j = 1, \dots, k, l = 1, \dots, n.$$

Proof. Use Fact 2.4.1 together with the results of [12, Section IV.3] (which are there established under the further assumptions that M, N are transitive, but can be proved just assuming the weaker assumption that (M, E) , (N, E) are models of ZFC^- with N a transitive subset of M). We leave the details to the reader. \square

3.2. Levy absoluteness. We need a generalization of Levy's absoluteness in most proofs of the remainder of this paper. We state and prove the Lemma under the assumption that the model of ZFC we work in is transitive; but this assumption is unnecessary. Here and in other places of this paper we just need that the models in question satisfy ZFC^- or slightly more.

Lemma 3.1. *Let (V, \in_{Δ_0}) be a model of ZFC_{Δ_0} and $\lambda > \kappa$ be infinite cardinals for V with λ regular. Assume $\phi_1(\vec{x}_1), \dots, \phi_k(\vec{x}_k), \psi_1(\vec{x}_1, y), \dots, \psi_n(\vec{x}_n, y)$ are \in -formulae which are in $\Delta_1(\text{ZFC}^-)$ and*

$$\text{ZFC}^- \models \forall \vec{x} \exists! y \psi_i(\vec{x}_i, y)$$

for $i = 1, \dots, n$. Then the structure

$$(H_\lambda, \in_{\Delta_0}^{H_\lambda}, R_{\phi_j}^{H_\lambda} : j = 1, \dots, k, f_{\psi_l}^{H_\lambda} : l = 1, \dots, n, A : A \subseteq \mathcal{P}(\kappa)^k, k \in \mathbb{N})$$

is Σ_1 -elementary in

$$(V, \in_{\Delta_0}^V, R_{\phi_j}^V : j = 1, \dots, k, f_{\psi_l}^V : l = 1, \dots, n, A : A \subseteq \mathcal{P}(\kappa)^k, k \in \mathbb{N}),$$

where R_{ϕ_j} and f_{ψ_l} are interpreted by means of axioms $\text{Ax}_{\phi_j}^0$ and $\text{Ax}_{\psi_l}^1$ for $j = 1, \dots, k$, $l = 1, \dots, n$ in both structures.

Its proof is a variant of the classical result of Levy (which is the above theorem stated just for the signature \in_{Δ_0}); it is a slight expansion of [20, Lemma 5.3]; we include it here since it is not literally the same:

Proof. Let τ be the signature $\in_{\Delta_0} \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$, $\phi(\vec{x}, y)$ be a quantifier free formula for the signature under consideration where only predicates A_1, \dots, A_k appears²⁰, and $\vec{a} \in H_\lambda$ be such that

$$(V, \tau^V, A_1, \dots, A_k) \models \exists y \phi(\vec{a}, y).$$

Let $\alpha > \kappa$ be large enough so that for some $b \in V_\alpha$

$$(V, \tau^V, A_1, \dots, A_k) \models \phi(\vec{a}, b).$$

Then

$$(V_\alpha, \tau^{V_\alpha}, A_1, \dots, A_k) \models \phi(\vec{a}, b)$$

(since $(V_\alpha, \tau^{V_\alpha}, A_1, \dots, A_k) \sqsubseteq (V, \tau^V, A_1, \dots, A_k)$ by Fact 3.1.7). By the downward Lowenheim-Skolem theorem, we can find $X \subseteq V_\alpha$ which is the domain of a $\tau \cup \{A_1, \dots, A_k\}$ -elementary substructure of

$$(V_\alpha, \tau^{V_\alpha}, A_1, \dots, A_k)$$

such that X is a set of size κ containing κ and such that $A_1, \dots, A_k, \kappa, b, \vec{a} \in X$. Since $|X| = \kappa \subseteq X$, a standard argument shows that $H_\lambda \cap X$ is a transitive set, and that κ^+ is

²⁰Note that $\exists x \in y A(y)$ is not a quantifier free formula, and is actually equivalent to the Σ_1 -formula $\exists x(x \in y) \wedge A(y)$.

the least ordinal in X which is not contained in X . Let M be the transitive collapse of X via the Mostowski collapsing map π_X .

We have that the first ordinal moved by π_X is κ^+ and π_X is the identity on $H_{\kappa^+} \cap X$. Therefore $\pi_X(a) = a$ for all $a \in H_{\kappa^+} \cap X$. Moreover for $A \subseteq \mathcal{P}(\kappa)^n$ in X

$$(14) \quad \pi_X(A) = A \cap M.$$

We prove equation (14):

Proof. Since $X \cap V_{\kappa+1} \subseteq X \cap H_{\kappa^+}$, π_X is the identity on $X \cap H_{\kappa^+}$, and $A \subseteq \mathcal{P}(\kappa) \subseteq V_{\kappa+1}$, we get that

$$\pi_X(A) = \pi_X[A \cap X] = \pi_X[A \cap X \cap V_{\kappa+1}] = A \cap M \cap V_{\kappa+1} = A \cap M.$$

□

It suffices now to show that

$$(15) \quad (M, \tau^M, \pi_X(A_1), \dots, \pi_X(A_k)) \sqsubseteq (H_\lambda, \tau^{H_\lambda}, A_1, \dots, A_k).$$

Assume 15 holds; since π_X is an isomorphism and $\pi_X(A_j) = \pi_X[A_j \cap X]$, we get that

$$(M, \tau^M, \pi_X(A_1), \dots, \pi_X(A_k)) \models \phi(\pi_X(b), \vec{a})$$

since

$$(X, \tau^V, A_1 \cap X, \dots, A_k \cap X) \models \phi(b, \vec{a}).$$

By (15) we get that

$$(H_\lambda, \tau^{H_\lambda}, A_1, \dots, A_k) \models \phi(\pi_X(b), \vec{a})$$

and we are done.

We prove (15):

Proof. since (M, \in) is a transitive model of ZFC^- with $M \subseteq H_\lambda$, any atomic τ -formula holds true in (M, τ^M) if and only if it holds in $(H_\lambda, \tau^{H_\lambda})$ (again by Fact 3.1.7). It remains to argue that the same occurs for the formulae of type $A_j(x)$, i.e. that $A_j \cap M = \pi_X(A_j)$ for all $j = 1, \dots, n$; which is the case by (14). □

□

3.3. Replacement holds in $\text{MC}(T, A)$ for any $T \supseteq \text{ZFC}$ and any $A \in \text{spec}_{\text{MC}}(T)$ such that $\in_A \supseteq \in_{\Delta_0}$. We show that for T extending ZFC and all $A \in \text{spec}_{\text{MC}}(T)$ such that $\in_A \supseteq \in_{\Delta_0}$, $\text{MC}(T, A)$ is a model of all \in_A -axioms holding in some H_λ with λ regular (i.e. all ZFC axioms with the exception of powerset, and with replacement holding for all \in_A -formulae).

The following Lemma shows that existential substructures of an \in_{Δ_0} -model of set theory preserve a great deal of set theoretic concepts.

Lemma 3.3.1. *Let $\tau \supseteq \in_{\Delta_0}$ and $T \supseteq \text{ZF}_\tau^-$. Assume $\mathcal{N} \models T$. Let $\mathcal{M} \prec_1 \mathcal{N}$. Then:*

- (1) \mathcal{M} models $\text{Z}_{\Delta_0}^-$.
- (2) $\mathcal{M} \models \text{Rep}(\phi)$ for any existential τ -formula ϕ .
- (3) Furthermore if $T \ni \text{AC}$, $\mathcal{M} \models \text{AC}$.

Therefore if $R \supseteq \text{ZF}$ is an \in -theory and $A \subseteq \text{Form}_\tau \times 2$ is such that $\in_A \supseteq \in_{\Delta_0}$, $R + T_{\tau, A}$ admits a model companion R' , then R' extends ZF_A^- (or ZFC_A^- if $\text{AC} \in R$).

Proof.

- (1) All axioms of $\text{Z}_{\Delta_0}^-$ hold in \mathcal{N} and are reflected to \mathcal{M} , since $\mathcal{M} \prec_1 \mathcal{N}$ and those axioms are at most Π_2 -sentences of \in_{Δ_0} .

(2) Let $\phi(x, y, z_1, \dots, z_m)$ be an existential \in_A -formula such that

$$(16) \quad \mathcal{M} \models \forall x \left[x \in C \rightarrow \exists! y \phi(x, y, \vec{b}) \right].$$

By Lemma 2.4.2, we can extend τ to $\tau_1 = \tau \cup \{f_\phi\}$ and uniquely extend \mathcal{M}, \mathcal{N} to τ_1 -structures

$$\mathcal{M}^* \prec_1 \mathcal{N}^*$$

both satisfying

$$(17) \quad \forall z_1, \dots, z_m, x \\ \left[(\exists! y \phi(x, y, z_1, \dots, z_m) \rightarrow \phi(x, f_\phi(x, z_1, \dots, z_m), z_1, \dots, z_m)) \right. \\ \left. \wedge (\neg \exists! y \phi(x, y, z_1, \dots, z_m) \rightarrow f_\phi(x, z_1, \dots, z_m) = \emptyset) \right].$$

Since $\mathcal{N}^* \models \text{ZF}_\tau^-$, it is immediate to check that $\mathcal{N}^* \models \text{ZF}_{\tau_1}^-$. By replacement in \mathcal{N}^* applied to C, \vec{b}, ϕ ,

$$\mathcal{N}^* \models \exists G \left[\text{dom}(G) = C \wedge \forall x \in C \langle x, f_\phi(x, \vec{b}) \rangle \in G \right],$$

hence so does \mathcal{M}^* , since $\mathcal{M}^* \prec_1 \mathcal{N}^*$.

So pick $F \in \mathcal{M}$ such that

$$\mathcal{M}^* \models \text{dom}(F) = C \wedge \forall x \in C \langle x, f_\phi(x, \vec{b}) \rangle \in F.$$

By 17 in \mathcal{M}^* and by 16, we conclude that

$$\mathcal{M} \models \phi(a, F(a), \vec{b})$$

whenever $\mathcal{M} \models a \in C$.

Therefore F witnesses in \mathcal{M} the required instance of the replacement axiom for C, \vec{b}, ϕ .

(3) It is immediately checked that if $T \supseteq \text{ZFC}$, \mathcal{M} satisfies also the axiom of choice, as this is a Π_2 -sentence for \in_{Δ_0} which holds in \mathcal{N} and thus reflects to \mathcal{M} .

Finally if $R + T_{\tau, A}$ admits a model companion R' , R' is model complete; hence every \in_A -formula is R' -equivalent to an existential \in_A -formula. Since any model of R' is a Σ_1 -substructure of a model of R , the conclusion follows by the previous items.

The Lemma is proved in all its parts. \square

This proves Theorem 1.6(i).

3.4. The theory of H_{κ^+} as the model companion of set theory. In this section we prove Thm. 1.6(ii). To motivate the result we first show the following:

Lemma 3.4.1. *Assume $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$ is such that $(H_{\kappa^+}^{\mathcal{N}}, \tau^{\mathcal{N}}) \prec_1 (\mathcal{N}, \tau^{\mathcal{N}})$ whenever \mathcal{N} models $S \supseteq \text{ZFC}_\tau + \kappa$ is a cardinal. Then every S -ec structure satisfies*

$$\forall x \exists f (f : \kappa \rightarrow x \text{ is a surjection}).$$

In particular for any τ, S as in the Lemma, the S -ec models satisfy ZC_{Δ_0} , Replacement for existential τ -formulae, and the Π_2 -sentence stating that all sets have size at most κ , e.g. they provide a reasonable class of models describing a theory of H_{κ^+} .

Proof. Let \mathcal{M} be S -ec and $\mathcal{N} \supseteq \mathcal{M}$ be a model of S . Since $(H_{\kappa^+}^{\mathcal{N}}, \tau^{\mathcal{N}}) \prec_1 (\mathcal{N}, \tau^{\mathcal{N}})$, the two structures share the same Π_1 -theory (which is clearly Π_1 -complete); hence by Lemma 2.1.2, there is $\mathcal{P} \supseteq \mathcal{N}$ satisfying the theory of $(H_{\kappa^+}^{\mathcal{N}}, \tau^{\mathcal{N}})$, and in particular the universal theory of S and the Π_2 -sentence $\forall x \exists f (f : \kappa \rightarrow x \text{ is a surjection})$. Since \mathcal{M} is S -ec, the latter sentence reflects to \mathcal{M} and we are done. \square

We now give existence results stating that there are many signatures $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$ so that $\text{ZFC}_\tau + \kappa$ is a cardinal admits as AMC the theory

$$\text{ZFC}_\tau^- + (\kappa \text{ is a cardinal}) + \forall x \exists f (f : \kappa \rightarrow x \text{ is a surjection}).$$

3.4.1. *By-interpretability of the first order theory of H_{κ^+} with the first order theory of $\mathcal{P}(\kappa)$.* Let's compare the first order theory of the structure

$$(\mathcal{P}(\kappa), \in_{\Delta_0}^V)$$

with that of the \in -theory of H_{κ^+} in models (V, \in) of ZFC . We show that they are ZFC_{Δ_0} -provably by-interpretible with a by-interpretation translating H_{κ^+} in a Π_1 -definable subset of $\mathcal{P}(\kappa^2)$ (in signature \in_{Δ_0}) and the \in -relation into a Σ_1 -relation over this set (in signature \in_{Δ_0}). This result is the key to the proof of Thm. 1.6(i) and is just outlining the model theoretic consequences of the well-known fact that sets can be coded by well-founded extensional graphs.

Definition 3.4.2. Given $a \in H_{\kappa^+}$, $R \in \mathcal{P}(\kappa^2)$ codes a , if R codes a well-founded extensional relation on some $\alpha \leq \kappa$ with top element 0 so that the transitive collapse mapping of (α, R) maps 0 to a .

- WFE_κ is the set of $R \in \mathcal{P}(\kappa^2)$ which are a well founded extensional relation with domain $\alpha \leq \kappa$ and top element 0.
- $\text{Cod}_\kappa : \text{WFE}_\kappa \rightarrow H_{\kappa^+}$ is the map assigning a to R if and only if R codes a .

The following theorem shows that the structure (H_{κ^+}, \in) is interpreted by means of “imaginaries” in the structure $(\mathcal{P}(\kappa), \in_{\Delta_0}^V)$ by means of:

- a universal $\in_{\Delta_0} \cup \{\kappa\}$ -formula (with quantifiers ranging over subsets of $\kappa^{<\omega}$) defining a set $\text{WFE}_\kappa \subseteq \mathcal{P}(\kappa^2)$.
- an equivalence relation \cong_κ on WFE_κ defined by an existential $\in_{\Delta_0} \cup \{\kappa\}$ -formula (with quantifiers ranging over subsets of $\kappa^{<\omega}$)
- A binary relation E_κ on WFE_κ invariant under \cong_κ representing the \in -relation as the extension of an existential $\in_{\Delta_0} \cup \{\kappa\}$ -formula (with quantifiers ranging over subsets of $\kappa^{<\omega}$)²¹.

Notation 3.4.3. Recall Notation 3.1.1.

$$\phi_{\text{Card}}(x)$$

is the Π_1 -formula for \in_{Δ_0}

$$(x \text{ is a cardinal}) \wedge \omega \subseteq x$$

- Z_κ^- is the $\in_{\Delta_0} \cup \{\kappa\}$ -theory $Z_{\Delta_0}^- \cup \{\phi_{\text{Card}}(\kappa)\}$.
- Accordingly we define the $\in_{\Delta_0} \cup \{\kappa\}$ -theories $\text{ZF}_\kappa^-, \text{ZFC}_\kappa^-, \text{ZF}_\kappa, \text{ZFC}_\kappa$.

Theorem 3.4.4. Assume ZFC_κ^- . The following holds²²:

- (1) The map Cod_κ and WFE_κ are defined by $\{\in, \kappa\}$ -formulae which are $\Delta_1(\text{ZFC}_\kappa^-)$. Moreover $\text{Cod}_\kappa : \text{WFE}_\kappa \rightarrow H_{\kappa^+}$ is surjective (provably in ZFC_κ^-), and WFE_κ is defined by a universal $\in_{\Delta_0} \cup \{\kappa\}$ -formula with quantifiers ranging over subsets of $\kappa^{<\omega}$.

²¹See [10, Section 25] for proofs of the case $\kappa = \omega$; in particular the statement and proof of Lemma 25.25 and the proof of [10, Thm. 13.28] contain all ideas on which one can elaborate to draw the conclusions of Thm. 3.4.4. Note that the map $x \mapsto x^{<\omega}$ has a Δ_0 -graph in models of ZF^- . Therefore quantification over κ or over $\kappa^{<\omega}$ are the same modulo an \in_{Δ_0} -term. In the sequel we might be sloppy and identify at our convenience κ with $\kappa^{<\omega}$.

²²Many transitive supersets of H_{κ^+} are $\in_{\Delta_0} \cup \{\kappa\}$ -models of ZFC_κ^- for κ an infinite cardinal (see [12, Section IV.6]). To simplify notation we assume to have fixed a transitive $\in_{\Delta_0} \cup \{\kappa\}$ -model \mathcal{N} of ZFC_κ^- with domain $N \supseteq H_{\kappa^+}$. The reader can easily realize that all these statements holds for an arbitrary model \mathcal{N} of ZFC_κ^- replacing H_{κ^+} with its version according to \mathcal{N} .

- (2) There are existential $\in_{\Delta_0} \cup \{\kappa\}$ -formulae (with quantifiers ranging over subsets of $\kappa^{<\omega}$), $\phi_{\in}, \phi_{=}$ such that for all $R, S \in \text{WFE}_{\kappa}$, $\phi_{=}(R, S)$ if and only if $\text{Cod}_{\kappa}(R) = \text{Cod}_{\kappa}(S)$ and $\phi_{\in}(R, S)$ if and only if $\text{Cod}_{\kappa}(R) \in \text{Cod}_{\kappa}(S)$. In particular letting

$$E_{\kappa} = \{(R, S) \in \text{WFE}_{\kappa} : \phi_{\in}(R, S)\},$$

$$\cong_{\kappa=} = \{(R, S) \in \text{WFE}_{\kappa} : \phi_{=}(R, S)\},$$

\cong_{κ} is a ZFC_{κ}^{-} -provably definable equivalence relation, E_{κ} respects it, and

$$(\text{WFE}_{\kappa}/\cong_{\kappa}, E_{\kappa}/\cong_{\kappa})$$

is isomorphic to $(H_{\kappa+}, \in)$ via the map $[R] \mapsto \text{Cod}_{\kappa}(R)$.

Proof. A detailed proof requires a careful examination of the syntactic properties of Δ_0 -formulae, in line with the one carried in Kunen's [12, Chapter IV]. We outline the main ideas, following (as we already did) Kunen's book terminology for certain set theoretic operations on sets, functions and relations (such as $\text{dom}(f), \text{ran}(f), \text{Ext}(R)$, etc). To simplify the notation, we prove the results for a transitive ZFC^{-} -model (N, \in) which is then extended to a structure $(N, \in_{\Delta_0}^N, \kappa^N)$ which models ZFC_{κ}^{-} , and whose domain contains $H_{\kappa+}$. The reader can verify by itself that the argument is modular and works for any other model of ZFC_{κ}^{-} (transitive or ill-founded, containing the "true" $H_{\kappa+}$ or not).

- (1) This is proved in details in [12, Chapter IV]. To define WFE_{κ} by a universal $\in_{\Delta_0} \cup \{\kappa\}$ -property over subsets of κ and Cod_{κ} by a Δ_1 -property for $\in_{\Delta_0} \cup \{\kappa\}$ over $H_{\kappa+}$, we proceed as follows:

- R is an extensional relation with top element 0 is defined by the \in_{Δ_0} -atomic formula $\psi_{\text{EXT}}(R)$ $\text{ZF}_{\Delta_0}^{-}$ -provably equivalent to the Δ_0 -formula:

$$(18) \quad \begin{aligned} & (R \text{ is a binary relation}) \wedge (0 \in \text{Ext}(R)) \wedge \\ & \wedge \forall z, w \in \text{Ext}(R) [\forall u \in \text{Ext}(R) (u R z \leftrightarrow u R w) \rightarrow (z = w)] \wedge \\ & \wedge \forall z \in \text{Ext}(R) (0 \neq z \rightarrow \exists y \in \text{Ext}(R) z R y). \end{aligned}$$

- WFE_{κ} is defined by the universal $\in_{\Delta_0} \cup \{\kappa\}$ -formula $\phi_{\text{WFE}_{\kappa}}(R)$ (quantifying only over subsets of $\kappa^{<\omega}$)

$$\begin{aligned} & \psi_{\text{EXT}}(R) \wedge \\ & (\text{Ext}(R) \in \kappa \vee \text{Ext}(R) = \kappa) \wedge \\ & \wedge [\forall f (f : \omega \rightarrow \text{Ext}(R) \text{ is a function} \rightarrow \exists n \in \omega \neg (\langle f(n+1), f(n) \rangle \in R))]. \end{aligned}$$

- To define Cod_{κ} , consider the \in_{Δ_0} -atomic formula $\psi_{\text{Cod}}(G, R)$ provably equivalent to the \in_{Δ_0} -formula:

$$\begin{aligned} & \psi_{\text{EXT}}(R) \wedge \\ & \wedge (G \text{ is a function}) \wedge \\ & \wedge (\text{dom}(G) = \text{Ext}(R)) \wedge (\text{ran}(G) \text{ is transitive}) \wedge \\ & \wedge \forall \alpha, \beta \in \text{Ext}(R) [\alpha R \beta \leftrightarrow G(\alpha) \in G(\beta)]. \end{aligned}$$

Then $\text{Cod}_\kappa(R) = a$ can²³ be defined either by the existential \in_{Δ_0} -formula²⁴

$$\exists G (\psi_{\text{Cod}}(G, R) \wedge G(0) = a)$$

or by the universal \in_{Δ_0} -formula

$$\forall G (\psi_{\text{Cod}}(G, R) \rightarrow G(0) = a).$$

- (2) The equality relation in H_{κ^+} is transferred to the isomorphism relation between elements of WFE_κ : if X, Y are well-founded extensional on κ with a top-element, the Mostowski collapsing theorem entails that $\text{Cod}_\kappa(X) = \text{Cod}_\kappa(Y)$ if and only if $(\text{Ext}(R), R) \cong (\text{Ext}(S), S)$. Isomorphism of the two structures $(\text{Ext}(X), X) \cong (\text{Ext}(Y), Y)$ is expressed by the Σ_1 -formula for τ_κ :

$$\phi_=(X, Y) \equiv \exists f (f \text{ is a bijection of } \kappa \text{ onto } \kappa \text{ and } \alpha X \beta \text{ if and only if } f(\alpha) Y f(\beta)).$$

In particular we get that $\phi_=(X, Y)$ holds in H_{κ^+} for $X, Y \in \text{WFE}_\kappa$ if and only if $\text{Cod}_\kappa(X) = \text{Cod}_\kappa(Y)$.

