PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

LECTURE 8

14/6/2022

WHAT HAS BEEN DONE SO FAR (1)

(1) - MODEL COMPANIONSHIP (MC)

- ABSOLUTE MODEL COMPANIONSHIP (AMC)
- WEAKENINGS OF THESE NOTIONS
- ~~RE~~ DEPENDENCE ON THE SIGNATURE OF THESE NOTIONS

Def T is the MC of S if

- $T_\forall = S_\forall$
- T is model complete

$\downarrow$

Def T is model complete if

- $\cancel{\forall}$ $\underset{\models T}{\mathcal{M}} \sqsubseteq \underset{\models T}{\mathcal{N}} \Rightarrow \mathcal{M} \prec \mathcal{N}$

By embeddability Lemmas

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Lemma 1 Let T, S be τ-theories.

TFAE

1. $T^\tau_\forall \supseteq S^\tau_\forall = \{\varphi : \varphi$ is universal τ-sent, $S \models \varphi\}$

2. if $\mathfrak{M} \models T$ there is $\mathfrak{N} \supseteq \mathfrak{M}$, $\mathfrak{N} \models S$.

Lemma 2 Let T, S be τ-theories.

TFAE

1. $T^{\tau}_{\forall\vee\exists} \supseteq S^{\tau}_{\forall\vee\exists} = \{\psi : \psi \text{ is a bool. comb. univ.}, \; S \models \psi\}$

2. for all $\mathfrak{M} \models T$ there is $\mathfrak{N} \models S$

   $\mathfrak{N} \sqsupseteq \mathfrak{M}$ and $\mathfrak{N} \equiv_2 \mathfrak{M}$

Def. Let $\mathcal{M}$ be a τ-structure and T be a τ-theory.

$\mathcal{M}$ is T-ec if

(i) there is $\mathcal{N} \supseteq \mathcal{M}$ $\mathcal{N} \models T$,

(ii) for all $\underset{\models T}{\mathcal{N}} \supseteq \mathcal{M}$, $\mathcal{M} \preceq_1 \mathcal{N}$.

Example: T is Fields in signature $\{+, \cdot, 0, 1\}$

$\mathcal{M}$ is an ACF.

Key Fact: Let T be a τ-theory

TFAE:

(i) $\mathcal{M}$ is T-ec

(ii) $\mathcal{M}$ is $T_{\forall}$-ec

(iii) $\mathcal{M}$ is $T_{\forall\exists}$-ec

Key Fact: for any $\mathcal{M} \models T$ there is $\mathcal{N} \supseteq \mathcal{M}$ $\mathcal{N}$ is T-ec

Def. T is the AMC of S if (7)

- $T_{\forall\vee\exists} = S_{\forall\vee\exists}$
- T is model complete

Thm (Robinson's test) Let T be a τ-theory TFAE:

(i) T is model complete.

(ii) every model of T is $T_\forall$-ec

(iii) every existential formula is T-equivalent to a universal formula

(iv) as (iii) but any formula

Fact Assume $\mathcal{M}$ is T-ec

then $\mathcal{M}$ reflects the Π_2-sentences which are true in some $\mathcal{N} \supseteq \mathcal{M}$.

$$\mathcal{N} \models T$$

Def. ~~V~~ Given a τ-theory T

$KH(T) = \{\psi : \psi$ is a Π_2-sentences for τ and ψ holds in all T-ec models$\}$

Thm. (Cor. of Rob. Test) if S is the MC of T

$S = KH(T)$.

Def. ψ is strongly $T_{\forall\vee\exists}$-consistent if $\psi + R_{\forall\vee\exists}$ is consistent for any $R \supseteq T$

$$SCH(T) = \{\psi : \psi \text{ is } \Pi_2 \text{ and } \psi \text{ is strongly } T_{\forall\vee\exists}\text{-consistent}\}$$

~~Thm~~ **Fact**: $SCH(T) \subseteq KH(T)$

Thm. (Rob. Test Corollary) TFAE

- S is the AMC of T
- $S = SCH(T)$ and S is model complete.

