PHD COURSE ON

MODEL* COMPANIONSHIP RESULTS

FOR SET THEORY

LECTURE 6

18/5/2022

Lemma Assume T, S are τ-theories (1)
and T is the AMC of S. Then for any
$R \supseteq S$ $T + R_\forall$ is the AMC of R.

Pf: $T + R_\forall$ is model complete (using Robinson's test)

$(T+R)_{\forall\vee\exists} \subseteq R_{\forall\vee\exists}$: take $\mathfrak{M} \models R$

by second embeddability Lemma and $S_{\forall\vee\exists} = T_{\forall\vee\exists}$

there exists $\mathfrak{N} \supseteq \mathfrak{M}$ s.t. $\mathfrak{N} \equiv_1 \mathfrak{M}$ $\mathfrak{N} \models T$

$\mathfrak{N} \models T + R_{\forall\vee\exists}$

$(T+R_\forall)_{\forall\vee\exists} \supseteq R_{\forall\vee\exists}$: Take $\mathfrak{M} \models T+R_\forall$ (2)

Then $(T+R_\forall)_\forall \supseteq R_\forall$. By first byembeddability Lemma $\exists\ \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{N} \models R \supseteq S$

$\mathfrak{M} \prec_1 \mathfrak{N}$ (since $\mathfrak{M}$ is $T_\forall \simeq S_\forall$-ec) $\Rightarrow$ $\mathfrak{N} \equiv_1 \mathfrak{M}$. Then conclusion follows by second byembeddability Lemma.

$\boxed{\mathfrak{M}}$

$\in_{\Delta_0} = \{\in\} \cup \{R_\phi : \phi \text{ a } \Delta_0\text{-formula}\} \cup \{f_\psi : \psi \text{ a } \Delta_0\text{-formula}$
which describes
the graph of a
Gödel operation$\}$

③

$\cup \{\phi, \omega\}$

Σ_{Δ_0}: Ext, found,

- $\forall x \in \phi\, (\neg x = x)$
- $\forall x \in \omega\, (x = \phi \vee \exists y \in \omega\; x = y+1)$
- $\forall \vec{x}\, (\neg R_\phi(\vec{x}) \leftrightarrow R_{\neg\phi}(\vec{x}))$
- $\forall \vec{x}\, (R_{\varphi\wedge\psi}(\vec{x}) \leftrightarrow R_\varphi(\vec{x}) \wedge R_\psi(\vec{x}))$
- $\forall \vec{x}\, (\exists y\, (y \in z \wedge R_\psi(y, z, \vec{x})) \leftrightarrow R_{\exists y \in z \psi}(z, \vec{x}))$
- $\forall \vec{x}, y\, (R_\psi(\vec{x}, y) \leftrightarrow f_\psi(\vec{x}) = y)$

if $\tau \supseteq \epsilon_{\Delta_0}$

$\cancel{Z}$ $Z^-_\tau = Z^-_{\Delta_0} + \cancel{Repl}_\tau \text{ Powerset} + Compr_\tau$

$ZC_\tau = Z_\tau + AC$

$ZF^-_\tau = Z^-_{\Delta_0} + Repl_\tau$

$$\boxed{\begin{aligned} ZFC^-_\tau &= ZF^-_\tau + AC \\ ZFC_\tau &= ZFC^-_\tau + \text{Powerset} \end{aligned}}$$

Def: Let (M, E) be an $\in$-structure and $N \subseteq M$. N is a E-transitive subset of M if $\forall a \in N$ $\mathrm{Pred}_E^M(a) \subseteq M$

Fact 1: if φ is Δ_0 and $N \subseteq M$ is E-trans. for all $\vec{a} \in N^{<\omega}$ $(N, E \cap N^2) \models \varphi(\vec{a})$ iff $(M, E) \models \varphi(\vec{a})$. Hence the substructure relation is preserved by adding $\{R_\varphi : \varphi \text{ a } \Delta_0\text{-formula}\}$

Fact 2 assume $(M,E) \models ZF^-$ and $N \subseteq M$ ⑥ is transitive, ~~Then~~ and $(N,E) \models ZF^-$ as well.

Then if $\mathcal{M}$ is the unique expansion of (M,E) to an $\in_{\Delta_0}$-model of $Z^-_{\Delta_0}$ and same for $\mathcal{N}$

$\mathcal{N} \sqsubseteq_{\in_{\Delta_0}} \mathcal{M}$ $T \supseteq ZF^-$

Fact 3 Assume $\psi(\vec{x}, y)$ is provably $\Delta_1(ZFC^-)$ and $ZFC^- \models \forall \vec{x} \exists! y\, \psi(\vec{x}, y)$. Let $(M,E) \models ZFC^-$ and $(N,E) \models ZFC^-$ with N trans. subset of M.