Similarly one can express $\text{Cod}_\kappa(X) \in \text{Cod}_\kappa(Y)$ by the Σ_1 -property ϕ_\in in τ_κ stating that $(\text{Ext}(X), X)$ is isomorphic to $(\text{pred}_Y(\alpha), Y)$ for some $\alpha \in \kappa$ with $\alpha Y 0$, where $\text{pred}_Y(\alpha)$ is given by the elements of $\text{Ext}(Y)$ which are Y -connected by a finite path to α .

Moreover letting $\cong_\kappa \subseteq \text{WFE}_\kappa^2$ denote the isomorphism relation between elements of WFE_κ and $E_\kappa \subseteq \text{WFE}_\kappa^2$ denote the relation which translates into the \in -relation via Cod_κ , it is clear that \cong_κ is a congruence relation over E_κ , i.e.: if $X_0 \cong_\kappa X_1$ and $Y_0 \cong_\kappa Y_1$, $X_0 E_\kappa Y_0$ if and only if $X_1 E_\kappa Y_1$.

This gives that the structure $(\text{WFE}_\kappa / \cong_\kappa, E_\kappa / \cong_\kappa)$ is isomorphic to (H_{κ^+}, \in) via the map $[X] \mapsto \text{Cod}_\kappa(X)$ (where $\text{WFE}_\kappa / \cong_\kappa$ is the set of equivalence classes of \cong_κ and the quotient relation $[X] E_\kappa / \cong_\kappa [Y]$ holds if and only if $X E_\kappa Y$).

This isomorphism is defined via the map Cod_κ , which is by itself defined by a ZFC_κ^- -provably Δ_1 -property for $\in_{\Delta_0} \cup \{\kappa\}$.

The very definition of $\text{WFE}_\kappa, \cong_\kappa, E_\kappa$ show that

$$\text{WFE}_\kappa = \phi_{\text{WFE}_\kappa}^N,$$

$$\cong_\kappa = (\phi_{\text{WFE}_\kappa}(x) \wedge \phi_{\text{WFE}_\kappa}(y) \wedge \phi_=(x, y))^N,$$

$$E_\kappa = (\phi_{\text{WFE}_\kappa}(x) \wedge \phi_{\text{WFE}_\kappa}(y) \wedge \phi_\in(x, y))^N.$$

Note that we crucially use the axiom of choice to prove the surjectivity of Cod_κ on H_{κ^+} . \square

3.4.2. Model completeness for the theory of H_{κ^+} . The following definition isolates those signatures extending \in_{Δ_0} with predicates defining a family of subsets of $\bigcup_{n \in \omega} \mathcal{P}(\kappa)^n$ closed under finite unions, complementations, and projections.

Definition 3.4.5. Let τ be a signature extending $\in_{\Delta_0} \cup \{\kappa\}$ only with predicate symbols.

τ is κ -projectively closed for some τ -theory²⁵ $T \supseteq \text{ZFC}_\tau^-$ if the following holds:

(1)

$$T \models (\kappa \text{ is a cardinal})$$

²³Note that Cod_κ can be defined without any reference to κ . This reference appears once we decide to restrict the domain of Cod_κ to WFE_κ .

²⁴Given an R such that $\psi_{\text{EXT}}(R)$ holds, R is a well founded relation holds in a model of ZFC_κ^- if and only if Cod_κ is defined on R . In the theory ZFC_κ^- , WFE_κ can be defined using a universal property by a $\in_{\Delta_0} \cup \{\kappa\}$ -formula quantifying only over subsets of κ . On the other hand if we allow arbitrary quantification over elements of H_{κ^+} , we can express the well-foundedness of R also using the existential formula $\exists G \psi_{\text{Cod}_\kappa}(G, R)$. This is why WFE_κ is defined by a universal $\in_{\Delta_0} \cup \{\kappa\}$ -property in the structure $(\mathcal{P}(\kappa), \in_{\Delta_0}^V, \kappa)$, while the graph of Cod_κ can be defined by a Δ_1 -property for $\in_{\Delta_0} \cup \{\kappa\}$ in the structure $(H_{\kappa^+}, \in_{\Delta_0}^V, \kappa^V)$.

²⁵Recall Notation 3.1.1 for $\text{ZFC}_\tau, \text{ZF}_\tau^-, \dots$

(2)

$$T \models \forall x_0, \dots, x_n (R(x_0, \dots, x_n) \rightarrow \bigwedge_{i=0}^n x_i \subseteq \kappa)$$

for all predicate symbols $R \in \tau \setminus \in_{\Delta_0}$.(3) For all quantifier free τ -formulae ϕ there exists $R_\phi \in \tau \setminus \in_{\Delta_0}$ such that

$$T \models \forall x_0, \dots, x_n \left[\left(\bigwedge_{i=0}^n y_i = \text{Cod}_\kappa(x_i) \wedge \phi(x_0, \dots, x_n) \right) \leftrightarrow R_\phi(y_0, \dots, y_n) \right]$$

(4) For all $R, S \in \tau \setminus \in_{\Delta_0}$ there is some U also in $\tau \setminus \in_{\Delta_0}$ such that

$$T \models \forall \vec{x}, \vec{y} [(R(\vec{x}) \vee S(\vec{y})) \leftrightarrow U(\vec{x}, \vec{y})].$$

(5) For all $R, S \in \tau \setminus \in_{\Delta_0}$ there is some U also in $\tau \setminus \in_{\Delta_0}$ such that

$$T \models \forall \vec{x}, \vec{y} [(R(\vec{x}) \wedge S(\vec{y})) \leftrightarrow U(\vec{x}, \vec{y})].$$

(6) For all $R \in \tau \setminus \in_{\Delta_0}$ there is some U also in $\tau \setminus \in_{\Delta_0}$ such that

$$T \models \forall x_0, \dots, x_n \left[\bigwedge_{i=0}^n x_i \subseteq \kappa \rightarrow (R(x_0, \dots, x_n) \leftrightarrow \neg U(x_0, \dots, x_n)) \right]$$

(7) For all $R \in \tau \setminus \in_{\Delta_0}$ and $i = 0, \dots, n$ there is some U also in $\tau \setminus \in_{\Delta_0}$ such that

$$T \models \forall x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n [U(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \leftrightarrow \exists x_i (x_i \subseteq \kappa \wedge R(x_0, \dots, x_n))].$$

We just write *projectively closed* if $\kappa = \omega$.

The following is a trivial but fundamental remark:

Fact 3.4.6. *Let τ be a signature extending $\in_{\Delta_0} \cup \{\kappa\}$ only with predicate symbols. Axioms 1 to 7 are all Π_2 -sentences for τ .***Lemma 3.4.7.** *Let $\tau \supseteq \in_{\Delta_0}$, $T \supseteq \text{ZFC}_\tau$ be a τ -theory, and κ be a T -definable cardinal.**Assume ϕ is a τ -formula which defines in any model (V, τ^V) of T a transitive model $M_\phi \subseteq V$ of ZF_τ^- which contains $H_{\kappa^+}^V$. Let $A = \bar{A} \times \{0\}$, where*

$$\bar{A} = \left\{ \theta^{M_\phi}(x_0, \dots, x_n) \wedge \bigwedge_{i=0}^n x_i \subseteq \kappa : \theta \text{ a } \tau\text{-formula} \right\}.$$

*Then τ_A is κ -projectively closed for $T + T_{\tau, A}$.**Proof.* This is an almost immediate consequence of the fact that whenever (V, τ^V) models T , $M_\phi \supseteq H_{\kappa^+}^V$ is a transitive model of ZF_τ^- , hence such that $\text{Cod}_\kappa^V = \text{Cod}_\kappa^M$ and $\mathcal{P}(\kappa)^V = \mathcal{P}(\kappa)^M$. Using this fact together with the fact that (M_ϕ, τ^V) is a model of ZF_τ^- , we get that the collection of subsets of $\mathcal{P}(\kappa)^V$ definable in M_ϕ without (or even with) parameters defines a family which is closed under projections (e.g. Axiom 7), finite unions and complementations (e.g. Axioms 4, 5, 6), and satisfies also axiom 3.The conclusion follows. \square **Theorem 3.4.8.** *Let $\tau \supseteq \in_{\Delta_0}$ be a κ -projectively closed signature for some τ -theory $T \supseteq \text{ZFC}_\tau$.**Let S be the τ -theory extending ZFC_τ^- with the axioms 1, ..., 7 which follow from T and the axiom*

$$\forall x \exists f [f \text{ is a surjection} \wedge \text{dom}(f) = \kappa \wedge \text{ran}(f) = x].$$

Then S is model complete.

Proof. To simplify notation, we conform to the assumption of Thm. 3.4.4, i.e. we assume that the model (N, τ^N) of S on which we work is a transitive superstructure of H_{κ^+} .

The statement *every set has size at most κ* is satisfied by a \mathbf{ZF}^- -model (N, \in^N, κ^N) with $N \supseteq H_{\kappa^+}$ if and only if $N = H_{\kappa^+}$. From now on we proceed assuming this equality.

It suffices to show that for all τ -formulae $\phi(\vec{x})$

$$S \models \forall \vec{x} (\phi(\vec{x}) \leftrightarrow \psi_\phi(\vec{x})),$$

for some universal τ -formula ψ_ϕ .

By Axiom 2 applied to the atomic τ -formulae $x = x$, $x \in y$ and $x = y$, we obtain predicate symbols $R_{\mathbf{WFE}_\kappa}$, $R_=$, R_\in in $\tau \setminus \in_{\Delta_0}$ such that:

$$R_{\mathbf{WFE}_\kappa}^{H_{\kappa^+}} = \mathbf{WFE}_\kappa,$$

$$R_{x=y}^{H_{\kappa^+}} = \cong_{\kappa} = \{(X, Y) \in (\mathbf{WFE}_\kappa)^2 : \text{Cod}_\kappa(X) = \text{Cod}_\kappa(Y)\},$$

$$R_{x \in y}^{H_{\kappa^+}} = E_\kappa = \{(X, Y) \in (\mathbf{WFE}_\kappa)^2 : \text{Cod}_\kappa(X) \in \text{Cod}_\kappa(Y)\}.$$

Now by assumptions on τ , we get that for any quantifier free τ -formula $\phi(\vec{x})$, there is some predicate $R_\phi \in \tau \setminus \in_{\Delta_0}$ such that:

$$S \models R_\phi(x_0, \dots, x_n) \leftrightarrow \exists y_0, \dots, y_n \left[\phi(y_0, \dots, y_n) \wedge \bigwedge_{i=0}^n y_i = \text{Cod}_\kappa(x_i) \right].$$

Now we proceed to define $R_\psi(\vec{x})$ for any τ -formula ϕ letting:

- $R_{\psi \wedge \psi}(\vec{x})$ be $U(\vec{x})$ for the U given by Axiom 5 applied to R_ψ, R_ϕ ,
- $R_{\neg \psi}(\vec{x})$ be $U(\vec{x})$ for the U given by Axiom 6 applied to R_ψ, R_ϕ ,
- $R_{\exists y \psi(y, \vec{x})}(\vec{x})$ be $U(\vec{x})$ for the U given by Axiom 7 applied to R_ψ .

An easy induction on the complexity of the τ -formulae $R_\phi(\vec{x})$ gives that for any τ -definable subset X of $(H_{\kappa^+})^n$ which is the extension of some τ -formula $\phi(x_1, \dots, x_n)$

$$\{(Y_1, \dots, Y_n) \in (\mathbf{WFE}_\kappa)^n : (\text{Cod}_\kappa(Y_1), \dots, \text{Cod}_\kappa(Y_n)) \in X\} = R_\phi^{H_{\kappa^+}},$$

with the further property that $R_\phi^{H_{\kappa^+}} \subseteq (\mathbf{WFE}_\kappa)^n$ respects the \cong_{κ} -relation.

Then for any τ_A -formula $\phi(x_1, \dots, x_n)$ $(H_{\kappa^+}, \tau_A^{H_{\kappa^+}}) \models \phi(a_1, \dots, a_n)$ if and only if

$$(\mathbf{WFE}_\kappa / \cong_{\kappa}, E_\kappa / \cong_{\kappa}, R_\phi^{H_{\kappa^+}} : \phi \text{ a } \tau\text{-formula}) \models \phi([X_1], \dots, [X_n])$$

whenever $\text{Cod}_\kappa(X_i) = a_i$ for $i = 1, \dots, n$, if and only if

$$(H_{\kappa^+}, \tau^{H_{\kappa^+}}) \models \forall X_1, \dots, X_n \left[\left(\bigwedge_{i=1}^n R_{\mathbf{WFE}_\kappa}(X_i) \wedge \text{Cod}_\kappa(X_i) = a_i \right) \rightarrow R_\phi(X_1, \dots, X_n) \right].$$

Since this argument can be repeated verbatim for any model of S , we have proved the following:

Claim 2. *For any τ -formula $\phi(x_1, \dots, x_n)$, S proves*

$$\forall x_1, \dots, x_n [\phi(x_1, \dots, x_n) \leftrightarrow \forall y_1, \dots, y_n \left[\left(\bigwedge_{i=1}^n R_{\mathbf{WFE}_\kappa}(y_i) \wedge \text{Cod}_\kappa(y_i) = x_i \right) \rightarrow R_\phi(y_1, \dots, y_n) \right]].$$

But $\text{Cod}_\kappa(y) = x$ is expressible provably in S by the existential τ -formula:

$$\exists g [(g \text{ is a bijection}) \wedge \text{dom}(g) = \text{Ext}(y) \wedge (\text{ran}(g) \text{ is transitive}) \wedge g(\emptyset) = x].$$

Therefore

$$\forall y_1, \dots, y_n \left[\left(\bigwedge_{i=1}^n R_{\mathbf{WFE}_\kappa}(y_i) \wedge \text{Cod}_\kappa(y_i) = x_i \right) \rightarrow R_\phi(y_1, \dots, y_n) \right]$$

is a universal τ -formula, and we are done. \square

3.4.3. *Proof of Thm. 1.6(i).* Conforming to the notation of Thm. 1.6(i), it is clear that if T extends ZFC and κ is a T -definable cardinal

$$\bar{A}_\kappa = \left\{ \phi^{\mathcal{P}(\kappa)}(x_1, \dots, x_n) \wedge \bigwedge_{i=1}^n x_i \subseteq \kappa : \phi \text{ an } \in\text{-formula} \right\},$$

we get that

$$A_\kappa = \{(\phi, 0) : \phi \in \bar{A}_\kappa\} \cup D$$

(where D is the set used in Notation 3.1.1) is such that \in_{A_κ} is κ -projectively closed for ZFC.

Therefore the following result completes the proof of Thm. 1.6(i).

Theorem 3.4.9. *Let τ be κ -projectively closed for some τ -theory $T \supseteq \text{ZFC}_\tau$.*

Then T has an AMC T' in signature τ .

Proof. By Thm. 3.4.8, any τ -theory extending

$$\text{ZFC}_\tau^- + \text{every set has size at most } \kappa$$

with the axioms 1, ..., 7 which follow from T is model complete.

Now observe that for any τ -structure \mathcal{M} which models T ,

$$H_{\kappa^+}^{\mathcal{M}} \prec_1 \mathcal{M}.$$

This occurs since any new predicate symbol in $\tau \setminus \in_{\Delta_0}$ defines a subset of $\mathcal{P}(\kappa)^{<\omega}$, hence we can apply Lemma 3.1. Therefore:

- $H_{\kappa^+}^{\mathcal{M}}$ models ZFC_τ^- , by the standard properties of H_{κ^+} and the regularity of κ^+ (any function F with domain in H_{κ^+} and range contained in H_{κ^+} is in H_{κ^+}).
- Clearly $H_{\kappa^+}^{\mathcal{M}}$ models *every set has size at most κ* .
- $H_{\kappa^+}^{\mathcal{M}}$ models the axioms 1, ..., 7 which follow from T , as those are all expressible by Π_2 -sentences for τ holding in \mathcal{M} .

Therefore

$$T' = \{ \phi : H_{\kappa^+}^{\mathcal{M}} \models \phi, \mathcal{M} \models T \}$$

is model complete.

Clearly $T'_{\forall\forall\exists} = T_{\forall\forall\exists}$, since $H_{\kappa^+}^{\mathcal{M}} \prec_1 \mathcal{M}$ whenever $\mathcal{M} \models T$.

The theorem is proved. □

Remark 3.4.10. Note that there is no reason to expect that the family of models $\{H_{\kappa^+}^{\mathcal{M}} : \mathcal{M} \models T\}$ we used to define T' is an elementary class for τ .

By Lemma 3.4.7 for any \in -theory $S \supseteq \text{ZFC}$ and any S -definable cardinal κ there is plenty of $A \supseteq \text{Form}_\in \times 2$ such that \in_A is κ -projectively closed for $\text{ZFC}_{\tau_A}^-$.

However linking model companionship results for set theory to forcibility as we do in Theorem 1.7 requires much more care in the definition of the signature. We will pursue this matter in more details in the next sections.

3.5. Absolute codings of the continuum in type ω_2 . We prove Theorems 1.9 and 1.10 assuming Thm. 1.7. Here we really use that our focus is on the possible AMCs of set theory rather than on its possible model companions. Modulo certain complications, slightly stronger results can be proved replacing the notion of absolute model companionship with that of strong consistency hull (recall Def. 2.2.4). We prove these stronger result in the case of Thm. 1.9. We refrain to do it for Thm. 1.10.

We start with Thm. 1.9. We need the following:

Proposition 3.5.1. *Assume $\tau \supseteq \in_{\Delta_0}$ and $S \supseteq \text{ZFC}_\tau$ is a τ -theory such that $S + \neg\text{CH}$ is consistent. Then CH is not in $\text{SCH}(S)$.*

Proof. Note that $\neg\text{CH}$ as formalized in 1.4 is expressible in signature τ as the conjunction of the purely Π_2 -sentence for \in_{Δ_0} “every function with domain \aleph_1 is not surjective on the reals” with the the purely Σ_2 -sentence for \in_{Δ_0} “ \aleph_1 exists”.

Either $\text{SCH}(S)$ does not model “ \aleph_1 exists” in which case there is a model of $\text{SCH}(S) + \neg\text{CH}$ (e.g. a model of $\text{SCH}(S) +$ “all sets are countable”), or $\text{SCH}(S)$ models “ \aleph_1 exists”. In this case we note that CH holds in $\text{SCH}(S)$ if and only if it $\text{SCH}(S)$ models the Σ_2 -sentence ψ for \in_{Δ_0} stating that “there is a bijection f with domain ω_1 and whose range contains all subsets of ω ”, and also that $S + \neg\text{CH}$ is consistent and logically equivalent to $S + \neg\psi$. By Fact 2.2.5 we can find a model of $\text{SCH}(S) + \neg\psi$, hence also in this case $\text{SCH}(S)$ does not model CH . \square

We can now prove Thm. 1.9:

Proof. 1 is immediate by the previous proposition, since the AMC of a theory S overlaps with its strong consistency hull when the model companion of S exists. 2 follows by part (b) of Thm. 1.7. More precisely: let us choose B so that \in_B is the signature $\in_{\text{NS}_{\omega_1, \mathcal{A}}}$ introduced in Notation 5.0.3 (it works for Thm. 1.7 by Thm. 5.4). Now $((x := \aleph_1), 1) \in B$ ensures that there is a constant κ of $\in_{\text{NS}_{\omega_1, \mathcal{A}}}$ such that $\kappa := \aleph_1$ is a boolean combination of universal \in_B -sentences true in $S \cup \{\text{Ax}_{\phi}^i : (\phi, i) \in B\}$; the latter implies in pure first order logic “ \aleph_1 exists”. Now all \in_B -models \mathcal{M} of $S + \text{MM}^{++} + \{\text{Ax}_{\phi}^i : (\phi, i) \in B\}$ are such that the Π_2 -sentence for \in_{Δ_0} “every function with domain \aleph_1 is not surjective on the reals” and the boolean combination of universal \in_B -sentences $\kappa := \aleph_1$ both hold in $H_{\aleph_2}^{\mathcal{M}}$; hence their conjunction is an axiom of $\text{AMC}(S, B)$, and $\neg\text{CH}$ as formalized in 1.4 is a consequence in pure first order logic of this axiom. \square

The rest of this section is dedicated to the proof of Thm. 1.10.

Definition 3.5.2. Let $\psi_{\aleph_1}(x)$ be the boolean combination of universal formulae for \in_{Δ_0} whose unique solution is the first uncountable cardinal²⁶. $c : [\omega_1]^2 \rightarrow \omega$ is a *ladder system on ω_1* if for every α limit point of ω_1 $c \upharpoonright \{\alpha\} \times \alpha \rightarrow \omega$ is surjective and monotone increasing.