AMC is stronger than MC

Fields has ACF as MC in signature $\{+, \cdot, 0, 1\}$ but not as its AMC

$\forall x\,\neg(x^2+1=0)$ is not in ACF

ACF $\models \exists x\,(x^2+1$

Fields $+ \forall x\,\neg(x^2+1=0) \Dashv \mathbb{Q}$

KH(Fields) $\supsetneq$ SCH(Fields)

MC of fields

this is not

Model companionship results depend on the signature 11

ACF is the MC of Domains

in signature $\{+, \cdot, 0, 1\}$

but not in $\{+, \cdot, 0, 1, {}^{-1}\}$

$\mathbb{C} \sqsubseteq \underset{T}{\mathbb{C}[X]}$ for $\{+, \cdot, 0, 1, {}^{-1}\}$

$\exists \; A_1 \; \cancel{\forall y (\neg y \cdot X = 1)} \; \exists X (X^{-1} = 0 \wedge X \neq 0$

~~Σ-signature~~

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Def Given τ signature ~~xx~~ and $\varphi(\vec{x})$

$\psi(\vec{x}, y)$ τ-formulae

$$Ax^0_\varphi := \forall \vec{x} \left(\varphi(\vec{x}) \longleftrightarrow R_\varphi(\vec{x}) \right)$$ is in $\tau \cup \{R_\varphi\}$

$$Ax^1_\psi := \forall \vec{x} \left[\left(\exists ! y\, \psi(\vec{x}, y) \wedge f_\psi(\vec{x}) = y \right) \vee \left(\neg \exists ! y\, \psi(\vec{x}, y) \wedge f_\psi(\vec{x}) = c_\tau \right) \right]$$

is in $\tau \cup \{f_\psi, c_\tau\}$

for c_τ

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~~For set theory~~

<u>Def</u> for $A \subseteq \mathrm{Form}_\tau \times 2$ and T a τ-theory

$$SCH(A,T) = SCH(T + Ax^{\nu}_{\varphi} : (\varphi,\nu) \in A)$$

$$KH(A,T) = KH(\quad \text{—} \quad \text{—} \quad \text{—} \quad)$$

The (A)MC-spectrum of T is given by those $A \subseteq \mathrm{Form}_\tau \times 2$ s.t

$SCH(A,T)$ $(KH(A,T))$ is model complete.

(14)

Def: $\in_{\Delta_0} = \in \cup \{R_\varphi : \varphi \text{ a } \Delta_0\text{-formula}\}$
$\cup \{f_\varphi : \varphi \ \Delta_0\text{-formula describing the graph of a rudimentary function Goedel operation}\}$

Z_{Δ_0} is ext., found., infinity

together with Π_2-sentences forcing correct interpretation of the new symbols

Thm 1 if $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$

$T \supseteq ZFC_\tau$ vs $Z_{\Delta_0} + AC +$ Repl. for τ-formulae

(i) $M \prec_1 N \models ZFC_\tau + \kappa$ is a cardinal

$$\Downarrow$$

$M \models Z_{\Delta_0} + AC + R$ (a certain amount of repl.)

(ii) if $H^M_{\kappa^+} \prec_1 M$ whenever $M \models ZFC_\tau$

then $SCH(ZFC_\tau) \supseteq \forall X \exists f. (f: \kappa \to X$ is surjective$)$

(iii) if $\tau \subseteq \bigcup_{n \in \omega} P(P(\kappa)^n) \cup \in_{\Delta_0} \cup \{\kappa\}$ then $H^M_{\kappa^+} \prec_1 M$ if $M \models ZFC_\tau$

Thm 2 it is plenty of τ s.t. (16)

any $T \supseteq ZFC_\tau$ has an AMC

$\{\varphi : \varphi \models \Sigma_\tau$ and $H_{\kappa^+} \models \varphi^M$ for any $M \models T\}$

for example $\tau = \in_{\Delta_0} \cup \{\kappa\} \cup \{R_\varphi : \cancel{\varphi}\ \varphi$ is an $\in$-formula / $\in\cup\{\kappa\}$-formula

+ κ is a cardinal

$ZFC_\tau \vdash \forall x_1 \dots x_n \left(\varphi(x_1 \dots x_n) \to \bigwedge_{i=1}^{n} x_i \subseteq \kappa \right)$ $\}$

$\in\cup\{\kappa\}$-formula

in $\in\cup\{\kappa\}$

ZFC_τ Σ_{Δ_0} + Rep for all τ-formulae

$\varphi^M \subseteq (P(\kappa)^n)^M$

~~Lemma~~ Lemma Assume $\tau \supseteq E_{\Delta_0}$ and $T \supseteq ZFC_\tau$ (17)

is s.t. $T_{\forall\exists} + \neg CH$ is consistent.