Then as above for unique extension of M, N to models of Ax^1_ψ

<u>Thm</u> (Strong Levy Absoluteness).

Assume $(V, \in) \models ZFC$. Let $\lambda \geq \kappa$ be ~~regular~~ cardinals, with λ regular. Assume $A_1 \ldots A_k$ are s.t. $A_i \subseteq P(\kappa)^{n_i}$. Then

$$(H_\lambda, \in_{\Delta_1}, A_1 \ldots A_k) \prec_1 (V, \in_{\Delta_1}, A_1 \ldots A_k)$$

<u>Proof</u>: Substructure relation holds in view of Facts 1, 2, 3 and observation that $P(\kappa)^n \subseteq H_\lambda$ for any n.

Let $\exists x \psi(x, \vec{a})$ be true in $V = (V, \in_{\Delta_1}, A_1 \ldots A_k)$. (8)

~~Let~~ Let $X \prec V$ s.t.

$|X| < \lambda$ and $K \subseteq X$ $\vec{a} \in X$ $trcl(\vec{a}) \subseteq X$

$(X, \in_{\Delta_1}, A_1 \cap X, \ldots A_m \cap X) \models \exists x \psi(x, \vec{a})$

Let $\pi_X : X \to N \subseteq H_\lambda$

$(N, \in_{\Delta_1}, \pi_X(A_1), \ldots \pi_X(A_m)) \models \exists x \psi(x, \vec{a})$

$\pi_X(K) = K$

$$(N, \in_{\Delta_2}, \pi_X(A_1) \dots \pi_X(A_k)) \sqsubseteq (H_X, \in_{\Delta_2}, A_1 \dots A_k)$$

Need to check that $A_1 \cap N = \pi_X(A_1)$

observe that

$$\pi_X(A_1) = \pi_X[A_1] = \pi_X[A_1 \cap X] = A_1 \cap X =$$

$$= A_1 \cap N = \{b \in A_1 : b \subseteq k \ b \in N\} = \{b \in A_1 : b \subseteq k, \ b \in X\}$$

$$\parallel$$

$$A_1 \cap X$$

Why Lemma Fails if one does not pick subsets whose elements have bounded rank below κ (10)

unary predicate symbol for countable ordinals

$$(H_{\omega_2}, \in_{\Delta_1}, \omega_1) \sqsubseteq (V, \in_{\Delta_1}, \omega_1)$$

$$\forall x (Ord(x) \to \omega_1(x)) \not\models_1 \exists x\, Ord(x) \wedge \neg\omega_1(x)$$

take $X \prec (V, \in_{\Delta_1}, \omega_1)$ X countable;

Collapse X to N. Then

$$\pi_X(\omega_2) = \omega_1^N < N \cap \omega_2,$$

hence $(N, \in_{\Delta_1}, \pi_X(\omega_1)) \not\sqsubseteq (H_{\omega_1}, \in_{\Delta_1}, \omega_1)$

<u>Lemma</u> Let $\tau \supseteq \in_{\Delta_0}$ and $T \supseteq ZFC^-_\tau$.

Assume $\mathcal{N} \models T$ and $\mathcal{M} \prec_1 \mathcal{N}$. Then

(1) $\mathcal{M} \models Z^-_{\Delta_0}$

(2) $\mathcal{M} \models AC$

(3) $\mathcal{M} \models Rep(\phi)$ for all existential τ-formulae ϕ.

<u>Pf</u> (1) ~~by~~ Since $Z^-_{\Delta_0}$ is axiomatized by Π_2-sent.

(2) Since AC is Π_2 in $\in_{\Delta_0}$

$$\forall X \exists f \cancel{\alpha} \underbrace{(f: X \to \alpha \wedge f \text{ is injection} \wedge \alpha \in Ord)}_{\Delta_0}$$

③ Let $\phi(x, y, z_1 \ldots z_m)$ be Σ_1-formula for τ. (12)

Assume $\mathcal{M} \models \forall x\, (x \in C \to \exists! y\, \phi(x, y, b_1 \cdots b_m))$

for $b_1 \ldots b_m = \vec{b} \in M^m$, and $C \subseteq M$.

Need the following Lemma:

__Lemma__ Assume $\mathcal{M} \prec_1 \mathcal{N}$ are τ-structures.

Let $\phi(\vec{x}, y)$ be a Σ_1-formula for τ.

Let $\mathcal{M}^*, \mathcal{N}^*$ be the unique expansions of $\mathcal{M}, \mathcal{N}$ to $\tau \cup \{f_\phi\} = \tau_1$ satisfying Ax_ϕ^1. Then $\mathcal{M}^* \prec_1 \mathcal{N}^*$

(Note that a priori it is not even clear that

$\mathcal{M}^* \subseteq_{\tau_1} \mathcal{N}^*$!)