$A : \omega_1 \rightarrow \omega$ is a *partition of ω_1 in countably many stationary sets* if $A^{-1}[n]$ is stationary for all $n \in \omega$ and A is surjective.

Fact 3.5.3. *There is a Δ_0 -formula $\psi_{\text{ladder}}(x, y)$ such that any solution of $\psi_{\text{ladder}}(x, a)$ in any $\in_{\Delta_0} \cup \{a\}$ -model of*

$$\text{ZF}_{\Delta_0} + \psi_{\aleph_1}(a)$$

defines a ladder system on ω_1 according to the model.

Accordingly there is a Π_1 -formula $\psi_{\text{countpartstat}}(x, y)$ for \in_{Δ_0} such that any solution of $\psi_{\text{countpartstat}}(x, a)$ in any model of

$$\text{ZF}_{\Delta_0} + \psi_{\aleph_1}(a)$$

defines a partition of ω_1 in countably many stationary sets according to the model.

Furthermore

$$\text{ZFC}_{\Delta_0}^- + \psi_{\aleph_1}(a) \models \exists z \psi_{\text{ladder}}(z, a) \wedge \exists y \psi_{\text{countpartstat}}(y, a).$$

Theorem 3.5.4 (Moore [16]). *Let T be the $\in_{\Delta_0} \cup \{\omega_1, c, A\}$ -theory*

$$\text{ZFC}_{\Delta_0}^- + \psi_{\aleph_1}(\omega_1) + \psi_{\text{countpartstat}}(A, \omega_1) + \psi_{\text{ladder}}(c, \omega_1).$$

There is an \in_{Δ_0} -formula²⁷ $\psi_{\text{Moore}}(x, y, u_0, u_1, z_0, z_1, z_2)$ such that in any $\in_{\Delta_0} \cup \{\omega_1, c, A\}$ -model \mathcal{M} of T :

²⁶Previously we noted this formula by $(x := \aleph_1)$ however for the sake of readability, we introduce below a constant symbol ω_1 to denote the least uncountable cardinal and we prefer to write $\psi_{\aleph_1}(\omega_1)$ rather than $(\omega_1 := \aleph_1)$, other occasions showing that this new convention is convenient will arise along this section.

²⁷ $\psi_{\text{Moore}}(r, \alpha, f, C, \omega_1, c, A)$ states that $f : \omega_1 \rightarrow \alpha$ is a surjection, C is a club subset of ω_1 and for all $\alpha \in C$, if $A(\alpha) = n$, then for eventually all $\beta \in \alpha \cap C$, $c(\beta, \alpha) < c(\text{otp}(f[\beta]), \text{otp}(f[\alpha]))$ if and only if $r(n) = 1$.

(a) If

$$\mathcal{M} \models \exists r \subseteq \omega \psi_{\text{Moore}}(r, \alpha, f, C, \omega_1, c, A),$$

then

$$\mathcal{M} \models (\alpha \text{ is an ordinal}) \wedge (f : \alpha \rightarrow \omega_1 \text{ is an injection}) \wedge (C \subseteq \omega_1) \text{ is a club.}$$

(b) If

$$\mathcal{M} \models r, s \subseteq \omega \wedge \exists h (h : \alpha \rightarrow \omega_1 \text{ is an injection}),$$

then

$$\mathcal{M} \models \forall f, g, C, D [(\psi_{\text{Moore}}(s, \alpha, f, C, \omega_1, c, A) \wedge \psi_{\text{Moore}}(r, \alpha, g, D, \omega_1, c, A)) \rightarrow r = s].$$

(c) $T + \text{ZFC}_{\Delta_0}$ proves that

$$\forall r \subseteq \omega \exists \alpha, f, C \psi_{\text{Moore}}(r, \alpha, f, C, \omega_1, c, A)$$

is forcible by a proper forcing.

By item (c) of the Theorem (and Thm. 4.1), if S is $T + \text{ZFC}_{\Delta_0} +$ there are class many Woodin cardinals

$$\forall r \subseteq \omega \exists \alpha \psi_{\text{Moore}}(r, \alpha, \omega_1, c, A)$$

holds in a model of $S + R_{\forall}$ for any consistent $\in_{\Delta_0} \cup \{\omega_1, c, A\}$ -theory R extending S .

Corollary 3.5.5. Let θ_{Moore} be the Π_2 -sentence for \in_{Δ_0}

$$\forall x, y, z [(\psi_{\aleph_1}(x) \wedge \psi_{\text{ladder}}(y, x) \wedge \psi_{\text{countpartstat}}(z, x)) \rightarrow \forall r \subseteq \omega \exists \alpha, f, C \psi_{\text{Moore}}(r, \alpha, f, C, x, y, z)].$$

Then the following holds:

(1) Assume $S \supseteq \text{ZFC}$ and $A \in \text{spec}_{\text{AMC}}(S)$ are such that

$$\exists x \psi_{\aleph_1}(x), \theta_{\text{Moore}}$$

are both in $\text{AMC}(S, A)$. Let \mathcal{M} be a model of $\text{AMC}(S, A)$ such that $\psi_{\text{countpartstat}}(a, \omega_1^{\mathcal{M}})$, $\psi_{\text{ladder}}(c, \omega_1^{\mathcal{M}})$ holds in \mathcal{M} . Then $\exists f, C \psi_{\text{Moore}}(x, y, f, C, \omega_1^{\mathcal{M}}, c, a)$ defines the graph of a surjection of $\omega_2^{\mathcal{M}}$ onto $\mathcal{P}(\omega)^{\mathcal{M}}$.

(2) There is at least one (recursive set) $B \subseteq \text{Form}_{\in} \times 2$ with $\in_B \supseteq \in_{\Delta_0}$ such that for any R extending

$\text{ZFC} +$ there are class many Woodin cardinals and a supercompact cardinal,

$B \in \text{spec}_{\text{AMC}}(R)$ and

$$\exists x \psi_{\aleph_1}(x), \theta_{\text{Moore}}$$

are both in $\text{AMC}(R, B)$.

(3) For any $R \supseteq \text{ZFC}$ and $A \in \text{spec}_{\text{AMC}}(R)$ if

$$\exists x \psi_{\aleph_2}(x), \theta_{\text{Moore}}$$

are both in $\text{AMC}(R, A)$, then $\text{AMC}(R, A) \models 2^{\aleph_0} \leq \aleph_2$.

(4) Assume $R \supseteq \text{ZFC}$ and $A \in \text{spec}_{\text{AMC}}(R)$ are such that $(\psi_{\aleph_2}(x), 1) \in A$, $\in_A \supseteq \in_{\Delta_0}$, and

$$\text{ZFC} + (R + T_{\in, A})_{\forall \forall \exists} + \theta_{\text{Moore}}$$

is consistent.

Then $2^{\aleph_0} > \aleph_2 \notin \text{AMC}(R, A)$.

Proof.

1: See the last item and adjust the proof from it.

2: By Thm. 1.7.

3: See the proof below and observe that the argument which below is given for just one \mathcal{M} which models $\text{AMC}(R, A)$, in this case can be repeated for all models of $\text{AMC}(R, A)$.

4: Since $\text{AMC}(R, A)$ is the AMC of $R + T_{\in, A}$, we can find $\mathcal{M} \prec_1 \mathcal{N}$ with \mathcal{N} an \in_A -model of

$$T_0 = \text{ZFC} + (R + T_{\in, A})_{\forall\forall\exists} + \theta_{\text{Moore}}$$

and \mathcal{M} an \in_A -model of $\text{AMC}(R, A)$. Then

$$\mathcal{M} \models \theta_{\text{Moore}} + \text{ZFC}_A^-.$$

Fix $a, b, c, A \in \mathcal{M}$ such that \mathcal{M} models:

- $\psi_{\aleph_1}(a)$,
- $\psi_{\aleph_2}(b)$,
- $\psi_{\text{ladder}}(c, a)$,
- $\psi_{\text{countpartstat}}(A, a)$.

Note that \mathcal{N} models all of the above formulae with the same parameters, since $\mathcal{M} \prec_1 \mathcal{N}$.

Let T be the $\in_A \cup \{a, b\}$ -theory

$$T_0 + \psi_{\aleph_1}(a) + \psi_{\aleph_2}(b).$$

By Lemma 2.4.2 (noting that a, b are constant symbols interpreted according to the axioms $\text{Ax}_{\psi_{\aleph_i}}^1$ for $i = 0, 1$) \mathcal{M} and \mathcal{N} admits unique extensions to models $\mathcal{M}^*, \mathcal{N}^*$ of T such that $\mathcal{M}^* \prec_1 \mathcal{N}^*$.

This gives that

$$(19) \quad \mathcal{M}^* \models \forall r \subseteq \omega \exists \alpha \in b \exists f, C \psi_{\text{Moore}}(r, \alpha, f, C, a, c, A).$$

By Lemma 3.3.1 Replacement for $\in_A \cup \{a, b\}$ -formulae holds in \mathcal{M}^* . By Thm. 3.5.4(b), \mathcal{M} models

$$(20) \quad \forall \alpha \in b \exists! r \subseteq \omega [\exists f, C \psi_{\text{Moore}}(r, \alpha, f, C, a, c, A) \vee (\neg \exists s \subseteq \omega, f, C \psi_{\text{Moore}}(s, \alpha, f, C, a, c, A) \wedge r = \emptyset)]$$

Applying it to the existential $\in_A \cup \{a, b\}$ -formula $\exists f, C \psi_{\text{Moore}}(x, y, f, C, a, v, w)$ with v, w replaced by parameters c, A , we get an $F \in \mathcal{M}$ which (by 19 and 20) is a function with domain

$$\{\alpha \in \mathcal{M} : \mathcal{M} \models \alpha \in b\}$$

and range exactly given by

$$\{r \in \mathcal{M} : \mathcal{M} \models r \subseteq \omega\}.$$

Hence $\mathcal{P}(\omega)$ according to \mathcal{M} is in \mathcal{M} the set $\text{ran}(F)$, which is the surjective image (via F) of the second uncountable cardinal according to \mathcal{M} . Therefore $2^{\aleph_0} \leq \aleph_2$ holds in \mathcal{M} as witnessed by \mathcal{F} . □

Remark 3.5.6. We expect to be able to prove a result of a similar vein of Cor. 3.5.5 for any $A \subseteq \text{Form}_{\in} \times 2$ such that $\in_A \supseteq \in_{\Delta_0}$ and replacing $\text{AMC}(S, A)$ (which may not exist for many such A) with the strong consistency hull of $S + T_{\in, A}$. However the proof becomes rather intricate since we must check that the needed instances of replacement hold in the required models of $\text{SCH}(S + T_{\in, A})$. Replacement for Σ_1 -formulae holds in such structures. One then should argue that all instances of replacement used in the above proof are on Σ_1 -formulae.

4. GENERIC INVARIANCE RESULTS FOR SIGNATURES OF SECOND AND THIRD ORDER ARITHMETIC

We collect here generic absoluteness results needed to prove Thm. 1.7. We prove all these results working in “standard” models of ZFC, i.e. we assume the models are well-founded. This is a practice we already adopted in Section 3.4. We leave to the reader to remove this unnecessary assumption.

4.1. Universally Baire sets and generic absoluteness for second order number theory. We recall here the properties of universally Baire sets and the generic absoluteness results for second order number theory we need to prove Thm. 1.7.

4.1.1. *Universally Baire sets.* Assuming large cardinals, there is a very large sample of projectively closed families of subsets of $\mathcal{P}(\omega)$ which are “simple”, hence it is natural to consider elements of these families as atomic predicates.

The exact definition of what is meant by a “simple” subset of 2^ω is captured by the notion of universally Baire set.

Given a topological space (X, τ) , $A \subseteq X$ is nowhere dense if its closure has a dense complement, meager if it is the countable union of nowhere dense sets, with the Baire property if it has meager symmetric difference with an open set. Recall that (X, τ) is Polish if τ is a completely metrizable, separable topology on X .

Definition 4.1.1. (Feng, Magidor, Woodin) Given a compact Polish space (X, τ) , $A \subseteq X$ is *universally Baire* if for every compact Hausdorff space (Y, σ) and every continuous $f : Y \rightarrow X$ we have that $f^{-1}[A]$ has the Baire property in Y .

UB denotes the family of universally Baire subsets of X for some compact Polish space X .

We adopt the convention that UB denotes the class of universally Baire sets and of all elements of $\bigcup_{n \in \omega+1} (2^\omega)^n$ (since the singleton of such elements are universally Baire sets).

The list below outlines three simple examples of projectively closed families of universally Baire sets containing 2^ω .

Remark 4.1.2 (Woodin). Let:

- T_0 be the \in_{Δ_0} -theory $\text{ZFC}_{\Delta_0} + \text{there are infinitely many Woodin cardinals and a measurable above}$;
- T_1 be the \in_{Δ_0} -theory $\text{ZFC}_{\Delta_0} + \text{there are class many Woodin cardinals}$;
- T_2 be the \in_{Δ_0} -theory $\text{ZFC}_{\Delta_0} + \text{there are class many Woodin cardinals which are a limit of Woodin cardinals}$.

The following holds:

- (A) [14, Thm. 3.1.12, Thm. 3.1.19] Assume (V, \in_{Δ_0}) models T_0 . Then every projective subset of 2^ω is universally Baire.
- (B) [14, Thm. 3.3.3, Thm. 3.3.5, Thm. 3.3.6, Thm. 3.3.8, Thm. 3.3.13, Thm. 3.3.14] Assume $(V, \in_{\Delta_0}) \models T_1$. Then UB is projectively closed.
- (C) (Woodin, Unpublished) Assume $(V, \in_{\Delta_0}) \models T_2$. Then the family of subsets of 2^ω which are definable in $L(\text{UB})$ consists of universally Baire sets.

We now list some standard facts about universally Baire sets we will need:

- (i) [10, Thm. 32.22] $A \subseteq 2^\omega$ is universally Baire if and only if for each forcing notion P there are trees T_A, S_A on $2 \times \delta$ for some $\delta > |P|$ such that $A = p[[T_A]]$ (where $p : (2 \times \delta)^\omega \rightarrow 2^\omega$ denotes the projection on the first component and $[T]$ denotes the body of the tree T), and

$$P \Vdash T_A \text{ and } S_A \text{ project to complements,}$$

by this meaning that for all G V -generic for P

$$V[G] \models (p[[T_A]] \cap p[[S_A]] = \emptyset) \wedge (p[[T_A]] \cup p[[S_A]] = (2^\omega)^{V[G]})$$

- (ii) Any two Polish spaces X, Y of the same cardinality are Borel isomorphic [11, Thm. 15.6].
- (iii) Any Polish space is Borel isomorphic to a Borel subset of $[0; 1]^\omega$ [11, Thm. 4.14], hence also to a Borel subset of 2^ω (by the previous item).

- (iv) Given $\phi : \mathbb{N} \rightarrow \mathbb{N}$, $\prod_{n \in \omega} 2^{\phi(n)}$ is compact and Polish (it is actually homeomorphic to the union of 2^ω with a countable Hausdorff space) [11, Thm. 6.4, Thm. 7.4].

Hence it is not restrictive to focus just on universally Baire subsets of 2^ω and of its countable products, which is what we will do in the sequel.

Notation 4.1.3. Given G a V -generic filter for some forcing $P \in V$, $A \in \text{UB}^{V[G]}$ and H $V[G]$ -generic filter for some forcing $Q \in V[G]$,

$$A^{V[G][H]} = \left\{ r \in (2^\omega)^{V[G][H]} : V[G][H] \models r \in p[[T_A]] \right\},$$

where $(T_A, S_A) \in V[G]$ is any pair of trees as given in item (i) above such that $p[[T_A]] = A$ holds in $V[G]$, and (T_A, S_A) project to complements in $V[G][H]$.

4.2. Generic absoluteness for second order number theory. The version of generic absoluteness for second order number theory we need in this paper follows readily from [14, Thm. 3.1.2] and the assumptions that there exists class many Woodin limits of Woodin. The theorem below reduces these large cardinal assumptions to the existence of class many Woodin cardinals.

Theorem 4.2.1. *Assume in V there are class many Woodin cardinals. Let $\mathcal{A} \in V$ be a family of universally Baire sets of V , and G be V -generic for some forcing notion $P \in V$. Then*

$$(H_{\omega_1}, \in, A : A \in \mathcal{A}) \prec (H_{\omega_1}^{V[G]}, \in, A^{V[G]} : A \in \mathcal{A}).$$

The reader can find a proof in the author's webpage, it is an improvement of [23, Thm. 3.1].

It is now convenient to reformulate projective closure in a semantic way which is handy when dealing with a fixed complete first order axiomatization of set theory.

Definition 4.2.2. Let $\mathcal{A} \subseteq \bigcup_{n \in \omega} \mathcal{P}(\omega)^n$. \mathcal{A} is H_{ω_1} -closed if any definable subset of $\mathcal{P}(\omega)^n$ for some $n \in \omega$ in the structure

$$(H_{\omega_1}, \in, U : U \in \mathcal{A})$$

is in \mathcal{A} .

Given a family $\mathcal{X} \subseteq \bigcup_{n \in \omega} \mathcal{P}(\omega)^n$ its projective closure $\text{PC}(\mathcal{X})$ is the smallest H_{ω_1} -closed family of subsets of $\bigcup_{n \in \omega} \mathcal{P}(\omega)^n$ containing \mathcal{X} .

It is immediate to check that if T is the theory of (V, \in) and \mathcal{A} is a family of universally Baire subsets of V , \mathcal{A} is projectively closed for T for the signature $\in_{\Delta_0} \cup \mathcal{A}$ (according to Def. 3.4.5) if and only if it is H_{ω_1} -closed.

We get the following:

Corollary 4.2.3. *Assume (V, \in) models ZFC+there are class many Woodin cardinals. Let $\mathcal{A} \subseteq \text{UB}^V$ be H_{ω_1} -closed and $\in_{\mathcal{A}} = \in_{\Delta_0} \cup \mathcal{A}$ be the signature in which each element of \mathcal{A} contained in $\mathcal{P}(\omega)^k$ is a predicate symbol of arity k . Then for any G V -generic for some forcing $P \in V$ the $\in_{\mathcal{A}}$ -theory of $H_{\omega_1}^V$ is the AMC of the $\in_{\mathcal{A}}$ -theory of $V[G]$ and $\{A^{V[G]} : A \in \mathcal{A}\}$ is $H_{\omega_1}^{V[G]}$ -closed.*

Proof. The assumptions grant that

$$(H_{\omega_1}^V, \in_{\Delta_0}^V, A : A \in \mathcal{A}) \prec (H_{\omega_1}^{V[G]}, \in_{\Delta_0}^{V[G]}, A^{V[G]} : A \in \mathcal{A}) \prec_1 (V[G], \in_{\Delta_0}^{V[G]}, A : A^{V[G]} \in \mathcal{A})$$

(by Thm. 4.2.1 and by Lemma 3.1 applied in $V[G]$). Now the theory of $H_{\omega_1}^V$ in signature $\in_{\mathcal{A}}$ is complete and model complete, and is also the $\in_{\mathcal{A}}$ -theory of $H_{\omega_1}^{V[G]}$. We conclude that it is the AMC of the $\in_{\mathcal{A}}$ -theory of $V[G]$. It is also easy to check that $\{A^{V[G]} : A \in \mathcal{A}\}$ is $H_{\omega_1}^{V[G]}$ -closed. \square

4.3. Generic invariance of the universal fragment of the theory of V with predicates for the non-stationary ideal and for universally Baire sets. The results of this section are the key to establish Thm. 1.7. The proofs require some familiarity with the basics of the \mathbb{P}_{\max} -technology and with Woodin's stationary tower forcing.

Notation 4.3.1.

- $\in_{\mathbf{NS}_{\omega_1}}$ is the signature $\in_{\Delta_0} \cup \{\omega_1\} \cup \{\mathbf{NS}_{\omega_1}\}$ with ω_1 a constant symbol, \mathbf{NS}_{ω_1} a unary predicate symbol.
- $T_{\mathbf{NS}_{\omega_1}}$ is the $\in_{\mathbf{NS}_{\omega_1}}$ -theory given by T_{Δ_0} together with the axioms

ω_1 is the first uncountable cardinal,

$$\forall x [(x \subseteq \omega_1 \text{ is non-stationary}) \leftrightarrow \mathbf{NS}_{\omega_1}(x)].$$

- $\text{ZFC}_{\mathbf{NS}_{\omega_1}}^-$ is the $\in_{\mathbf{NS}_{\omega_1}}$ -theory

$$\text{ZFC}_{\Delta_0}^- + T_{\mathbf{NS}_{\omega_1}}.$$

- Accordingly we define $\text{ZFC}_{\mathbf{NS}_{\omega_1}}$.

Clearly the above axioms are of the form Ax_{ϕ}^0 for ϕ the formula defining the non-stationary ideal on ω_1 , and Ax_{ϕ}^1 for ϕ the formula defining the first uncountable cardinal. Furthermore the above axioms are Π_2 -sentences of the relevant signature.

Theorem 4.1. *Assume (V, \in) models $\text{ZFC}+$ there are class many Woodin cardinals. Let $\phi_1(\vec{x}_1), \dots, \phi_k(\vec{x}_k), \psi_1(\vec{x}_1, y), \dots, \psi_n(\vec{x}_n, y)$ be \in -formulae which are in $\Delta_1(\text{ZFC}^-)$ and such that*

$$\text{ZFC}^- \models \forall \vec{x} \exists ! y \psi_i(\vec{x}_i, y)$$

for $i = 1, \dots, n$. Then the Π_1 -theory of V for the language

$$\in_{\mathbf{NS}_{\omega_1}} \cup \text{UB}^V \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$$

is invariant under set sized forcings²⁸.

Asperó and Veličkovič provided the following basic counterexample to the conclusion of the theorem if large cardinal assumptions are dropped.

Remark 4.3.2. Let $\phi(y)$ be the Δ_1 -property in $\in_{\mathbf{NS}_{\omega_1}}$

$$\exists y (y = \omega_1 \wedge L_{y+1} \models y = \omega_1).$$

Then L models this property, while the property fails in any forcing extension of L which collapses ω_1^L to become countable.

Remark 4.3.3. Note that for any T extending $\text{ZFC}+$ there are class many Woodin cardinals and any signature τ extending \in by predicates and function symbols as prescribed by Thm. 4.1, we obtain that a Π_2 -sentence ψ for τ such that $\psi^{H_{\aleph_2}}$ is forcible over any model of T is in the strong consistency hull of T , hence it is realized in any T -ec model. In particular the Kaiser hull of $\text{ZFC}_{\mathbf{NS}_{\omega_1}}+$ there are class many Woodin cardinals in signature $\in_{\mathbf{NS}_{\omega_1}}$ contains $\neg\text{CH}$ and the definable version of $2^{\aleph_0} = \aleph_2$ implied by θ_{Moore} .

To see that this is the case note that:

- If V is a model of T , R is the τ -theory of V and G is V -generic for some forcing such that ψ holds in $H_{\aleph_2}^{V[G]}$, ψ is consistent with $R_{\forall\exists}$ since $H_{\aleph_2}^{V[G]}$ is Σ_1 -elementary with respect to $V[G]$ in signature τ and V and $V[G]$ share the same Π_1 -theory for τ . We conclude that such a ψ is strongly consistent with T .

²⁸Here we consider any $A \subseteq (2^\omega)^k$ in UB^V as a predicate symbol of arity k and we interpret R_{ϕ_i} and f_{ψ_j} in V and $V[G]$ using suitable axioms Ax_{θ}^i for (θ, i) .

- $\neg\text{CH}$ and the Π_2 -sentence θ_{Moore} are Π_2 -sentences in signature $\in_{\Delta_0} \cup \{\omega_1\}$. The theorem applies to $\text{ZFC}_{\Delta_0} + \text{there are class many Woodin cardinals}$ enriched with the sentence ω_1 is the first uncountable cardinal in signature $\in_{\Delta_0} \cup \{\omega_1\}$.

It will be much harder to find a signature τ extending $\in_{\text{NS}_{\omega_1}}$ with recursively many predicates and function symbols as prescribed Thm. 4.1 and such that set theory enriched with large cardinals as axiomatized in this signature admits an AMC.

In order to prove the Theorem we need to recall some basic terminology and facts about iterations of countable structures.

4.3.1. Generic iterations of countable structures.

Definition 4.3.4. [13, Def. 1.2] Let M be a transitive countable model of ZFC. Let γ be an ordinal less than or equal to ω_1 . An iteration \mathcal{J} of M of length γ consists of models $\langle M_\alpha : \alpha \leq \gamma \rangle$, sets $\langle G_\alpha : \alpha < \gamma \rangle$ and a commuting family of elementary embeddings

$$\langle j_{\alpha\beta} : M_\alpha \rightarrow M_\beta : \alpha \leq \beta \leq \gamma \rangle$$

such that:

- $M_0 = M$,
- each G_α is an M_α -generic filter for $(\mathcal{P}(\omega_1) / \text{NS}_{\omega_1})^{M_\alpha}$,
- each $j_{\alpha\alpha}$ is the identity mapping,
- each $j_{\alpha\alpha+1}$ is the ultrapower embedding induced by G_α ,
- for each limit ordinal $\beta \leq \gamma$, M_β is the direct limit of the system $\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\}$, and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.

We adopt the convention to denote an iteration \mathcal{J} just by $\langle j_{\alpha\beta} : \alpha \leq \beta \leq \gamma \rangle$, we also stipulate that if X denotes the domain of $j_{0\alpha}$, X_α or $j_{0\alpha}(X)$ will denote the domain of $j_{\alpha\beta}$ for any $\alpha \leq \beta \leq \gamma$.

Definition 4.3.5. Let A be a universally Baire sets of reals. M is A -iterable if:

- (1) M is transitive and such that $H_{\omega_1}^M$ is countable.
- (2) $M \models \text{ZFC} + \text{NS}_{\omega_1}$ is precipitous.
- (3) Any iteration

$$\{j_{\alpha\beta} : \alpha \leq \beta \leq \gamma\}$$

of M is well founded and such that $A \cap M_\beta = j_{\alpha\beta}(A \cap M_0)$ for all $\beta \leq \gamma$.

4.3.2. Proof of Theorem 4.1.

Proof. Let ϕ be a Π_1 -sentence for

$$\in_{\text{NS}_{\omega_1}} \cup \text{UB}^V \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$$

as in the theorem.

Assume ϕ holds in V but for some forcing notion P , ϕ fails in $V[h]$ with h V -generic for P . By forcing over $V[h]$ with the appropriate stationary set preserving (in $V[h]$) forcing notion (using a Woodin cardinal γ of $V[h]$), we may assume that $V[h]$ is extended to a generic extension $V[g]$ such that $V[g]$ models NS_{ω_1} is saturated²⁹. Since $V[g]$ is an extension of $V[h]$ by a stationary set preserving forcing and there are in $V[h]$ class many Woodin cardinals, we get that $V[h] \sqsubseteq V[g]$ with respect to the signature $\in_{\text{NS}_{\omega_1}} \cup \text{UB}^V \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$ (we can use again Fact 3.1.7 to handle the relations and function symbols $G_i, R_{\phi_j}, f_{\psi_l}$ for $i = 1, \dots, 10, j = 1, \dots, k, l = 1, \dots, n$). Since Σ_1 -properties are upward absolute and $\neg\phi$ holds in $V[h]$, ϕ fails in $V[g]$ as well.

Let δ be inaccessible in $V[g]$ and let $\gamma > \delta$ be a Woodin cardinal.

²⁹A result of Shelah whose outline can be found in [17, Chapter XVI], or [24], or in an handout of Schindler available on his webpage.

Let G be V -generic for $\mathcal{T}_\gamma^{\omega_1}$ (the countable tower $\mathbb{Q}_{<\gamma}$ according to [14, Section 2.7]) and such that $g \in V[G]$. Let $j_G : V \rightarrow \text{Ult}(V, G)$ be the induced ultrapower embedding.

Now remark that $V_\delta[g] \in \text{Ult}(V, G)$ is $B^{V[G]}$ -iterable for all $B \in \text{UB}^V$ (since $V_\eta[g] \in \text{Ult}(V, G)$ for all $\eta < \gamma$, and this suffices to check that $V_\delta[g]$ is $B^{V[G]}$ -iterable for all $B \in \text{UB}^V$, see [13, Thm. 4.10]).

By [13, Lemma 2.8] applied in $\text{Ult}(V, G)$, there exists in $\text{Ult}(V, G)$ an iteration $\mathcal{J} = \left\{ j_{\alpha\beta} : \alpha \leq \beta \leq \gamma = \omega_1^{\text{Ult}(V, G)} \right\}$ of $V_\delta[g]$ such that $\text{NS}_{\omega_1}^{X_\gamma} = \text{NS}_{\omega_1}^{\text{Ult}(V, G)} \cap X_\gamma$, where $X_\alpha = j_{0\alpha}(V_\delta[g])$ for all $\alpha \leq \gamma = \omega_1^{\text{Ult}(V, G)}$.

This gives that $X_\gamma \subseteq \text{Ult}(V, G)$ for

$$\in \text{NS}_{\omega_1} \cup \text{UB}^V \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$$

(again appealing to Fact 3.1.7 for the relation and function symbols R_{ϕ_j}, f_{ψ_j} or for those in \in_{Δ_0} , and to the iterability of $V_\delta[g]$ for the relation symbols given by elements of \mathcal{A}). Since $V_\delta[g] \models \neg\phi$, so does X_γ , by elementarity. But $\neg\phi$ is a Σ_1 -sentence, hence it is upward absolute for superstructures, therefore $\text{Ult}(V, G) \models \neg\phi$. This is a contradiction, since $\text{Ult}(V, G)$ is elementarily equivalent to V for $\in \text{NS}_{\omega_1} \cup \text{UB}^V$, and $V \models \phi$.

A similar argument shows that if V models a Σ_1 -sentence ϕ for $\in \text{NS}_{\omega_1} \cup \text{UB}^V$ this will remain true in all of its generic extensions:

Assume $V[h] \models \neg\phi$ for some h V -generic for some forcing notion $P \in V$. Let $\gamma > |P|$ be a Woodin cardinal, and let g be V -generic for³⁰ \mathcal{T}_γ with $h \in V[g]$ and $\text{crit}(j_g) = \omega_1^V$ (hence there is in g some stationary set of V_γ concentrating on countable sets). Then $V[g] \models \phi$ since:

- $V_\gamma \models \phi$, since $V_\gamma \prec_1 V$ for $\in \text{NS}_{\omega_1} \cup \text{UB}^V \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$ by Lemma 3.1;
- $V_\gamma^{\text{Ult}(V, g)} = V_\gamma^{V[g]}$, since $V[g]$ models that $\text{Ult}(V, g)^{<\gamma} \subseteq \text{Ult}(V, g)$;
- $V_\gamma^{\text{Ult}(V, g)} \models \phi$, by elementarity of j_g , since $j_g(V_\gamma) = V_\gamma^{\text{Ult}(V, g)}$;
- $V_\gamma^{V[g]} \prec_1 V[g]$ with respect to $\in \text{NS}_{\omega_1} \cup \text{UB}^V \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$, again by Lemma 3.1 applied in $V[g]$.

Now repeat the same argument as before for the Π_1 -property $\neg\phi$, with $V[h]$ in the place of V and $V[g]$ in the place of $V[h]$. \square

5. MODEL COMPANIONSHIP VERSUS GENERIC ABSOLUTENESS FOR THE THEORY OF H_{\aleph_2}

Let UB denote the family of universally Baire sets; for $\mathcal{A} \subseteq \text{UB}$ $L(\mathcal{A})$ denotes the smallest transitive model of ZF which contains \mathcal{A} (see for details Section 4.1.1).

We will be interested in (what we will call *generically tame*) families \mathcal{A} of universally Baire sets which are constructibly closed (i.e. $\mathcal{P}(2^\omega)^{L(\mathcal{A})} = \mathcal{A}$), countably closed (e.g. $\mathcal{A}^\omega \subseteq L(\mathcal{A})$), and generically invariant (e.g. the theory of $L(\mathcal{A})$ cannot be changed by set sized forcing). One example will be given by $\mathcal{A} = \mathcal{P}(2^\omega)^{L(\text{Ord}^\omega)}$ assuming large cardinals³¹, another by the class UB itself as computed in any generic extension of V collapsing a supercompact to countable.

We aim to prove two model companionship results relating the theory of V to that of H_{ω_2} . First of all we must include a certain finite and explicit set of properties and definable functions which are $\Delta_1(\text{ZFC}^-)$ to \in_{Δ_0} (in order to be able to express by means of quantifier free formulae certain absolute concepts of set theory). We may call this extended signature \in_{Δ_1} . This is harmless in view of Fact 3.1.7, since we will only consider \in -structures which

³⁰ \mathcal{T}_γ is the full stationary tower of height γ whose conditions are stationary sets in V_γ , denoted as $\mathbb{P}_{<\gamma}$ in [14], see in particular [14, Section 2.5].

³¹This remarkable result of Woodin is to my knowledge unpublished. There are some handwritten notes of a proof sketch by Larson, and it is mentioned in [14, Remark 3.3.12].

are models of ZFC^- . We will then consider signatures extending $\in_{\Delta_1} \cup \{\mathbf{NS}_{\omega_1}, \omega_1\}$ with predicate symbols for certain elements of a generically tame \mathcal{A} : one with predicate symbols for *all* elements of \mathcal{A} , the other just for the *lightface definable* elements of \mathcal{A} .

- The first result establishes the equivalence between the conditional version of Woodin’s axiom $(*)$ to the model $L(\mathcal{A})$ and the assertion that the theory of V is the model companion of the theory of H_{ω_2} in signature³² $\in_{\Delta_1} \cup \mathcal{A} \cup \{\mathbf{NS}_{\omega_1}, \omega_1\}$.

This result however makes sense to a platonist but not to a formalist since it subsumes the existence of V and it is stated for a signature which is highly non constructive (it includes predicate symbols for the —at least continuum many— elements of \mathcal{A} in V).

- The second result is a model companionship result for $\text{ZFC} + \textit{large cardinals}$ with respect to the signature $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ extending $\in_{\Delta_1} \cup \{\mathbf{NS}_{\omega_1}, \omega_1\}$ only with predicate symbols for those elements of \mathcal{A} which are provably the extension of an \in -formula $\phi(x)$ in no parameters. This is a recursive signature and we can give a recursive axiomatization of the model companion theory. This model companion theory is the common chore of the $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ -theory of $H_{\omega_2}^V$ as (V, \in) ranges over the models of $\text{MM}^{++} + \textit{suitable large cardinal axioms}$; moreover the Π_1 -fragment of this theory is forcing invariant.

We will actually need a generic invariance property for \mathcal{A} which is delicate to formulate as \mathcal{A}^V and $\mathcal{A}^{V[G]}$ share their defining \in -formula but, in general, have a trivial intersection (since for most V -generic filters G on the one hand there could be completely new universally Baire existing in $\mathcal{A}^{V[G]} \setminus \mathcal{A}^V$ — even if G does not add new reals, on the other hand $\mathcal{A}^{V[G]} \neq \mathcal{A}^V$ for any uncountable universally Baire set $A \in \mathcal{A}^V$ — in case G adds a new real). These difficulties lead us to formulate the above properties by means of the somewhat convoluted syntactic definitions given below.

Notation 5.0.1. Let (V, \in) be a model of ZFC .

A family \mathcal{A} of subsets of³³ $\mathcal{P}(2^\omega)$ is:

- *constructibly closed* if

$$\mathcal{P}(2^\omega)^{L(\mathcal{A})} = \mathcal{A},$$

and every binary relation R on $2^\omega \times 2^\omega$ in \mathcal{A} can be uniformized by a function $f : 2^\omega \rightarrow 2^\omega$ in \mathcal{A} .

- *countably closed* if $\mathcal{A}^\omega \subseteq L(\mathcal{A})$,
- *lightface definable* if it is the extension in V of some \in -formula $\phi_{\mathcal{A}}(x)$ without parameters.

Woodin calls $\text{MAX}(\mathcal{A})$ as defined below the “sealing theorem” for \mathcal{A} .

Definition 5.0.2. $\text{MAX}(\mathcal{A})$ is the conjunction of the following three first order \in -sentences construed from some \in -formula $\phi_{\mathcal{A}}(x)$ in one free variable (together with the axioms of set theory needed to make sense of them³⁴):

- (1) The extension \mathcal{A} of $\phi_{\mathcal{A}}(x)$ in any generic extension $V[G]$ of a ZFC -model (V, \in) is a constructibly closed family of universally Baire sets of $V[G]$.
- (2) As above but replacing *constructibly closed* with *countably closed*.

³²We consider an element $A \subseteq (2^\omega)^k$ of \mathcal{A} as a k -ary predicate symbol, \mathbf{NS}_{ω_1} as a unary predicate symbol, ω_1 as a constant symbol.

³³For the remainder of the paper to avoid heavy notation we feel free to identify when needed 2^ω with $\mathcal{P}(\omega)$, $\mathcal{P}(\omega^{<\omega})$ or any variation of these sets which is clearly a canonical representation of the Cantor space.

³⁴To make sense of $\text{MAX}(\mathcal{A})$ it suffices that (V, \in) satisfies all axioms of ZFC with the exception of replacement and comprehension, with the latter replaced by the Σ_n -replacement schema for some large enough n . It is well known that this theory is finitely axiomatizable for each $n \in \omega$.

- (3) Whenever G is V -generic for some forcing notion $P \in V$ and H is $V[G]$ -generic for some forcing notion $Q \in V[G]$ there are class of ordinals $I_{V[G]}, I_{V[G][H]}$ such that any order preserving map of $I_{V[G]}$ into $I_{V[G][H]}$, combined with the map $A^{V[G]} \mapsto A^{V[G][H]}$ extends uniquely to an elementary embedding of $(L(\mathcal{A}^{V[G]}), \in)$ into $(L(\mathcal{A}^{V[G][H]}), \in)$ which is the identity on $H_{\omega_1}^{V[G]}$.
- An \in -formula $\phi_{\mathcal{A}}(x)$ in one free variable is *generically tame for T* if $T \models \mathbf{MAX}(\mathcal{A})$.
 - $\mathcal{A} \in V$ is *generically tame for T* if \mathcal{A} is the extension of some \in -formula $\phi_{\mathcal{A}}(x)$ in (V, \in) such that:
 - $T \models \mathbf{MAX}(\mathcal{A})$,
 - $(V, \in) \models T$.
 - $\mathcal{A} \in V$ is *generically tame* if it is generically tame for the theory of (V, \in) .

Note that if $(V, \in) \models T$, $T \models \mathbf{MAX}(\mathcal{A})$ and G is V -generic for some forcing $P \in V$, we may have that $V[G] \not\models T$ nonetheless $V[G] \models \mathbf{MAX}(\mathcal{A})$ holds for sure. In particular $\mathbf{MAX}(\mathcal{A})$ is a consequence of T which is preserved by any forcing over any model of T , while other axioms of T may not (for example $T \supseteq \mathbf{ZFC}$ might have either one of \mathbf{CH} or $\neg\mathbf{CH}$ among its axioms, and neither of them is preserved through forcing extensions over models of T).

By [14, Remark 3.3.12, Thm. 3.3.14, Thm. 3.3.19, Thm 3.4.18] and [11, Thm. 36.9] we get the following:

- (i) Assume T_0 extends \mathbf{ZFC} +*there are class many Woodin cardinals which are a limit of Woodin cardinals*.

Then the \in -formula $\phi_0(x)$ defining³⁵ $\mathcal{P}(2^\omega)^{L(\text{Ord}^\omega)}$ defines a generically tame family for T_0 ; e.g. $T_0 \models \mathbf{MAX}(\mathcal{A})$ for $\mathcal{A} = \mathcal{P}(2^\omega)^{L(\text{Ord}^\omega)}$.

- (ii) Assume T_1 extends T_0 with the axiom³⁶

Any model (V, \in) of T_1 is a generic extension of some inner model (W, \in) where some countable ordinal δ of V is a supercompact cardinal in W .

Then the universally Baire sets are a generically tame family for T_1 ; e.g. $T_1 \models \mathbf{MAX}(\text{UB})$ for $\phi_{\text{UB}}(x)$ an \in -formula defining the universally Baire sets.

Let us now be precise in our definition of the relevant signatures of this Section:

Notation 5.0.3. We expand \in_{Δ_0} to a signature \in_{Δ_1} which includes symbols R_{ϕ_i}, f_{ψ_j} for finitely many \in -formulae ϕ_i, ψ_j which are $\Delta_1(\mathbf{ZFC}^-)$ and such that $\mathbf{ZFC}^- \models \forall \vec{x} \exists! y \psi_j(\vec{x}, y)$ for all ψ_j . T_{Δ_1} is T_{Δ_0} enriched with axioms $\text{Ax}_{\phi_i}^0, \text{Ax}_{\psi_j}^1$ for all the needed³⁷ ϕ_i, ψ_j .

Given an \in -formula $\phi_{\mathcal{A}}(x)$:

- $\in_{\Delta_0, \mathcal{A}}$ is the extension of \in_{Δ_0} in which we add an n -ary relation symbol S_ϕ for any \in -formula ϕ of arity n ;
- $T_{\Delta_0, \mathcal{A}}$ is the extension of $\mathbf{ZFC} + T_{\Delta_0}$ by the axioms

$$\forall \vec{x} \left[S_\phi(\vec{x}) \leftrightarrow (\phi^{L(\mathcal{A})}(\vec{x}) \wedge \bigwedge_{i=1}^n x_i \in \mathcal{P}(\omega)) \right]$$

as ϕ ranges over the \in -formulae of some arity n ;

³⁵The *Chang model* $L(\text{Ord}^\omega)$ is the smallest transitive model of \mathbf{ZF} containing all the countable sequences of ordinals.

³⁶This is first order expressible by [15].

³⁷The exact definition of which R_{ϕ_i}, f_{ψ_j} we need to add into \in_{Δ_1} is left to the reader. We certainly need to be able to express the $\Delta_1(\mathbf{ZFC}^-)$ -property³⁸ “ \mathcal{J} is an iteration of length γ ” by a quantifier free formula of type $R_{\phi_i}(\mathcal{J}, \gamma)$. This is needed in the proof of Lemma 5.5.4 and to argue that each instance of (3) of Thm. 5.5.3 is formalized by a Π_2 -sentence in the relevant signature. It will also be transparent that only a finite number of such predicate and function symbols for $\Delta_1(\mathbf{ZFC}^-)$ -properties need to be included in \in_{Δ_1} in order to grant the correctness of all arguments of this section based on considerations on \in_{Δ_1} .

- $\in_{\Delta_1, \mathcal{A}} = \in_{\Delta_0, \mathcal{A}} \cup \in_{\Delta_1}$;
- $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}} = \in_{\Delta_1, \mathcal{A}} \cup \{\mathbf{NS}_{\omega_1}, \omega_1\}$;
- $T_{\in_{\Delta_1}, \mathcal{A}} = T_{\Delta_0, \mathcal{A}} + T_{\Delta_1}$;
- $T_{\mathbf{NS}_{\omega_1}, \mathcal{A}} = T_{\Delta_1, \mathcal{A}} + T_{\mathbf{NS}_{\omega_1}}$ (recall Notation 4.3.1).