Now note that for $\psi(x_1 \ldots x_n)$ a Δ_0-formula and $t_1(\vec{x}, z) \ldots t_n(\vec{x})$ $\tau_1 = \tau \cup \{f_\phi\}$-terms $\psi(t_1(\vec{x}) \ldots t_n(\vec{x}))$ is quantifier free.

Hence for $\psi(t, u, v)$ the Δ_0-formula $(\mathrm{dom}(t) = u \wedge t$ is a function $\wedge \forall x \in u \ \langle x, v \rangle \in t)$ we have that

$(\mathrm{dom}(t) = u \wedge t$ is a function $\wedge \forall x \in u \ \langle x, f_\phi(x, \vec{z}) \rangle \in t)$

is quantifier free for τ_1.

Note also that

$\mathcal{N}^* \models ZFC^-_{\tau_1}$ where $\mathcal{N}^*$ is the unique expansion to τ_1 of $\mathcal{N}$ which models Ax^1_ϕ.

Hence

$$\mathcal{N}^* \models \exists G \underbrace{(\mathrm{dom}(G) = C \wedge G \text{ is a function} \wedge \forall x \in C \langle x, f_\phi(x, \vec{b}) \rangle \in G)}_{\text{quantifier free for } \tau_1}$$

Since $\mathcal{M}^* \prec_1 \mathcal{N}^*$, we get that for some

~~M~~ $F \in \mathcal{M}$ $\mathcal{M}^* \models \mathrm{dom}(F) = C \wedge F \text{ is a function} \wedge$
$\forall x \in C \langle x, f_\phi(x, \vec{b}) \rangle \in F$

Then $\mathfrak{M}^* \models \forall x \in C\, \exists! y\, \phi(x, y, \vec{b})$

hence $\mathfrak{M}^* \models \forall x \in C\, \phi(x, f_\phi(x, \vec{b}), \vec{b})$

and $\mathfrak{M}^* \models \forall x \in C\, f_\phi(x, \vec{b}) = F(x)$

We conclude that

$$\mathfrak{M}^* \models \forall x \in C\, \phi(x, F(x), \vec{b})$$

which being a τ-formula holds in $\mathfrak{M}$ as well.

The proof is completed $\blacksquare$

Corollary 1 Assume T, S are τ-theories with $\tau \supseteq \in_{\Delta_0}$, $S \supseteq ZFC^{(-)}_{\tau}$, T is the MC of S. Then $T \models ZFC^{-}_{\tau}$.

Pf: Since T is model complete $Rep_{\tau} \models$ is equivalent to $\{Rep(\phi) : \phi\ \Sigma_1\text{-formula for } \tau\}$

Corollary 2: Assume T, S as above and $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$ and $S \supseteq ZFC^{(-)}_{\tau} + \kappa$ is a cardinal, and

$$\forall \mathcal{M}\quad \mathcal{M} \models S \Rightarrow \boxed{(H^{\mathcal{M}}_{\kappa^+}, \tau^{\mathcal{M}})} \prec_{\Sigma_1} \mathcal{M} \models S$$

Then $T \models ZFC^{-}_{\tau} + \forall X \exists F (F : \kappa \to X \text{ is a surjection})$

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Pf Let $\mathfrak{M} \models T$, Then $\exists\, \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{N} \models S$ & $\mathfrak{M} \prec_1 \mathfrak{N}$ $\Rightarrow$ $\mathfrak{M} \equiv_1 \underbrace{\mathfrak{N} \equiv_1}_{} H^{\mathfrak{N}}_{\kappa^+} \models \forall X \exists F (F : \kappa \to X$ is a surj.)

ψ Π_2 in $\in_{\Delta_0} \cup \{\kappa\} \subseteq \tau$

$\Rightarrow$ $\mathfrak{M} \models \forall X \exists F (F : \kappa \to X$ is surj.)

because the latter belongs to $SCH(S) \subseteq KH(S)$

$\cup$

T

$R = Th(\mathfrak{M})$ $\quad R_{\forall \vee \exists} + \psi$ is cons.

and $R_{\forall \vee \exists} = (R_{\forall \vee \exists} + \psi)_{\forall \vee \exists}$ $\Rightarrow$ $\exists\, \mathfrak{N} \supseteq \mathfrak{M}$ $\mathfrak{N} \models R_{\forall \vee \exists} + \psi$

$\cup$

$S_\forall$

$\mathfrak{M} \models \psi$ $\Leftarrow$ $\mathfrak{M} \prec_1 \mathfrak{N}$