Remark 5.0.4. $T_{\Delta_0, \mathcal{A}}$ makes the subsets of $\mathcal{P}(\omega)$ definable in no parameters in $(L(\mathcal{A}), \in)$ the extension of an atomic formula of $\in_{\Delta_0, \mathcal{A}}$. It is clear that $\in_{\Delta_0, \mathcal{A}}, \in_{\Delta_1, \mathcal{A}}, \in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ can all be expressed as \in_E for suitably chosen $E \subseteq \text{Form}_\in \times 2$.

Note the difference between $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ which is a countable recursive signature, and $\in_{\mathbf{NS}_{\omega_1}} \cup \mathcal{A}^V$ for (V, \in) a model of ZFC. The latter has predicate symbols for the at least continuum (in the sense of V) many elements of \mathcal{A} existing in V .

Furthermore note that for any \in -model (M, E) of ZFC⁻ its unique extensions to \in_{Δ_i} -models of T_{Δ_i} (for $i = 0, 1$) have exactly the same algebra of definable sets. Moreover if N is a transitive subset of M (according to Notation 3.1.6) such that $(N, E \cap N^2)$ also satisfies ZFC⁻, we get that (for both $i = 0, 1$) their unique extensions to \in_{Δ_i} -structures which satisfy T_{Δ_i} are \in_{Δ_i} -substructures (in view of Fact 3.1.7).

Key to all results of this section is an analysis of the properties of generic extensions by \mathbb{P}_{\max} of $L(\mathcal{A})$ for \mathcal{A} generically tame. Generic tameness of \mathcal{A} is used to argue (among other things) that most of the results established in [13] on the properties of \mathbb{P}_{\max} for $L(\mathbb{R})$ can be also asserted for $L(\mathcal{A})$. We refrain to define the \mathbb{P}_{\max} -forcing rightaway and we will introduce it when needed in our proofs (see Def. 5.5.6). Meanwhile we assume the reader is familiar with \mathbb{P}_{\max} or can accept as a blackbox its existence as a certain forcing notion; our reference on this topic is [13].

We now give a precise definition of $(*)$ - \mathcal{A} (modulo the definition of \mathbb{P}_{\max}).

Definition 5.1. $(*)$ - \mathcal{A} holds in some ZFC-model (V, \in) if in V :

- there are class many Woodin cardinals;
- $\mathbf{MAX}(\mathcal{A})$ holds for some \in -formula $\phi_{\mathcal{A}}(x)$;
- \mathbf{NS}_{ω_1} is saturated³⁹;
- there exists a filter G on \mathbb{P}_{\max} meeting all the dense subsets of \mathbb{P}_{\max} definable in $L(\mathcal{A})$ (where $\mathcal{A} \in V$ is the extension of the formula $\phi_{\mathcal{A}}(x)$).

$(*)$ -UB holds if $(*)$ - \mathcal{A} holds for $\mathcal{A} = \text{UB}$.

Woodin's axiom $(*)$ as defined in [13, Def. 7.5] is $(*)$ - \mathcal{A} for $\mathcal{A} = \mathcal{P}(2^\omega)^{L(\mathbb{R})}$ with $\mathbf{MAX}(\mathcal{A})$ omitted.

This is the first main result of this section:

Theorem 5.1. *Assume (V, \in) models*

ZFC + there is a supercompact cardinal and class many Woodin cardinals

and $\mathcal{A} \in V$ is generically tame.

TFAE:

- (1) (V, \in) models $(*)$ - \mathcal{A} ;

³⁹See [14, Section 1.6, pag. 39] for a discussion of saturated ideals on ω_1 .

- (2) \mathbf{NS}_{ω_1} is precipitous⁴⁰ in V and the $\in_{\mathbf{NS}_{\omega_1}} \cup \mathcal{A}^V$ -theory⁴¹ of V has as model companion the $\in_{\mathbf{NS}_{\omega_1}} \cup \mathcal{A}^V$ -theory of H_{ω_2} .

We obtain as immediate corollaries the following:

Corollary 5.2. *Let (V, \in) be a model of*

ZFC + there is a supercompact cardinal and class many Woodin limit of Woodins,
and $\mathcal{A} = \mathcal{P}(2^\omega)^{L(\text{Ord}^\omega)}$. TFAE:

- (1) (V, \in) models $(*)$ - \mathcal{A} ;
(2) \mathbf{NS}_{ω_1} is precipitous and the $\in_{\mathbf{NS}_{\omega_1}} \cup \mathcal{A}^V$ -theory of V has as model companion the $\in_{\mathbf{NS}_{\omega_1}} \cup \mathcal{A}^V$ -theory of H_{ω_2} .

Corollary 5.3. *Let (V, \in) be a model of*

ZFC + $\mathbf{MAX}(\text{UB})$ + there is a supercompact cardinal and class many Woodin cardinals,
where UB denote the family of universally Baire sets in V . TFAE:

- (1) (V, \in) models $(*)$ - UB ;
(2) \mathbf{NS}_{ω_1} is precipitous and the $\in_{\mathbf{NS}_{\omega_1}} \cup \text{UB}$ -theory of V has as model companion the $\in_{\mathbf{NS}_{\omega_1}} \cup \text{UB}$ -theory of H_{ω_2} .

An objection to Thm. 5.1 is that it subsumes the Platonist standpoint that there exists a definite universe of sets. We can prove a version of Thm. 5.1 which makes perfect sense also to a formalist and from which we immediately derive Thm. 1.7. This is one of the reasons we paid attention to give a syntactically meaningful definition of generic tameness and of $\mathbf{MAX}(\mathcal{A})$.

Theorem 5.4. *Assume T is an \in -theory extending⁴²*

ZFC + $\mathbf{MAX}(\mathcal{A})$ + *there is a supercompact cardinal and class many Woodin cardinals.*

Let $B \subseteq \text{Form} \times 2$ be such that \in_B is exactly $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$. Then $B \in \mathbf{spec}_{\text{AMC}}(T)$.

Moreover TFAE for any for any Π_2 -sentence ψ for $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$:

- (A) $\text{AMC}(T, B) \vdash \psi$.
(B) $(V[G], \in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}^{V[G]}) \models \psi^{H_{\omega_2}}$ whenever $(V, \in) \models T$, $V[G]$ is a forcing extension of V ,
and $(V[G], \in) \models (*)$ - \mathcal{A} .
(C) T proves⁴³

$$\exists P (P \text{ is a stationary set preserving partial order} \wedge \Vdash_P \psi^{\dot{H}_{\omega_2}}).$$

- (D) T proves

$$\exists P (P \text{ is a partial order} \wedge \Vdash_P \psi^{\dot{H}_{\omega_2}}).$$

⁴⁰See [14, Section 1.6, pag. 41] for a definition of precipitousness and a discussion of its properties.

A key observation is that \mathbf{NS}_{ω_1} being precipitous is independent of CH (see for example [14, Thm. 1.6.24]), while $(*)$ - \mathcal{A} entails $2^{\aleph_0} = \aleph_2$ (for example by the results of [13, Section 6]).

Another key point is that we stick to the formulation of \mathbb{P}_{max} as in [13] so to be able in its proof to quote verbatim from [13] all the relevant results on \mathbb{P}_{max} -preconditions we will use. It is however possible to develop \mathbb{P}_{max} focusing on Woodin's countable tower rather than on the precipitousness of \mathbf{NS}_{ω_1} to define the notion of \mathbb{P}_{max} -precondition. Following this approach in all its scopes, one should be able to reformulate Thm. 5.1(2) omitting the request that \mathbf{NS}_{ω_1} is precipitous. We do not explore this venue any further.

⁴¹E.g. we regard $A \subseteq (2^\omega)^k$ as a k -ary predicate symbol for any $A \in \mathcal{A}$.

⁴²With $\mathbf{MAX}(\mathcal{A})$ predicated for the \in -formula $\phi_{\mathcal{A}}(x)$.

⁴³ \dot{H}_{ω_2} denotes a canonical P -name for H_{ω_2} as computed in generic extension by P . $\Vdash_P \psi^{\dot{H}_{\omega_2}}$ stands for:

$$(V[G], \in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}^{V[G]}) \models \psi^{H_{\omega_2}}$$

whenever G is V -generic for P .

(E) T proves⁴⁴

$$L(\mathcal{A}) \models [\mathbb{P}_{\max} \Vdash \psi^{\dot{H}_{\omega_2}}].$$

(F) If $(V, \in) \models T$ and ψ is $\forall x \exists y \phi(x, y)$ with ϕ quantifier free $\in \mathbf{NS}_{\omega_1, \mathcal{A}}$ -formula, then for all⁴⁵ $A \in H_{\omega_2}^V$

$$\exists y \phi(A, y) \text{ is honestly consistent according to } V.$$

(G) For any consistent $\in \mathbf{NS}_{\omega_1, \mathcal{A}}$ -theory

$$S \supseteq T + T_{\in, B},$$

$S_{\forall \exists} + \psi$ is consistent.

Note that even if $T \models \text{CH}$, $\neg \text{CH}$ is in $\text{AMC}(T, B)$ (for example by (E) above). In particular the model companion $\text{AMC}(T, B)$ of T describes a theory of H_{\aleph_2} which can be completely unrelated to that given by models of T . Moreover recall again that CH is not expressible as a boolean combination of Π_1 -sentences in $\in \mathbf{NS}_{\omega_1, \mathcal{A}}$ for models of T : it is not preserved by forcing, while $T_{\forall \exists}$ is.

The rest of this section is devoted to proof of Theorems 5.1 and 5.2. Crucial to their proof is the recent breakthrough of Asperó and Schindler [2] establishing that $(*)\text{-}\mathcal{A}$ follows from MM^{++} for any generically tame \mathcal{A} . First of all it is convenient to detail more on $\text{MAX}(\mathcal{A})$ and its use in our proofs.

5.1. **MAX**(\mathcal{A}). From now on we will need in several occasions that **MAX**(\mathcal{A}) holds in V for a generically tame \mathcal{A} (recall Def. 5.0.2). We will always explicitly state where this assumption is used, hence if a statement does not mention it in the hypothesis, the assumption is not needed for its thesis.

We will use all three properties of **MAX**(\mathcal{A}) crucially: (1) and (2) are used in the proof of Lemma 5.4.6; (3) in the proof of Fact 5.4.7. Similarly they are essentially used in Remark 5.5.2. Specifically we will need (1) and (2) of **MAX**(\mathcal{A}) to prove that certain countable families of subsets of H_{ω_1} simply definable using an existential \in -formula quantifying over $L(\mathcal{A})$ with parameters in $H_{\omega_1} \cup \mathcal{A}$ are coded by a universally Baire set in \mathcal{A} , and (3) to prove that this coding is absolute between generic extensions: i.e. for any family $\{\phi_n : n \in \mathbb{N}\}$ of \in -formulae⁴⁶ with parameters in $H_{\omega_1}^V \cup \mathcal{A}$, if

$$A_n = \{x \in H_{\omega_1}^V : (H_{\omega_1} \cup \mathcal{A}, \in^V) \models \phi_n(x)\}$$

and $\{A_n : n \in \mathbb{N}\}$ is coded by $A \in \mathcal{A}$, letting

$$A_n^{V[G]} = \left\{ x \in H_{\omega_1}^{V[G]} : (H_{\omega_1}^{V[G]} \cup \mathcal{A}^{V[G]}, \in^{V[G]}) \models \phi_n(x) \right\}$$

⁴⁴ $L(\mathcal{A}) \models [\mathbb{P}_{\max} \Vdash \psi^{\dot{H}_{\omega_2}}]$ stands for:

$$(L(\mathcal{A})[G], \in_{\mathbf{NS}_{\omega_1, \mathcal{A}}}^{L(\mathcal{A})[G]}) \models \psi^{H_{\omega_2}}$$

whenever G is $L(\mathcal{A})$ -generic for P .

⁴⁵See [1, Def. 1.8] for the notion of honest consistency. It can be equivalently stated as: *For some κ and G V -generic for $\text{Coll}(\omega, \kappa)$, there is a transitive set $M \in V[G]$ such that:*

$$\begin{aligned} (H_{\omega_2}^V, \in_{\Delta_1}^V, \mathbf{NS}_{\omega_1}^V, \mathbf{UB}^V) \sqsubseteq (M, \in_{\Delta_1}^M, \mathbf{NS}_{\omega_1}^M, B^{V[G]} \cap M : B \in \mathbf{UB}^V), \\ (M, \in_{\Delta_1}^M, B^{V[G]} \cap M : B \in \mathbf{UB}^V) \sqsubseteq (V[G], \in_{\Delta_1}^{V[G]}, B^{V[G]} : B \in \mathbf{UB}^V), \end{aligned}$$

and

$$(M, \in_{\Delta_1}^M, \mathbf{NS}_{\omega_1}^M, B^{V[G]} \cap M : B \in \mathbf{UB}^V) \models \exists y \phi(A, y).$$

⁴⁶Note that the structures $(H_{\omega_1} \cup \mathcal{A}, \in)$, $(H_{\omega_1} \cup \mathcal{A}, \in_{\Delta_0}^V)$, $(H_{\omega_1} \cup \mathcal{A}, \in_{\Delta_1}^V)$ have the same algebra of definable sets, hence we will use one or the other as we deem most convenient, since any set definable by some formula in one of these structures is also defined by a possibly different formula in the other. The formulation of **MAX**(\mathcal{A}) is unaffected if we choose any of the two structures as the one for which we predicate it.

$\{A_n^{V[G]} : n \in \mathbb{N}\}$ is coded by $A^{V[G]} \in \mathcal{A}^{V[G]}$.

It is useful to outline what is the different expressive power of the structures $(H_{\omega_1}, \in_{\Delta_0}^V, A : A \in \mathcal{A}^V)$ and $(H_{\omega_1} \cup \mathcal{A}^V, \in_{\Delta_0}^V)$. The latter can be seen as a second order extension of H_{ω_1} , where we also allow formulae to quantify over the family of universally Baire sets given by \mathcal{A} ; in the former quantifiers only range over elements of H_{ω_1} , but we can use the subsets of H_{ω_1} whose universally Baire code is in \mathcal{A} as parameters. This is in exact analogy between the comprehension scheme for the Morse-Kelley axiomatization of set theory (where formulae with quantifiers ranging over classes are allowed) and the comprehension scheme for Gödel-Bernays axiomatization of set theory (where just formulae using classes as parameters and quantifiers ranging only over sets are allowed). To appreciate the difference between the two set-up, note that that the axiom of determinacy for universally Baire sets in \mathcal{A} is expressible in

$$(H_{\omega_1} \cup \mathcal{A}, \in_{\Delta_0}^V)$$

by the Π_2 -sentence for \in_{Δ_0}

For all $A \subseteq 2^\omega$ there is a winning strategy for one of the players in the game with payoff A ,

while in

$$(H_{\omega_1}, \in_{\Delta_0}^V, A : A \in \mathcal{A}^V)$$

it is expressed by the axiom schema of Σ_2 -sentences for $\in_{\Delta_0} \cup \{A\}$

There is a winning strategy for some player in the game with payoff A

as A ranges over the universally Baire sets in \mathcal{A} .

We will crucially use the stronger expressive power of the structure $(H_{\omega_1} \cup \mathcal{A}, \in_{\Delta_0})$ to define certain universally Baire sets as the extension in $(H_{\omega_1} \cup \mathcal{A}, \in_{\Delta_0}^V)$ of lightface definable properties (according to the Levy hierarchy); properties which require an existential quantifier ranging over all universally Baire sets in \mathcal{A} .

5.2. A streamline of the proofs of Theorems 5.1, 5.2. Let us give a general outline of these proofs before getting into details. From now on we assume the reader is familiar with the basic theory of \mathbb{P}_{\max} as exposed in [13].

Much of our efforts will be now devoted to establish the model completeness of the theory of $(H_{\omega_2}^V, \in_{\Delta_1}^V, \mathbf{NS}_{\omega_1}^V, A : A \in \mathcal{A}^V)$ (assuming $(*)$ - \mathcal{A} in V).

We will then leverage on Levy absoluteness to infer that in models of $(*)$ - \mathcal{A} the theory of $(H_{\omega_2}^V, \in_{\Delta_0^1}^V, \mathbf{NS}_{\omega_1}^V, A : A \in \mathcal{A}^V)$ is the absolute model companion of the theory of $(V, \in_{\Delta_1}^V, \mathbf{NS}_{\omega_1}^V, A : A \in \mathcal{A}^V)$.

A similar strategy will work for the AMC-result for the signature $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$. To prove this model completeness result we use Robinson's test and we show the following:

Assuming $\mathbf{MAX}(\mathcal{A})$ there is a *special* universally Baire set $\bar{D}_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ which belongs to \mathcal{A} and is defined by an \in -formula (*in no parameters*) relativized to $L(\mathcal{A})$ (hence represented by a relation symbol of $\in_{\Delta_1, \mathcal{A}}$) coding a family of \mathbb{P}_{\max} -preconditions with the following fundamental property:

For any existential $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}} \cup \{B_1, \dots, B_k\}$ -formula $\psi(x_1, \dots, x_n)$ mentioning the universally Baire predicates⁴⁷ $B_1, \dots, B_k \in \mathcal{A}$, there is an algorithmic procedure which finds a universal $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}} \cup \{B_1, \dots, B_k\}$ -formula $\theta_\psi(x_1, \dots, x_n)$ mentioning just the universally Baire predicates $B_1, \dots, B_k, \bar{D}_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ such that

$$(H_{\omega_2}^{L(\mathcal{A})[G]}, \in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}^{L(\mathcal{A})[G]}, B_1, \dots, B_k) \models \forall \vec{x} (\psi(x_1, \dots, x_n) \leftrightarrow \theta_\psi(x_1, \dots, x_n))$$

whenever G is $L(\mathcal{A})$ -generic for \mathbb{P}_{\max} .

⁴⁷Some of these B_j may not appear in $\in_{\Delta_1, \mathcal{A}}$ being not definable by a lightface property.

Moreover the definition and properties of $\bar{D}_{\mathbf{NS}_{\omega_1, \mathcal{A}}}$ and the computation of $\theta_\psi(x_1, \dots, x_n)$ from $\psi(x_1, \dots, x_n)$ are just based on the assumption that (V, \in) is a model of $\mathbf{MAX}(\mathcal{A})$, hence can be replicated mutatis-mutandis in any model of $\mathbf{ZFC} + \mathbf{MAX}(\mathcal{A})$. We will need that (V, \in) is a model of $\mathbf{MAX}(\mathcal{A}) + (*)\text{-}\mathcal{A}$ just to argue that in V there is an $L(\mathcal{A})$ -generic filter G for \mathbb{P}_{\max} such that⁴⁸ $H_{\omega_2}^{L(\mathcal{A})[G]} = H_{\omega_2}^V$. Since in all our arguments we will only use that (V, \in) is a model of $\mathbf{MAX}(\mathcal{A})$ and (in some of them also of $(*)\text{-}\mathcal{A}$), we will be in the position to conclude easily for the truth of Theorem 5.1 and 5.2.

We condense the above information in the following:

Theorem 5.2.1. *Let $\phi_{\mathcal{A}}(x)$ be an \in -formula such that in some model (V, \in) of \mathbf{ZFC}*

$$(V, \in) \models \mathbf{MAX}(\mathcal{A}) + \text{there are class many Woodin cardinals}$$

with \mathcal{A}^V being the extension in V of $\phi_{\mathcal{A}}(x)$.

Then there is an \in -formula $\phi_{\mathbf{NS}_{\omega_1, \mathcal{A}}}(x)$ in one free variable⁴⁹ such that:

- (1) *Whenever $(V, \in) \models \mathbf{ZFC} + \mathbf{MAX}(\mathcal{A}) + \text{there are class many Woodin cardinals}$,*

$$\{z \in V : (V, \in) \models \phi_{\mathbf{NS}_{\omega_1, \mathcal{A}}}(z)\}$$

is the extension of some element of \mathcal{A}^V .

- (2) *Given predicate symbols B_1, \dots, B_k of arity n_1, \dots, n_k , and the theory T_{B_1, \dots, B_k} in signature $\in_{\mathbf{NS}_{\omega_1, \mathcal{A}}} \cup \{B_1, \dots, B_k\}$ extending*

$$\mathbf{ZFC} + \mathbf{MAX}(\mathcal{A}) + \text{there are class many Woodin cardinals}$$

by the axioms⁵⁰:

$$\exists! y \subseteq (2^\omega)^{n_j} \left[\forall z_1, \dots, z_{n_j} \left(\langle z_1, \dots, z_{n_j} \rangle \in y \leftrightarrow B_j(z_1, \dots, z_{n_j}) \right) \wedge \phi_{\mathcal{A}}(y) \right],$$

there is a recursive procedure assigning to any existential formula $\varphi(x_1, \dots, x_n)$ for $\in_{\mathbf{NS}_{\omega_1, \mathcal{A}}} \cup \{B_1, \dots, B_k\}$ a universal formula $\theta_\varphi(x_1, \dots, x_n)$ for $\in_{\mathbf{NS}_{\omega_1, \mathcal{A}}} \cup \{B_1, \dots, B_k\}$ (mentioning just the predicate symbols occurring in φ and $S_{\phi_{\mathbf{NS}_{\omega_1, \mathcal{A}}}}$) such that T_{B_1, \dots, B_k} proves that⁵¹

$$\mathbb{P}_{\max} \Vdash \left[\left(H_{\omega_2}^{L(\mathcal{A})[\dot{G}]} \in_{\mathbf{NS}_{\omega_1, \mathcal{A}}}^{L(\mathcal{A})[\dot{G}]} \check{B}_1, \dots, \check{B}_k \right) \models \forall \vec{x} \left(\varphi(x_1, \dots, x_n) \leftrightarrow \theta_\varphi(x_1, \dots, x_n) \right) \right]$$

5.3. Proofs of Thm. 5.2, and of (1) \rightarrow (2) of Thm. 5.1.

Theorem 5.2, and (1) \rightarrow (2) of Theorem 5.1 are immediate corollaries of the above theorem combined with:

- Asperò and Schindler's proof that $\mathbf{MM}^{++} + \mathbf{MAX}(\mathcal{A}) + \text{there are class many Woodin cardinals}$ implies $(*)\text{-}\mathcal{A}$,
- Theorem 4.1.

We start with the proof of (1) \rightarrow (2) of Thm. 5.1 assuming Thm. 5.2.1 and Thm. 4.1:

Proof. Assume (V, \in) models $(*)\text{-}\mathcal{A}$. Then there is a \mathbb{P}_{\max} -filter $G \in V$ such that $H_{\omega_2}^{L(\mathcal{A})[G]} = H_{\omega_2}^V$. By Thm. 5.2.1 and Robinson's test, we get that the first order $\in_{\mathbf{NS}_{\omega_1, \mathcal{A}}} \cup \mathcal{A}^V$ -theory of $H_{\omega_2}^{L(\mathcal{A})[G]}$ is model complete. By Levy's absoluteness (Lemma 3.1), $H_{\omega_2}^{L(\mathcal{A})[G]}$ is a Σ_1 -elementary substructure of V also according to the signature $\in_{\mathbf{NS}_{\omega_1, \mathcal{A}}} \cup \mathcal{A}$. We conclude since the two theories are Π_1 -complete and share the same Π_1 and Σ_1 fragments. \square

⁴⁸It is this part of our argument where the result of Asperò and Schindler establishing the consistency of $(*)\text{-}\mathcal{A}$ relative to a supercompact is used in an essential way. We will address again the role of Asperò and Schindler's result in all our proofs in some closing remarks.

⁴⁹Whose canonical interpretation in models of $\mathbf{MAX}(\mathcal{A})$ will be the magic set $\bar{D}_{\mathbf{NS}_{\omega_1, \mathcal{A}}}$. Note also that $\phi_{\mathbf{NS}_{\omega_1, \mathcal{A}}}(x)$ is computable in terms of $\phi_{\mathcal{A}}$.

⁵⁰Axioms stating that B_j is an element of \mathcal{A} for all the new predicate symbols B_1, \dots, B_k .

⁵¹ $\dot{G} \in L(\mathcal{A})$ is the canonical \mathbb{P}_{\max} -name for the generic filter.

The proof of the converse implication requires more information on $\bar{D}_{\mathbf{NS}_{\omega_1, \mathcal{A}}}$ than what is conveyed in Thm. 5.2.1. We defer it to a later stage.

We now prove Thm. 5.2:

Proof. Let R be the theory given by the Π_2 -sentences ψ for $\in \mathbf{NS}_{\omega_1, \mathcal{A}}$ which hold in every model of the form $(H_{\omega_2}^{V[G]}, \in_{\mathbf{NS}_{\omega_1, \mathcal{A}}}^{V[G]})$ obtained by forcing over some model (V, \in) of T with $V[G]$ a generic extension of (V, \in) such that $(V[G], \in) \models (*)\text{-}\mathcal{A}$. Recall that $B \subseteq \text{Form}_{\{\in\}} \times 2$ is such that $\in_{\mathbf{NS}_{\omega_1, \mathcal{A}}} = \in_B$.

We show that $\bar{R} = \text{AMC}(T, B)$.

R is consistent: by Schindler and Asperó's result [2]

$\text{ZFC} + \mathbf{MAX}(\mathcal{A}) + \text{MM}^{++} + \text{there are class many Woodin cardinals}$

implies $(*)\text{-}\mathcal{A}$.

MM^{++} is forcible over a model of

$\text{ZFC} + \text{there is a supercompact}$

and

$\text{ZFC} + \mathbf{MAX}(\mathcal{A}) + \text{there are class many Woodin cardinals}$

is preserved in forcing extensions.

R is model complete:

- for any existential $\in \mathbf{NS}_{\omega_1, \mathcal{A}}$ -formula $\phi(\vec{x})$, the Π_2 -sentence for $\in \mathbf{NS}_{\omega_1, \mathcal{A}}$

$$\forall \vec{x} (\phi(\vec{x}) \leftrightarrow \theta_\phi(\vec{x}))$$

is in R : it holds in all the structures used to define R (by Thm. 5.2.1).

- By Robinson's test those axioms suffice to establish the model completeness of R .

$(T + T_{\in, B})_{\forall \forall \exists}$ **and** $R_{\forall \forall \exists}$ **are the same:** By the very definition of R , we get that $R_{\forall \forall \exists}$ is equal to

(21)

$$\{\psi \in (\in_B)_{\forall \forall \exists} : \forall (V, \in), \mathbf{B} \in V \left[((V, \in) \models T + \mathbf{B} \text{ is a cba} + \llbracket (*)\text{-}\mathcal{A} \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}] \rightarrow \llbracket \psi^{H_{\omega_2}} \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}} \right] \}.$$

In view of Thm. 4.1 and Lemma 3.1, the formulae ψ in the set displayed in 21 are exactly the same ψ which belong to $(T + T_{\in, B})_{\forall \forall \exists}$: by Thm. 4.1 the Π_1 -theory of (V, \in_B^V) for (V, \in) a model of T is exactly the same Π_1 -theory of $(V[G], \in_B^{V[G]})$ for G V -generic for some cba forcing $(*)\text{-}\mathcal{A}$ over (V, \in) ; by Lemma 3.1 applied in $V[G]$ this Π_1 -theory is exactly equal to that of the structure of $(H_{\omega_2}^{V[G]}, \in_B^{V[G]})$.

This immediately gives $(\mathbf{A}) \iff (\mathbf{G})$ for $(T + T_{\in, B})$ and R .

We are left with the proof of the remaining equivalences between (\mathbf{A}) , (\mathbf{B}) , (\mathbf{C}) , (\mathbf{D}) , (\mathbf{E}) , (\mathbf{F}) , (\mathbf{G}) .

(A) \implies (B): By definition of R .

(B) \implies (C): Given an \in -model (V, \in) of T , by the results of [8], we can find a stationary set preserving forcing extension $V[G]$ of V which models MM^{++} . By the key result of Asperó and Schindler [2], $V[G] \models (*)\text{-}\mathcal{A}$. By (B) $(V[G], \in_{\mathbf{NS}_{\omega_1, \mathcal{A}}}^{V[G]})$ models $\psi^{H_{\omega_2}^{V[G]}}$, and we are done.

(C) \implies (D): Trivial.

(D) \implies (E): By⁵² [13, Thm. 7.3], if some P forces $\psi^{\dot{H}_{\omega_2}}$, we get that $L(\mathcal{A}) \models \left[\mathbb{P}_{\max} \Vdash \psi^{\dot{H}_{\omega_2}} \right]$.

(E) \iff (F): By [1, Thm. 2.7, Thm. 2.8].

⁵² $\mathbf{MAX}(\mathcal{A})$ implies that the same assumption used in the cited theorem for $L(\mathbb{R})$ holds for $L(\mathcal{A})$.

(E) \implies **(G)**: Given some complete $S \supseteq T + T_{\in, B}$, and a model \mathcal{M} of S , find \mathcal{N} forcing extension of \mathcal{M} which models $\psi^{H_{\omega_2}^{\mathcal{N}}}$. By Thm. 4.1 and Levy's absoluteness Lemma 3.1, $H_{\omega_2}^{\mathcal{N}}$ models $\psi + S_{\forall\exists}$, and we are done. \square

5.4. Proof of Thm. 5.2.1. The rest of this section is devoted to the proof of Thm. 5.2.1. What we will do first is to sketch the key intuition on how to define $\bar{D}_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$.

5.4.1. More ideas on the proof of Thm. 5.2.1. Recall the notion of generic tameness introduced in Def. 5.0.2. Let M be a countable transitive model of $\mathbf{ZFC} + \mathbf{MAX}(\mathcal{A}) + \text{there are class many Woodin cardinals}$ for some generically tame \mathcal{A} . Then it will model that \mathcal{A}^M is generically tame for M .

Now assume that there is a countable family \mathcal{A}_M of universally Baire sets in $L(\mathcal{A})$ which is such that $\mathcal{A}^M = \{B \cap M : B \in \mathcal{A}_M\}$. Furthermore assume that the map $B \cap M \mapsto B$ extends the identity on $H_{\omega_1}^M$ to an elementary embedding of

$$(H_{\omega_1}^M \cup \mathcal{A}^M, \in_{\Delta_0}^M)$$

into

$$(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V).$$

The setup described above is quite easy to realize (for example M could be the transitive collapse of some countable $X \prec V_\theta$ for some large enough θ); in particular for any $a \in H_{\omega_1}$ and $B_1, \dots, B_k \in \mathcal{A}$, we can find M countable transitive model of a suitable fragment of \mathbf{ZFC} with $a \in H_{\omega_1}^M$ and $\mathcal{A}_M \supseteq \{B_1, \dots, B_k\}$ countable and H_{ω_1} -closed family of sets in \mathcal{A} such that $\mathcal{A}^M = \{B \cap M : B \in \mathcal{A}_M\}$.

Letting $B_M = \prod \mathcal{A}_M$, $B_M \in \mathcal{A}$ since \mathcal{A} is countably closed; hence $(L(\mathcal{A}), \in_{\Delta_0})$ is able to compute correctly whether B_M encodes a set \mathcal{A}_M such that the pair (\mathcal{A}_M, M) satisfies the above list of requirements, e.g.:

- M is a countable transitive model of $\mathbf{ZFC} + \mathbf{MAX}(\mathcal{A}) + \text{there are class many Woodin cardinals}$.
- the map $B \cap M \mapsto B$ extends the identity on $H_{\omega_1}^M$ to an elementary embedding of

$$(H_{\omega_1}^M \cup \mathcal{A}^M, \in_{\Delta_0}^M)$$

into

$$(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V).$$

In particular $(L(\mathcal{A}), \in_{\Delta_0})$ correctly computes⁵³ the set $D_{\mathcal{A}}$ of $M \in H_{\omega_1}$ such that there exists a universally Baire set $B_M = \prod \mathcal{A}_M$ with the property that the pair (M, \mathcal{A}_M) realizes the above set of requirements. By $\mathbf{MAX}(\mathcal{A})$, $\bar{D}_{\mathcal{A}} = \text{Cod}_\omega^{-1}[D_{\mathcal{A}}]$ is a universally Baire set in \mathcal{A} .

Note moreover that $\bar{D}_{\mathcal{A}}$ is defined by a \in -formula $\phi_{\mathcal{A}}^*(x)$ in no extra parameters; in particular for any model $\mathcal{W} = (W, E)$ of $\mathbf{ZFC} + \mathbf{MAX}(\mathcal{A})$, we can define $\bar{D}_{\mathcal{A}}$ in \mathcal{W} and all its properties outlined above will hold relativized to \mathcal{W} .

We will consider the set $D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ of $M \in D_{\mathcal{A}}$ such that:

- $(M, \mathbf{NS}_{\omega_1}^M)$ is a \mathbb{P}_{\max} -precondition which is B -iterable for all $B \in \mathcal{A}_M$ (according to [13, Def. 4.1]);
- $j_{0\omega_1}$ is a Σ_1 -elementary embedding of $(H_{\omega_2}^M \cup \mathcal{A}^M, \in_{\Delta_1}, \mathbf{NS}_{\omega_1}^M)$ into $(H_{\omega_2}^V \cup \mathcal{A}, \in_{\Delta_1}, \mathbf{NS}_{\omega_1}^V)$ whenever $\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1\}$ is an iteration of M with $j_{0\omega_1}(\mathbf{NS}_{\omega_1}^M) = \mathbf{NS}_{\omega_1}^V \cap j_{0\omega_1}(H_{\omega_2}^M)$.

It will take a certain effort to prove that (assuming $(*)$)- \mathcal{A}):

⁵³Note that $(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V)$ cannot define $D_{\mathcal{A}}$ since the notion that $(H_{\omega_1}^M \cup \mathcal{A}^M, \in_{\Delta_0}^M)$ is an elementary substructure of $(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V)$ cannot be defined in $(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V)$.

- for any $A \in H_{\omega_2}$ and $B \in \mathcal{A}$, we can find $M \in D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ with $B \in \mathcal{A}_M$, some $a \in H_{\omega_2}^M$, and an iteration $\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1\}$ of M with $j_{0\omega_1}(\mathbf{NS}_{\omega_1}^M) = \mathbf{NS}_{\omega_1}^V \cap j_{0\omega_1}(H_{\omega_2}^M)$ such that $j_{0\omega_1}(a) = A$.
- $D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ is correctly⁵⁴ computable in $(L(\mathcal{A}), \in_{\Delta_0})$ using $D_{\mathcal{A}}$.

This effort will pay off, since we will then be able to prove the model completeness of the $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ -theory⁵⁵

$$(H_{\omega_2}, \in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}^V)$$

using Robinson's test with $\bar{D}_{\mathbf{NS}_{\omega_1}, \mathcal{A}} = \text{Cod}_{\omega}^{-1}[D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}]$ being the extension of the formula $\phi_{\mathbf{NS}_{\omega_1}, \mathcal{A}}(x)$ mentioned in Thm. 5.2.1, and showing that for a quantifier free $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ -formula $\phi(\vec{x}, y)$ and $A \in H_{\omega_2}$ we have that

$$(H_{\omega_2}^V, \in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}^V) \models \exists \vec{x} \phi(\vec{x}, A)$$

if and only if

For all $M \in D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ and for all \mathcal{J} iteration of M of length ω_1 mapping correctly the nonstationary ideal, if there is $a \in M$ mapped via \mathcal{J} to A , then

$$(H_{\omega_2}^M, \in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}^M) \models \exists \vec{x} \phi(\vec{x}, a).$$

A bit of work shows that the latter quoted statement is formalized by a Π_1 -formula for $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ using the predicate symbol $\text{Cod}_{\omega}^{-1}[D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}]$ together with those appearing in ϕ .

We now get into the details.

5.4.2. \mathcal{A} -correct models.

Definition 5.4.1. Given M, N iterable structures, $M \geq N$ if $M \in (H_{\omega_1})^N$ and there is an iteration

$$\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \gamma = (\omega_1)^N\}$$

of M with $\mathcal{J} \in N$ such that

$$\mathbf{NS}_{\gamma}^{M_{\gamma}} = \mathbf{NS}_{\gamma}^N \cap M_{\gamma}.$$

Notation 5.4.2. Given a countable family $\mathcal{X} = \{B_n : n \in \omega\}$ of universally Baire sets with each $B_n \subseteq (2^{\omega})^{k_n}$, we say that $B_{\mathcal{X}} = \prod_{n \in \omega} B_n \subseteq \prod_n (2^{\omega})^{k_n}$ is a code for $\{B_n : n \in \omega\}$.

Clearly $B_{\mathcal{X}}$ is a universally Baire subset of the compact Polish space $\prod_n (2^{\omega})^{k_n}$.

Definition 5.4.3. Let \mathcal{A} be a generically tame family of V .

A transitive model of ZFC (M, \in) is \mathcal{A} -correct if there is \mathcal{A}_M countable family of universally Baire sets in V such that:

- The map

$$\begin{aligned} \Theta_M : \mathcal{A}_M &\rightarrow M \\ A &\mapsto A \cap M \end{aligned}$$

is injective.

- (M, \in) models that $\{A \cap M : A \in \mathcal{A}_M\}$ is the family of sets in M satisfying $\phi_{\mathcal{A}}(x)$ in (M, \in) .
- $(H_{\omega_1}^M \cup \{A \cap M : A \in \mathcal{A}_M\}, \in) \prec (H_{\omega_1}^V \cup \mathcal{A}^V, \in)$ via the identity on H_{ω_1} and $A \cap M \mapsto A$ on $\{A \cap M : A \in \mathcal{A}_M\}$.
- If M is countable, M is \mathcal{A} -iterable for all $A \in \mathcal{A}_M$.

M is absolutely \mathcal{A} -correct if for all $N \geq M$:

⁵⁴The key point is that we will be able to define by a formula in parameter $D_{\mathcal{A}}$ an element of \mathcal{A} which is exactly $D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ if $(*)$ - \mathcal{A} holds.

⁵⁵Note that aiming for the model completeness result for the theory of H_{ω_2} we are loosing a certain flavour of second order logic: $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ is a signature considered in a structure where universally Baire sets are considered only as parameters, but not as objects over which we can quantify.

N is \mathcal{A} -correct in V if and only if $(M, \in) \models N$ is \mathcal{A}^M -correct.

Notation 5.4.4. $D_{\mathcal{A}}$ denotes the set of countable absolutely \mathcal{A} -correct M ; $\bar{D}_{\mathcal{A}} = \text{Cod}_{\omega}^{-1}[D_{\mathcal{A}}]$.

For each $M \in D_{\mathcal{A}}$, \mathcal{A}_M is a witness that M is \mathcal{A} -correct and $B_{\mathcal{A}_M} = \prod \mathcal{A}_M$ is a universally Baire set in \mathcal{A} coding this witness.

For universally Baire sets $B_1, \dots, B_k \in \mathcal{A}$, $E_{\mathcal{A}, B_1, \dots, B_k}$ denotes the set of $M \in D_{\mathcal{A}}$ with $B_1, \dots, B_k \in \mathcal{A}_M$ for some witness \mathcal{A}_M that $M \in D_{\mathcal{A}}$; $\bar{E}_{\mathcal{A}, B_1, \dots, B_k} = \text{Cod}_{\omega}^{-1}[E_{\mathcal{A}, B_1, \dots, B_k}]$.

Fact 5.4.5. (V, \in) models M is countable and (absolutely) \mathcal{A} -correct as witnessed by⁵⁶ \mathcal{A}_M if and only if so does $(L(\mathcal{A}), \in)$.

Consequently the set $D_{\mathcal{A}}$ of countable absolutely \mathcal{A} -correct M is properly computed in $(L(\mathcal{A}), \in)$.

Therefore assuming $\mathbf{MAX}(\mathcal{A})$

$$\bar{D}_{\mathcal{A}} = \text{Cod}^{-1}[D_{\mathcal{A}}]$$

is universally Baire.

The same holds for $\bar{E}_{\mathcal{A}, B_1, \dots, B_k}$ for given universally Baire sets B_1, \dots, B_k .

Proof. The first part follows almost immediately by the definitions, since the assertion in parameters B, M :

$B = \prod_{n \in \omega} B_n$ codes an H_{ω_1} -closed family $\mathcal{A}_M = \{B_n : n \in \omega\}$ of sets such that

- M is \mathcal{A} -iterable for all $A \in \mathcal{A}_M$,
- M models that $\{A \cap M : A \in \mathcal{A}_M\}$ is the family of sets realizing $\phi_{\mathcal{A}}(x)$ in M and is H_{ω_1} -closed,
- $(H_{\omega_1}^M \cup \{A \cap M : A \in \mathcal{A}_M\}, \in_{\Delta_0}^M)$ embeds elementarily into $(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V)$ via the map extending the identity on $H_{\omega_1}^M$ by $A \cap M \mapsto A$,

gets the same truth value in (V, \in) and in $(L(\mathcal{A}), \in)$. Note that $B = \prod \mathcal{A}_M \in \mathcal{A}$ by countable closure of \mathcal{A} .

We conclude that $D_{\mathcal{A}}$ has the same extension in (V, \in) and in $(L(\mathcal{A}), \in)$. By $\mathbf{MAX}(\mathcal{A})$ $\bar{D}_{\mathcal{A}}$ is universally Baire.

The same argument can be replicated for $\bar{E}_{\mathcal{A}, B_1, \dots, B_k}$. □

Note that while the notion of being absolutely \mathcal{A} -correct is a priori not computable in $(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V)$, if $M \leq N$ are both absolutely \mathcal{A} -correct in V , there is an elementary embedding of $(H_{\omega_1}^N \cup \mathcal{A}^N, \in_{\Delta_0}^N)$ into $(H_{\omega_1}^M \cup \mathcal{A}^M, \in_{\Delta_0}^M)$ which belongs to M and is coded by a universally set of V which belongs to \mathcal{A}_M whenever the latter is a witness that M is \mathcal{A} -correct.

Lemma 5.4.6. Assume \mathbf{NS}_{ω_1} is precipitous and there are class many Woodin cardinals in V . Let δ be an inaccessible cardinal in V and G be V -generic for $\text{Coll}(\omega, \delta)$. Then V_{δ} is absolutely $\mathcal{A}^{V[G]}$ -correct in $V[G]$ as witnessed by $\prod \{B^{V[G]} : B \in \mathcal{A}^V\}$. Furthermore V_{δ} is $B^{V[G]}$ -iterable for all $B \in \mathcal{A}^V$.

Proof. $\{B^{V[G]} : B \in \mathcal{A}^V\}$ is a countable family of universally Baire sets in $\mathcal{A}^{V[G]}$ hence its product belongs to $\mathcal{A}^{V[G]}$ by countable closure of \mathcal{A} in $V[G]$.

By $\mathbf{MAX}(\mathcal{A})$ there is an elementary $j : L(\mathcal{A}^V) \rightarrow L(\mathcal{A}^{V[G]})$ which is the identity on H_{ω_1} and maps $B \in \mathcal{A}$ to $B^{V[G]}$ in $\mathcal{A}^{V[G]}$.

By Fact 5.4.5 V and $L(\mathcal{A}^V)$ (respectively $V[G]$ and $L(\mathcal{A}^{V[G]})$) agree on which elements of $H_{\omega_1}^V$ are \mathcal{A} -correct.

⁵⁶Note that the map $M \mapsto \mathcal{A}_M$ can be defined in V but possibly not in $L(\mathcal{A})$, however the binary relation $\{(M, B) : B \in \mathcal{A} \text{ is a witness that } M \text{ is } \mathcal{A}\text{-correct}\}$ is in $L(\mathcal{A})$. This suffices for all our proofs.

The above grants that the unique missing condition to get the $\mathcal{A}^{V[G]}$ -correctness of V_δ is to check that it is $B^{V[G]}$ -iterable for all $B \in \mathcal{A}^V$. This is a standard argument which can be reconstrued looking for example at the proof of [13, Thm. 4.10]. \square

Fact 5.4.7. (MAX(\mathcal{A})) *Assume \mathbf{NS}_{ω_1} is precipitous and MAX(\mathcal{A}) holds. Then for any iterable M , $B_1, \dots, B_k \in \mathcal{A}$, there is $N \leq M$ such that:*

- N is absolutely \mathcal{A} -correct as witnessed by \mathcal{A}_N ;
- $B_1, \dots, B_k \in \mathcal{A}_N$.

Proof. The assumptions grant that whenever G is $\text{Coll}(\omega, \delta)$ -generic for V , V_δ is absolutely $\mathcal{A}^{V[G]}$ -correct in $V[G]$ and $B^{V[G]}$ iterable for all $B \in \mathcal{A}^V$ (i.e. Lemma 5.4.6).

By [13, Lemma 2.8], for any iterable $M \in H_{\omega_1}^V$ there is in V an iteration $\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^V\}$ of M such that $\mathbf{NS}_{\omega_1}^V \cap M_{\omega_1} = \mathbf{NS}_{\omega_1}^{M_{\omega_1}}$.

By MAX(\mathcal{A})

$$(L(\mathcal{A}^V), \in_{\Delta_0}) \prec (L(\mathcal{A}^{V[G]}), \in_{\Delta_0})$$

via a map which is the identity on H_{ω_1} and maps B to $B^{V[G]}$ for $B \in \mathcal{A}$. Therefore we have that in $V[G]$ $\bar{E}_{\mathcal{A}, B_1, \dots, B_k}^{V[G]}$ is exactly $\bar{E}_{\mathcal{A}, B_1^{V[G]}, \dots, B_k^{V[G]}}$.

Also for any $B \in \mathcal{A}^V$, V_δ is $B^{V[G]}$ -iterable, and $B^{V[G]} \in \mathcal{A}_{V_\delta} = \{A^{V[G]} : A \in \mathcal{A}^V\}$;

Hence for each iterable $M \in H_{\omega_1}^V$ and $B_1, \dots, B_k \in \mathcal{A}^V$, $N = V_\delta \leq M$ belongs to $E_{\mathcal{A}, B_1, \dots, B_k}^{V[G]}$ as witnessed by $\mathcal{A}_{V_\delta} = \{U^{V[G]} : U \in \mathcal{A}\}$ and $B_1^{V[G]}, \dots, B_k^{V[G]} \in \mathcal{A}^{V[G]}$.

Note that the Σ_1 -formulae for \in_{Δ_0} we established to be true in $H_{\omega_1}^{V[G]} \cup \mathcal{A}^{V[G]}$ as witnessed by V_δ are in parameters $M \in H_{\omega_1}^V, B_1^{V[G]}, \dots, B_k^{V[G]}, D_{\mathcal{A}}^{V[G]}$ as B_1, \dots, B_k vary in \mathcal{A}^V . Hence these formulae reflect to $H_{\omega_1}^V \cup \mathcal{A}^V$.

The Lemma is proved. \square

5.5. Three characterizations of $(*)$ - \mathcal{A} . From now for \mathcal{X} a set of universally Baire sets, we let $\in_{\mathcal{X}} \in \in_{\Delta_0} \cup \mathcal{X}$ and $\in_{\mathbf{NS}_{\omega_1}, \mathcal{A}_M}$ is $\in_{\mathcal{A}_M}$ enriched with a e symbol for \mathbf{NS}_{ω_1} and a constant symbol for ω_1 .

Definition 5.5.1. An absolutely \mathcal{A} -correct M is $(\mathbf{NS}_{\omega_1}, \mathcal{A})$ -ec if (M, \in) models that \mathbf{NS}_{ω_1} is precipitous and there is a witness \mathcal{A}_M that M is \mathcal{A} -correct with the following property:

Assume an iterable $N \leq M$ is absolutely \mathcal{A} -correct with a witness \mathcal{A}_N such that $\prod \mathcal{A}_M \in \mathcal{A}_N$.

Then for all iterations

$$\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \gamma = \omega_1^N\}$$

in N witnessing $M \geq N$, we have that $j_{0\gamma}$ defines a Σ_1 -elementary embedding of

$$(H_{\omega_2}^M \cup \mathcal{A}^M, \in_{\Delta_1}^M, \mathbf{NS}_{\omega_1}^M)$$

into

$$(H_{\omega_2}^N \cup \mathcal{A}^N, \in_{\Delta_1}^N, \mathbf{NS}_{\omega_1}^N).$$

Remark 5.5.2. A crucial observation is that “ x is $(\mathbf{NS}_{\omega_1}, \mathcal{A})$ -ec” is a property correctly definable in $(H_{\omega_1} \cup \mathcal{A}, \in)$ using as parameter the set⁵⁷ $D_{\mathcal{A}}$. Therefore (assuming MAX(\mathcal{A}))

$$D_{\mathbf{NS}_{\omega_1}, \mathcal{A}} = \{M \in H_{\omega_1} : M \text{ is } (\mathbf{NS}_{\omega_1}, \mathcal{A})\text{-ec}\}$$

is such that $\bar{D}_{\mathbf{NS}_{\omega_1}, \mathcal{A}} = \text{Cod}_\omega^{-1}[D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}]$ is a universally Baire set in \mathcal{A} . Moreover letting for $V[G]$ a generic extension of V

$$D_{\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]}} = \left\{ M \in H_{\omega_1}^{V[G]} : M \text{ is } (\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]})\text{-ec} \right\},$$

⁵⁷Note however that $D_{\mathcal{A}}$ may not be lightface definable in $(H_{\omega_1} \cup \mathcal{A}, \in)$.

we have that

$$\bar{D}_{\mathbf{NS}_{\omega_1}, \mathcal{A}}^{V[G]} = \text{Cod}_{\omega}^{-1}[D_{\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]}}].$$

Theorem 5.5.3. *Assume V models $\mathbf{MAX}(\mathcal{A})$. The following are equivalent:*

- (1) *Woodin's axiom $(*)$ - \mathcal{A} holds (i.e. \mathbf{NS}_{ω_1} is precipitous, and there is an $L(\mathcal{A})$ -generic filter G for \mathbb{P}_{\max} such that $L(\mathcal{A})[G] \supseteq \mathcal{P}(\omega_1)^V$).*
- (2) *Let δ be inaccessible. Whenever G is V -generic for $\text{Coll}(\omega, \delta)$, V_δ is $(\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]})$ -ec in $V[G]$.*
- (3) *\mathbf{NS}_{ω_1} is precipitous and for all $A \in H_{\omega_2}$, $B \in \mathcal{A}$, there is an $(\mathbf{NS}_{\omega_1}, \mathcal{A})$ -ec M with witness \mathcal{A}_M , and an iteration $\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1\}$ of M such that:*
 - $A \in M_{\omega_1}$,
 - $B \in \mathcal{A}_M$,
 - $\mathbf{NS}_{\omega_1}^{M_{\omega_1}} = \mathbf{NS}_{\omega_1} \cap M_{\omega_1}$.

Theorem 5.5.3 is the key to the proofs of Theorem 5.2.1 and to the missing implication in the proof of Theorem 5.1.

5.5.1. *Proof of Theorem 5.2.1.* The theorem is an immediate corollary of the following:

Lemma 5.5.4. *Let B_1, \dots, B_k be new predicate symbols and $S_{\mathbf{NS}_{\omega_1}, \mathcal{A}, B_1, \dots, B_k}$ be the $\in \mathbf{NS}_{\omega_1}, \mathcal{A} \cup \{B_1, \dots, B_k\}$ -theory $\text{ZFC}_{\mathbf{NS}_{\omega_1}, \mathcal{A}} + (*)$ - \mathcal{A} enriched with the sentences asserting that B_1, \dots, B_k have as extension elements of \mathcal{A} .*

Let E_{B_1, \dots, B_k} consists of the set of $M \in D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ such that:

- M is B_j -iterable for all $j = 1, \dots, k$;
- there is \mathcal{A}_M witnessing $M \in D_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$ with $B_j \in \mathcal{A}_M$ for all j .

Let also $\bar{E}_{B_1, \dots, B_k} = \text{Cod}_{\omega}^{-1}[E_{B_1, \dots, B_k}]$.

Then $S_{\mathbf{NS}_{\omega_1}, \mathcal{A}, B_1, \dots, B_k}$ proves that $\bar{E}_{B_1, \dots, B_k}$ is in \mathcal{A} .

Moreover:

- *Let $S_{\mathbf{NS}_{\omega_1}, \mathcal{A}, B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}}$ be the natural extension of $S_{\mathbf{NS}_{\omega_1}, \mathcal{A}, B_1, \dots, B_k}$ adding a predicate symbol for $\bar{E}_{B_1, \dots, B_k}$ to $\in \mathbf{NS}_{\omega_1}, \mathcal{A} \cup \{B_1, \dots, B_k\}$ and the axioms stating that the interpretation of $\bar{E}_{B_1, \dots, B_k}$ is given by its definition.*
- *Let $T_{\mathbf{NS}_{\omega_1}, \mathcal{A}, B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}}$ be the family of Π_2 -sentences ψ for $\in \mathbf{NS}_{\omega_1}, \mathcal{A} \cup \{B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}\}$ such that $S_{\mathbf{NS}_{\omega_1}, \mathcal{A}, B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}}$ proves $\psi^{H_{\omega_2}}$.*

Then $T_{\mathbf{NS}_{\omega_1}, \mathcal{A}, B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}}$ proves that every existential formula for $\in \mathbf{NS}_{\omega_1}, \mathcal{A} \cup \{B_1, \dots, B_k\}$ is equivalent to a universal formula for $\in \mathbf{NS}_{\omega_1}, \mathcal{A} \cup \{B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}\}$.

Proof. $\bar{E}_{B_1, \dots, B_k}$ is universally Baire and in \mathcal{A} by $\mathbf{MAX}(\mathcal{A})$, since E_{B_1, \dots, B_k} is definable in $(H_{\omega_1} \cup \mathcal{A}, \in)$ with parameters the universally Baire sets $B_1, \dots, B_k, \bar{D}_{\mathbf{NS}_{\omega_1}, \mathcal{A}}$.

Given any Σ_1 -formula $\phi(\vec{x})$ for $\in \mathbf{NS}_{\omega_1}, \mathcal{A} \cup \{B_1, \dots, B_k\}$ mentioning the universally Baire predicates B_1, \dots, B_k , we want to find a universal formula $\psi(\vec{x})$ for $\in \mathbf{NS}_{\omega_1}, \mathcal{A} \cup \{B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}\}$ such that

$$T_{\{B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}\}, \mathbf{NS}_{\omega_1}} \models \forall \vec{x} (\phi(\vec{x}) \leftrightarrow \psi(\vec{x})).$$

Let $\theta_\phi(\vec{x})$ be the formula asserting:

For all $M \in E_{B_1, \dots, B_k}$, for all iterations $\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1\}$ of M such that:

- $\vec{x} = j_{0\omega_1}(\vec{a})$ for some $\vec{a} \in M$,
- $\mathbf{NS}_{\omega_1}^{j_{0\omega_1}(M)} = \mathbf{NS}_{\omega_1} \cap j_{0\omega_1}(M)$,
- $(H_{\omega_2}^M, \in_{\mathcal{A}, \mathbf{NS}_{\omega_1}}^M) \models \phi(\vec{a})$.

More formally:

$$\begin{aligned}
& \forall r \forall \mathcal{J} \{ \\
& \quad [\\
& \quad (r \in \bar{E}_{B_1, \dots, B_k}) \wedge \\
& \quad \wedge \mathcal{J} = \{j_\alpha \beta : \alpha \leq \beta \leq \omega_1\} \text{ is an iteration of } \text{Cod}(r) \wedge \\
& \quad \wedge \mathbf{NS}_{\omega_1}^{j_{0\omega_1}(\text{Cod}(r))} = \mathbf{NS}_{\omega_1} \cap j_{0\omega_1}(\text{Cod}(r)) \wedge \\
& \quad \wedge \exists \vec{a} \in \text{Cod}(r) (\vec{x} = j_{0\omega_1}(\vec{a})) \\
& \quad] \\
& \quad \rightarrow \\
& \quad (H_{\omega_2}^{\text{Cod}(r)}, \in_{\mathcal{A}^{\text{Cod}(r)}, \mathbf{NS}_{\omega_1}}) \models \phi(\vec{a}) \\
& \quad \}.
\end{aligned}$$

The above is a Π_1 -formula for $\in_{\Delta_1} \cup \{\omega_1, \mathbf{NS}_{\omega_1}\} \cup \{B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}\}$.

(We leave to the reader to check that the property

$$\mathcal{J} = \{j_\alpha \beta : \alpha \leq \beta \leq \omega_1\} \text{ is an iteration of } M \text{ such that } \mathbf{NS}_{\omega_1}^{j_{0\omega_1}(M)} = \mathbf{NS}_{\omega_1} \cap j_{0\omega_1}(M)$$

is definable by a Δ_1 -property in parameters M, \mathcal{J} in the signature $\in_{\Delta_0} \cup \{\omega_1, \mathbf{NS}_{\omega_1}\}$).

Now it is not hard to check that:

Claim 3. For all $\vec{A} \in H_{\omega_2}$

$$(H_{\omega_2}^V, \in_{\mathbf{NS}_{\omega_1}}^V, B_1, \dots, B_k) \models \phi(\vec{A})$$

if and only if

$$(H_{\omega_2}, \in_{\mathbf{NS}_{\omega_1}}^V, B_1, \dots, B_k, \bar{E}_{B_1, \dots, B_k}) \models \theta_\phi(\vec{A}).$$

Proof.

$\theta_\phi(\vec{A}) \rightarrow \phi(\vec{A})$: Take any M and \mathcal{J} satisfying the premises of the implication⁵⁸ in $\theta_\phi(\vec{A})$,

Then $(H_{\omega_2}^M, \in_{\mathbf{NS}_{\omega_1}, \mathcal{A}^M}^M) \models \phi(\vec{a})$ for some \vec{a} such that $j_{0, \omega_1}(\vec{a}) = \vec{A}$ and $B_j \cap M_{\omega_1} = j_{0\omega_1}(B_j \cap M)$ for all $j = 1, \dots, k$.

Since Σ_1 -properties are upward absolute and $(M_{\omega_1}, \in_{\mathbf{NS}_{\omega_1}}^{M_{\omega_1}}, B_j \cap M_{\omega_1} : j = 1, \dots, k)$ is a $\in_{\mathbf{NS}_{\omega_1} \cup \{B_1, \dots, B_k\}}$ -substructure of $(H_{\omega_2}, \in_{\mathbf{NS}_{\omega_1}}^V, B_j : j = 1, \dots, k)$ which models $\phi(\vec{A})$, we get that $\phi(\vec{A})$ holds for $(H_{\omega_2}, \in_{\mathbf{NS}_{\omega_1}}^V, B_1, \dots, B_k)$.

$\phi(\vec{A}) \rightarrow \theta_\phi(\vec{A})$: Assume

$$(H_{\omega_2}, \in_{\mathbf{NS}_{\omega_1}}^V, B_1, \dots, B_k) \models \phi(\vec{A}).$$

Take any $(\mathbf{NS}_{\omega_1}, \mathcal{A})$ -ec $M \in V$ and any iteration $\mathcal{J} = \{j_\alpha \beta : \alpha \leq \beta \leq \omega_1\}$ of M witnessing the premises of the implication in $\psi(\vec{A})$, in particular such that:

- $\vec{A} = j_{0\omega_1}(\vec{a}) \in M_{\omega_1}$ for some $\vec{a} \in M$,
- $\mathbf{NS}_{\omega_1}^{M_{\omega_1}} = \mathbf{NS}_{\omega_1} \cap M_{\omega_1}$,
- M is B_j -iterable for $j = 1, \dots, k$.

Such M and \mathcal{J} exists by Thm. 5.5.3(3) applied to $\bar{E}_{B_1, \dots, B_k}$ and \vec{A} .

Let G be V -generic for $\text{Coll}(\omega, \delta)$ with δ inaccessible. Then in $V[G]$, V_δ is $\mathcal{A}^{V[G]}$ -correct, by Lemma 5.4.6.

⁵⁸At least one such M exists by $(*)$ -A.

Therefore (since M is $(\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]})$ -ec also in $V[G]$ by $\mathbf{MAX}(\mathcal{A})$), $V[G]$ models that $j_{0\omega_1^V}$ is a Σ_1 -elementary embedding of⁵⁹

$$(H_{\omega_2}^M, \in_{\mathbf{NS}_{\omega_1}^M}, B \cap M : B \in \mathcal{A}_M)$$

into

$$(H_{\omega_2}^V, \in_{\mathbf{NS}_{\omega_1}^V}, B : B \in \mathcal{A}_M).$$

This grants that

$$(H_{\omega_2}^M, \in_{\mathbf{NS}_{\omega_1}^M}, B \cap M : B \in \mathcal{A}_M) \models \phi(\vec{a}),$$

as was to be shown. □

The Lemma is proved. □

Remark 5.5.5. Note that by essentially the same proof we can argue that the $\in_{\mathbf{NS}_{\omega_1}}$ -theory of $H_{\aleph_2} \cup \mathcal{A}$ in models of $\mathbf{ZFC}_{\mathbf{NS}_{\omega_1}} + \mathbf{MAX}(\mathcal{A})$ is model complete. However this result is not relevant for the AMC-spectrum results we are aiming for, since $(H_{\aleph_2} \cup \mathcal{A}, \in_{\Delta_1}, \mathbf{NS}_{\omega_1})$ is not a Σ_1 -substructure of $(V, \in_{\Delta_1}, \mathbf{NS}_{\omega_1})$.

5.5.2. Proof of (2) \rightarrow (1) of Theorem 5.1.

Proof. Assume δ is supercompact, P is a standard forcing notion to force \mathbf{MM}^{++} of size δ (such as the one introduced in [8] to prove the consistency of Martin's maximum), and G is V -generic for P ; then $(*)$ - \mathcal{A} holds in $V[G]$ by Asperó and Schindler's recent breakthrough [2]. By Thm. 4.1 V and $V[G]$ agree on the Π_1 -fragment of the $\in_{\mathbf{NS}_{\omega_1}} \cup \mathcal{A}^V$ -theory \bar{T} of V , therefore so do $H_{\omega_2}^V$ and $H_{\omega_2}^{V[G]}$ (by Lemma 3.1 applied in V and $V[G]$ respectively).

Since $P \in \mathbf{SSP}$

$$(H_{\omega_2}^V, \in_{\mathbf{NS}_{\omega_1}^V}, A : A \in \mathcal{A}^V) \sqsubseteq (H_{\omega_2}^{V[G]}, \in_{\mathbf{NS}_{\omega_1}^{V[G]}}, A^{V[G]} : A \in \mathcal{A}^V).$$

Now the model completeness of the $\in_{\mathbf{NS}_{\omega_1}} \cup \mathcal{A}^V$ -theory \bar{S} of $H_{\omega_2}^V$ grants that $H_{\omega_2}^V$ is \bar{T}_V -ec. This gives that:

$$(H_{\omega_2}^V, \in_{\mathbf{NS}_{\omega_1}^V}, \mathcal{A}^V) \prec_{\Sigma_1} (H_{\omega_2}^{V[G]}, \in_{\mathbf{NS}_{\omega_1}^{V[G]}}, A^{V[G]} : A \in \mathcal{A}^V).$$

Therefore any Π_2 -property for $\in_{\mathbf{NS}_{\omega_1}} \cup \mathcal{A}^V$ with parameters in $H_{\omega_2}^V$ which holds in

$$(H_{\omega_2}^{V[G]}, \in_{\mathbf{NS}_{\omega_1}^{V[G]}}, A^{V[G]} : A \in \mathcal{A})$$

also holds in $(H_{\omega_2}^V, \in_{\mathbf{NS}_{\omega_1}^V}, \mathcal{A}^V)$.

Hence in $H_{\omega_2}^V$ it holds characterization (3) of $(*)$ - \mathcal{A} given by Thm. 5.5.3 and we are done. □

5.5.3. Proof of Theorem 5.5.3.

Definition 5.5.6. [13, Def. 2.1] \mathbb{P}_{\max} is the subset of H_{ω_1} given by the pairs (M, a) such that

- M is iterable, countable, and models Martin's axiom.
- $a \in \mathcal{P}(\omega_1)^M \setminus L(\mathbb{R})^M$, and there exists $r \in \mathcal{P}(\omega) \cap M$ such that $\omega_1^M = \omega_1^{L[a,r]}$.

$(M, a) \leq (N, b)$ if there exists $\mathcal{J} = \langle j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^M \rangle$ in M iteration of N of length ω_1^M such that $j_{0\omega_1^M}(b) = a$ and (M, \in) models that \mathcal{J} is correct.

⁵⁹Actually is Σ_1 -elementary between the structures $(H_{\omega_2}^M \cup \mathcal{A}^M, \in_{\Delta_0}, \mathbf{NS}_{\omega_1}^M)$ and $(H_{\omega_2}^V \cup \mathcal{A}^V, \in_{\Delta_0}, \mathbf{NS}_{\omega_1}^V)$, but we only need the weaker form of Σ_1 -elementarity described in the proof.

Note that \mathbb{P}_{\max} is a definable class in (H_{ω_1}, \in) ; in particular it belongs to any transitive model of ZFC^- containing $\mathcal{P}(\omega)$.

Our definition of \mathbb{P}_{\max} is slightly different than the one given in [13], but it defines a dense subset of the poset defined in [13, Def. 2.1] in view of the following⁶⁰:

Fact 5.5.7. *Let \mathbb{P}_{\max}^0 be the forcing defined⁶¹ in [13, Def. 2.1]. Assume there are class many Woodin cardinals.*

Then for every condition (M, I, a) in \mathbb{P}_{\max}^0 there is a condition (N, b) in \mathbb{P}_{\max} and an iteration $\langle j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^N \rangle \in N$ of (M, I) according to [13, Def. 1.2] such that $j_{0\omega_1^N}(I) = \mathbf{NS}_{\omega_1^N}^N \cap j_{0\omega_1^N}[M]$ and $j_0(a) = b$. Hence $(N, \mathbf{NS}_{\omega_1^N}^N, b)$ refines (M, I, a) in \mathbb{P}_{\max}^0 .

Proof. Let $\gamma > \delta$ be two Woodin cardinals. Let $X \prec V_\gamma$ be countable with $\delta, (M, I, a) \in X$. Let N_0 be the transitive collapse of X , and N be a generic extension of N_0 by a forcing collapsing δ to become ω_2 and forcing \mathbf{NS}_{ω_1} is precipitous and Martin's axiom. Since γ is Woodin, there are class many measurables in V_γ , hence N_0 is iterable and so is N (by [13, Thm. 4.10]).

By [13, Lemma 2.8] there is in N the required iteration $\langle j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^N \rangle$ of (M, I) and we can set $b = j_{0\omega_1^N}(a)$. \square

In particular (at the prize of assuming the right large cardinal assumptions) the forcings \mathbb{P}_{\max} and \mathbb{P}_{\max}^0 are equivalent as the former sits inside the latter as a dense subset.

This is a key property of \mathbb{P}_{\max} we will need, and is based on Asperò and Schindler result that $\text{MM}^{++} + \mathbf{MAX}(\mathcal{A})$ implies $(*)\text{-}\mathcal{A}$:

Lemma 5.5.8. *Assume $\mathbf{MAX}(\mathcal{A})$ and there is a supercompact cardinal. Let $\dot{A} \in L(\mathbb{R})$ and $\dot{N} \in L(\mathcal{A})$ be the \mathbb{P}_{\max} -canonical names respectively for:*

- $\bigcup \{a : (N, a) \in G\}$,
- $H_{\omega_2}^{L(\mathcal{A})[G]} \cup \mathcal{A}$,

whenever G is a \mathbb{P}_{\max} -generic filter for $L(\mathcal{A})$.

For any quantifier free formula $\phi(x, y, z)$ for $\in_{\Delta_1} \cup \{\mathbf{NS}_{\omega_1}\}$ and $B \in \mathcal{A}^V$

$$L(\mathcal{A}^V) \models \left[(N, a) \Vdash \exists y \in \dot{N} \phi(\dot{A}, y, \dot{B}) \right]$$

if and only if the set D_ϕ of (M, e) such that

- M is B -iterable,
- $(H_{\omega_2}^M \cup \mathcal{A}^M, \in_{\Delta_1}^M, \mathbf{NS}_{\omega_1}^M) \models \exists y \phi(e, y, B \cap M)$,

is dense below (N, a) .

Proof. Let G be a \mathbb{P}_{\max} -generic filter for $L(\mathcal{A}^V)$ and $A = \dot{A}_G$. Then (by [13, Lemma 2.7])

$$G = \{(N, a) : \exists \mathcal{J}_N \text{ NS-correct iteration of } N \text{ mapping } a \text{ to } A\}.$$

If some $(M, a) \in G$ is such that:

- M is B -iterable,
- $(H_{\omega_2}^M \cup \mathcal{A}^M, \in_{\Delta_1}^M, \mathbf{NS}_{\omega_1}^M) \models \exists y \phi(a, y, B \cap M)$.

Let

$$\mathcal{J}_M = \{j_{\alpha,\beta} : M_\alpha \rightarrow M_\beta : \alpha \leq \beta \leq \omega_1\}.$$

Then

$$\mathcal{M} = (H_{\omega_2}^{M_{\omega_1}} \cup \mathcal{A}^{M_{\omega_1}}, \in_{\Delta_1}^{M_{\omega_1}}, \mathbf{NS}_{\omega_1}^{M_{\omega_1}}) \models \exists y \phi(A, y, B \cap M_{\omega_1}).$$

This yields that

$$\mathcal{N} = (H_{\omega_2}^{L(\mathcal{A})[G]} \cup \mathcal{A}, \in_{\Delta_1}^{L(\mathcal{A})[G]}, \mathbf{NS}_{\omega_1}^{L(\mathcal{A})[G]}) \models \exists y \phi(A, y, B),$$

⁶⁰Much weaker large cardinals assumptions are needed, we don't spell the optimal hypothesis.

⁶¹E.g. the forcing \mathbb{P}_{\max} according to the terminology of [13].

since it is not hard to check⁶² that $\mathcal{M} \sqsubseteq \mathcal{N}$ via the map which is the identity on $H_{\omega_2}^{L(\mathcal{A})[G]}$ and maps $B \cap M_{\omega_1}$ to B on $\mathcal{A}^{M_{\omega_1}}$.

Otherwise note that

$$D_\phi \cup \{(M, b) : \forall (N, c) \leq (M, b) (N, c) \notin D_\phi\}$$

is dense in \mathbb{P}_{\max} and belongs to $L(\mathcal{A})$. Hence for some $(M, a) \in G$, $L(\mathcal{A})$ models that for all $(N, c) \leq (M, a)$ which are B -iterable

$$(H_{\omega_2}^N \cup \mathcal{A}^N, \in_{\Delta_1}^N, \mathbf{NS}_{\omega_1}^N) \not\models \exists y \phi(c, y, B \cap N).$$

If the Lemma fails we can find $(N, c) \leq (M, a)$ in the above set such that

$$L(\mathcal{A}) \models \left[(N, c) \Vdash_{\mathbb{P}_{\max}} \exists y \in \dot{N} \phi(\dot{A}, y, \check{B}) \right].$$

In particular for any $(P, d) \leq (N, c)$ we have that

$$(22) \quad (H_{\omega_2}^P \cup \mathcal{A}^P, \in_{\Delta_1}^P, \mathbf{NS}_{\omega_1}^P) \not\models \exists y \phi(d, y, B \cap P),$$

and

$$(23) \quad L(\mathcal{A}) \models \left[(P, d) \Vdash_{\mathbb{P}_{\max}} \exists y \in \dot{N} \phi(\dot{A}, y, \check{B}) \right].$$

Fix in V

$$\mathcal{K} = \{k_{\alpha, \beta} : N_\alpha \rightarrow N_\beta : \alpha \leq \beta \leq \omega_1\}.$$

NS-correct iteration of N . Let $A = k_{0, \omega_1}(c)$.

Now let γ be a supercompact cardinal, $\delta > \gamma$ be inaccessible, and H be V -generic for $\text{Coll}(\omega, \delta)$. Then in $V[H]$ we can find K V -generic for some stationary set preserving forcing of V collapsing γ to become ω_2 , together with MM^{++} (and therefore $(*)$ - \mathcal{A} by Asperò and Schindler's result). Then $(V_\delta[K], A)$ is a \mathbb{P}_{\max} -condition in $V[H]$ refining (N, c) (as witnessed by \mathcal{K}), hence such that

$$(H_{\omega_2}^{V_\delta[K]} \cup \mathcal{A}^{V_\delta[K]}, \in_{\Delta_1}^{V_\delta[K]}, \mathbf{NS}_{\omega_1}^{V_\delta[K]}) \not\models \exists y \phi(A, y, B^{V_\delta[K]})$$

by 22. On the other hand since $(V_\delta[K], A)$ models $(*)$ - \mathcal{A} and $A \in \mathcal{P}(\omega_1) \setminus L(\mathcal{A}^{V[K]})$,

$$G_A = \{(P, d) : \exists \mathcal{J} \text{ NS-correct iteration of } P \text{ mapping } d \text{ to } A\}$$

is $L(\mathcal{A}^{V[K]})$ -generic for \mathbb{P}_{\max} , with (N, c) belonging to G_A ; therefore

$$(H_{\omega_2}^{L(\mathcal{A}^{V[K]})[G_A]} \cup \mathcal{A}^{V_\delta[K]}, \in_{\Delta_1}^{L(\mathcal{A}^{V[K]})[G_A]}, \mathbf{NS}_{\omega_1}^{L(\mathcal{A}^{V[K]})[G_A]}) \models \exists y \phi(A, y, B^{V_\delta[K]})$$

by 23. Since $H_{\omega_2}^{L(\mathcal{A}^{V[K]})[G_A]} = H_{\omega_2}^{V[K]}$, we have reached a contradiction. \square

We can now prove Thm. 5.5.3.

Proof. (1) implies (2): Let G be V -generic for $\text{Coll}(\omega, \delta)$. By Lemma 5.4.6, V_δ is absolutely $\mathcal{A}^{V[G]}$ -correct in $V[G]$ as witnessed by $\{B^{V[G]} : B \in \mathcal{A}^V\} = \mathcal{A}_V = \{B_n^{V[G]} : n \in \omega\}$ and $B^{V[G]}$ -iterable for all $B \in \mathcal{A}^V$.

Claim 4. V_δ is $(\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]})$ -ec in $V[G]$ as witnessed by \mathcal{A}_V .

⁶²If $\phi(x_1, \dots, x_n, y_1, \dots, y_k)$ is provably $\Delta_1(\text{ZFC}^-)$, one can prove by an induction on its syntactic complexity that for each $A_1, \dots, A_n \in H_{\omega_2}^{M_{\omega_1}}$ and each $B_1, \dots, B_k \in \mathcal{A}^{M_{\omega_1}}$ $\phi(A_1, \dots, A_n, B_1, \dots, B_k)$ holds in \mathcal{N} if and only if $\phi(A_1, \dots, A_n, B_1 \cap M_{\omega_1}, \dots, B_k \cap M_{\omega_1})$ holds in \mathcal{M} .

Proof. Let in $V[G]$ $B_{\mathcal{A}^V} = \prod_{n \in \omega} B_n^{V[G]}$ be the universally Baire set in $\mathcal{A}^{V[G]}$ coding \mathcal{A}^V .

Let (by Fact 5.4.7) $N \leq V_\delta$ in $V[G]$ be absolutely $\mathcal{A}^{V[G]}$ -correct, $B_{\mathcal{A}^V}$ -iterable, with $B_{\mathcal{A}^V} \in \mathcal{A}_N$ for some \mathcal{A}_N countable subset of $\mathcal{A}^{V[G]}$ witnessing that N is $\mathcal{A}^{V[G]}$ -correct.

Then it is not hard to check that

$$(24) \quad (H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_1}^V) \prec (H_{\omega_1}^N \cup \mathcal{A}^N, \in_{\Delta_1}^N)$$

via the map extending the identity on $H_{\omega_1}^V$ by $B \mapsto B^{V[G]} \cap N$ for $B \in \mathcal{A}^V$. This holds since $N \in D_{\mathcal{A}}^{V[G]}$ and $\{B^{V[G]} \cap N : B \in \mathcal{A}^V\}$ is in N .

Let

$$\mathcal{J} = \{j_{\alpha, \beta} : \alpha \leq \beta \leq \gamma = (\omega_1)^N\} \in N$$

be an iteration witnessing $V_\delta \geq N$ in $V[G]$.

We must show that

$$j_{0\gamma} : H_{\omega_2}^V \cup \mathcal{A}^V \rightarrow H_{\omega_2}^N \cup \mathcal{A}^N$$

is Σ_1 -elementary for $\in_{\Delta_1} \cup \{\mathbf{NS}_{\omega_1}\}$.

By simple coding tricks (e.g. coding any finite tuple of elements of H_{ω_2} by a subset of ω_1 via the map Cod_{ω_1} and any finite tuple of elements of \mathcal{A} by their product), it suffices to check that for any Σ_1 -formula $\phi(x, y)$ for $\in_{\Delta_1} \cup \{\mathbf{NS}_{\omega_1}\}$ $A \in \mathcal{P}(\omega_1)^V$, $B \in \mathcal{A}^V$

$$(H_{\omega_2}^N \cup \mathcal{A}^N, \in_{\Delta_1}^N, \mathbf{NS}_{\omega_1}^N) \models \phi(j_{0\gamma}(A), B^{V[G]} \cap N).$$

if and only if

$$(H_{\omega_2}^V \cup \mathcal{A}^V, \in_{\Delta_1}^V, \mathbf{NS}_{\omega_1}^V) \models \phi(A, B).$$

Now by (*)- \mathcal{A} in V ,

$$G_A = \{(N, a) : \exists \mathcal{J} \text{ NS-correct iteration of } N \text{ mapping } a \text{ to } A\}$$

is $L(\mathcal{A})$ -generic for \mathbb{P}_{\max} .

Recall the set D_ϕ defined in 5.5.8 and the dense set

$$E_\phi = D_\phi \cup \{(M, b) : \forall (N, c) \leq (M, b), (N, c) \notin D_\phi\}$$

which are both elements of $L(\mathcal{A})^V$, since both sets are the image under Cod_ω of universally Baire sets in \mathcal{A}^V . Hence (by 24) we can argue in $V[G]$ that $(N, j_{0\gamma}(A))$ witnesses that any $(M, b) \in G_A \cap E_\phi \subseteq E_\phi^{V[G]}$ is not in $E_\phi^{V[G]} \setminus D_\phi^{V[G]}$ (since $(M, b) \geq (V_\delta, A) \geq (N, j_{0\gamma}(A))$). Therefore any such $(M, b) \in G_A \cap E_\phi$ is in $D_\phi^{V[G]} \cap V = D_\phi$.

We conclude that

$$(H_{\omega_2}^V \cup \mathcal{A}^V, \in_{\Delta_1}^V, \mathbf{NS}_{\omega_1}^V) \models \phi(A, B),$$

by Lemma 5.5.8. □

(2) implies (3): Our assumptions grants that the set

$$D_{\mathcal{A}} = \{M \in H_{\omega_1}^V : M \text{ is absolutely } \mathcal{A}^V\text{-correct}\}$$

is coded by a universally Baire set $\bar{D}_{\mathcal{A}}$ in V . Moreover we also get that whenever G is V -generic for $\text{Coll}(\omega, \delta)$, the lift $\bar{D}_{\mathcal{A}}^{V[G]}$ of $\bar{D}_{\mathcal{A}}$ to $V[G]$ codes

$$D_{\mathcal{A}^{V[G]}}^{V[G]} = \left\{ M \in H_{\omega_1}^{V[G]} : M \text{ is absolutely } \mathcal{A}^{V[G]}\text{-correct} \right\}.$$

By (2) we get that $V_\delta \in D_{\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]}}^{V[G]}$.

By Fact 5.4.7

$(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V) \models$ for all iterable M there exists an absolutely \mathcal{A} -correct structure $\bar{M} \geq M$.

Again since

$$(H_{\omega_1}^V \cup \mathcal{A}^V, \in_{\Delta_0}^V) \prec (H_{\omega_1}^{V[G]} \cup \mathcal{A}^{V[G]}, \in_{\Delta_0}^{V[G]}),$$

and the latter is first order expressible in the predicate $\bar{D}_{\mathcal{A}} \in \mathcal{A}^V$, we get that

$(H_{\omega_1}^{V[G]} \cup \mathcal{A}^{V[G]}, \in_{\Delta_0}^{V[G]}) \models$ for all iterable M there exists an absolutely $\mathcal{A}^{V[G]}$ -correct structure $\bar{M} \geq M$.

So let $N \leq V_\delta$ be in $V[G]$ an absolutely $\mathcal{A}^{V[G]}$ -correct structure with $V_\delta \in H_{\omega_1}^N$ and $\{B^{V[G]} \cap N : B \in \mathcal{A}^V\} \in N$.

Let $\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \gamma = \omega_1^N\} \in H_{\omega_2}^N$ be an iteration witnessing $N \leq V_\delta$.

Now for any $A \in \mathcal{P}(\omega_1)^V$ and $B \in \mathcal{A}^V$

$$(H_{\omega_2}^N \cup \mathcal{A}^N, \in_{\Delta_1}^N, \mathbf{NS}_\gamma^N)$$

models

There exists an $(\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]})$ -ec structure M with $B^{V[G]} \cap N \in \mathcal{A}_M$ and an \mathbf{NS} -correct iteration $\bar{\mathcal{J}} = \{\bar{j}_{\alpha\beta} : \alpha \leq \beta \leq \gamma\}$ of M such that $\bar{j}_{0\gamma}(A) = j_{0\gamma}(A)$.

This statement is witnessed exactly by V_δ in the place of M (since $B = B^{V[G]} \cap V_\delta \in \mathcal{A}^V$ and $\mathcal{A}_{V_\delta}^{V[G]} = \{B^{V[G]} : B \in \mathcal{A}^V\}$), and \mathcal{J} in the place of $\bar{\mathcal{J}}$.

Since V_δ is $(\mathbf{NS}_{\omega_1}, \mathcal{A}^{V[G]})$ -ec in $V[G]$ we get that $j_{0\gamma} \upharpoonright H_{\omega_2}^V \cup \mathcal{A}^V$ is Σ_1 -elementary for $\in_{\Delta_1} \cup \{\mathbf{NS}_{\omega_1}\}$ between $H_{\omega_2}^V \cup \mathcal{A}^V$ and $H_{\omega_2}^N \cup \mathcal{A}^N$.

Hence

$$(H_{\omega_2}^V \cup \mathcal{A}^V, \in_{\Delta_1}^V, \mathbf{NS}_\gamma^V)$$

models

There exists an $(\mathbf{NS}_{\omega_1}^V, \mathcal{A}^V)$ -ec structure M with $B \in \mathcal{A}_M$ and an iteration $\bar{\mathcal{J}} = \{\bar{j}_{\alpha\beta} : \alpha \leq \beta \leq (\omega_1)^V\}$ of M such that $\bar{j}_{0\omega_1}(a) = A$ and $\mathbf{NS}_{\omega_1}^{\bar{j}_{0\omega_1}(M)} = \mathbf{NS}_{\omega_1}^V \cap \bar{j}_{0\omega_1}(M)$.

(3) implies (1): The key point is to prove that if M is $(\mathbf{NS}, \mathcal{A})$ -ec, $a \in \mathcal{P}(\omega_1)^M \setminus L(\mathbb{R})^M$, D is a dense open set of \mathbb{P}_{\max} such that $D = \text{Cod}_\omega[\bar{D}]$ for some $\bar{D} \in \mathcal{A}_M$, then there is some $(M_0, a_0) \geq (M, a)$ with $(M_0, a_0) \in D \cap M$.

Once this is achieved (3) gives immediately the desired conclusion.

So pick D, a as above. Find $(P, b) \leq (M, a) \in D$ and $(N, c) \leq (P, b)$ with N \mathcal{A} -ec and $\mathcal{A}_M \subseteq \mathcal{A}_N$.

Then N models (as witnessed by (P, b)) the Σ_1 -statement for $\in_{\Delta_1} \cup \{\mathbf{NS}_{\omega_1}\}$ in parameters \bar{D}, c :

There exists a \mathbf{NS} -correct iteration of some $(M_0, a_0) \in D^N$ which maps a_0 to c .

Since M is $(\mathbf{NS}, \mathcal{A})$ -ec and there is a unique \mathbf{NS} -correct iteration of M which maps a to c , we get that M models

There exists a \mathbf{NS} -correct iteration of some $(M_0, a_0) \in D^M = D \cap M$ which maps a_0 to a .

Now the rest of the argument is routine and is left to the reader. \square

SOME COMMENTS AND OPEN QUESTIONS

We believe there is still room to improve the model completeness results one can predicate from Woodin's axiom (*). Specifically we conjecture the following:

Conjecture 5.5.9. *Assume $\mathbf{MAX}(\mathbf{UB})$ and $(*)\text{-UB}$. Let Θ be the supremum of the ordinals α which are the surjective image of some $\phi : 2^\omega \rightarrow \text{Ord}$ which exists in $L(\mathbf{UB})$. Then the theory of $L_\Theta(\mathbf{UB}^{\omega_1})$ is model complete for the signature $\in_{\Delta_1} \cup \{\omega_1, \mathbf{NS}_{\omega_1}, \mathbf{UB}\}$ where \mathbf{UB} is a predicate symbol which detects which subsets of 2^ω are universally Baire.*

Note that the above conjecture does not say that Θ (the supremum of Wadge rank of universally Baire sets) is regular in V assuming V models $\mathbf{MAX}(\mathbf{UB}) + (*)\text{-UB}$ (it could certainly have cofinality ω_2 in V , and we conjecture it could not have cofinality ω_1). In fact an argument of Woodin combined with the results of [22] should bring that Θ cannot be regular in models of \mathbf{MM}^{+++} .

A subtle question is whether for ZFC the AMC-spectrum and the model companionship spectrum can be distinct. We conjecture the following:

Conjecture 5.5.10. *Assume $S + T_{\in, A}$ has a model companion for some \in -theory $S \supseteq \mathbf{ZFC}$ and some A with $\in_A \supseteq \in_{\Delta_0}$. Then $A \in \mathbf{spec}_{\text{AMC}}(S)$.*

Another set of open questions is whether the (generically invariant for suitable classes of forcings) theories of H_{\aleph_2} under bounded category forcing axioms (or under iterated resurrection axioms) isolated in [3, 4] produce model complete theories for some signature extending \in_{Δ_0} and for some theory extending $\mathbf{ZFC} + \text{large cardinals}$ with some Σ_2 -sentence not holding assuming $(*)\text{-UB}$ (for example the assertion that canonical functions are not dominating modulo clubs, or some other Σ_2 -sentence whose negation can only be forced using a stationary set preserving forcing which cannot be proper).

We believe it can also be interesting to investigate more the notions of model companionship spectrum or AMC-spectrum. For example: given a countable theory, analyze the descriptive set theoretic complexity of the partial order given by its AMC-spectrum under inclusion; can this be a useful measure to compare the complexity of countable theories? Are there other model-theoretic properties of a mathematical theory sensitive to the signature for which the spectrum makes sense? In which case what type of information can we extract from this spectrum?

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