Then $CH \notin SCH(T)$.

(under very mild assumptions on τ $\qquad \tau = E_{\Delta_0} \cup \{\omega_2\}$

$SCH(T)$ describes a theory of H_{κ^+})

Pf: $\neg CH$: $\underbrace{\exists x\,(x \text{ is } \aleph_1)}_{\Pi_1} \wedge \underbrace{\forall f\,(f \text{ is a function} \wedge \text{dom}(f) = \aleph_2 \Rightarrow \exists z \subseteq \omega \wedge z \notin \text{ran} f}_{\Pi_2}$

Σ

~~CH: $\exists x\,(x \text{ is } \aleph_1) \wedge \exists F\,(F: \ldots \to F \ldots$~~

$SCH(T) \models \forall x \, \exists F \, (F : \omega \to x \text{ is surj.})$

i.e. $\underbrace{SCH(T) \text{ describes a theory of } \aleph_1}_{\Sigma_2}$

$$CH: \exists F \, (F \text{ is a function} \wedge dom(F) = \aleph_2 \wedge \forall z \, (z \subseteq \omega \to z \in ran(F)) \wedge \underbrace{\exists X \, (X \text{ is } \aleph_2)}_{\Sigma_2})$$

false

$CH \notin SCH(T)$

Otherwise fund

$\mathfrak{M}_1 \equiv_1 \mathfrak{M}_0$

$\mathfrak{M} \models T_{\forall\vee\exists} + \neg CH$

$\mathfrak{M}_2 \equiv_1 \mathfrak{M}_1$

$$\overset{\shortparallel}{\mathfrak{M}_0} \sqsubseteq \underset{\underset{SCH(\cancel{\forall CH})}{\shortparallel}}{\mathfrak{M}_1} \sqsubseteq \underset{\underset{T_{\forall\vee\exists}+\neg CH}{\shortparallel}}{\mathfrak{M}_2} \sqsubseteq \ldots$$

$(\forall f\,(\mathrm{dom}(f) = \kappa_1 \Rightarrow \exists r\,(r \subseteq \omega \wedge r \notin \mathrm{ran} f))\}$

$$\mathfrak{M}_\omega = \bigcup_{n\in\omega} \mathfrak{M}_n \models \neg CH + SCH(T)$$

Thm.: ~~Assume~~ Let $S = ZFC + LC$

(LC stands here for there are class many Woodin cardinals)

Let $\tau = \in_{\Delta_1} \cup \{NS_{\omega_1}\} \cup \{\omega_1\} \cup \underbrace{\{\text{certain UB sets}\}}_{\text{forget for now}}$

$\downarrow$

$\{R_\varphi, f_\psi$ s.t. ψ, φ is ZFC$^-$ provably equivalent to a Δ_1-property

and $ZFC^- \vdash \forall \vec{x} \exists! y\, \psi(\vec{x}, y)\}$

Then for any $(V, \in) \models S$ $\quad (V, \tau^V) \equiv_1 (V[G], \tau^{V[G]})$

whenever G is V-generic for some forcing in V

take G V-generic for $\mathrm{Coll}(\omega, \omega_1^V)$

$$(V, \in_{\Delta_0}, \omega_1^V) \not\preceq (V[G], \in_{\Delta_0}, \omega_1^{V[G]})$$

but $\cancel{\forall}$ $\mathrm{Th}(V, \in_{\Delta_0}, \omega_1^V)_{\forall\vee\exists} \doteq$

$$\mathrm{Th}(V[G], \in_{\Delta_0}, \omega_1^{V[G]})_{\forall\vee\exists}$$

$\exists x \, (x \text{ is } \aleph_1)$

Let N be a ground of $V[G]$

Cor If ~~$\in_{\Delta_1} \cup \{\omega_2\} \supseteq \tau \supseteq$~~

$\in_{\Delta_1} \cup \{NS_{\omega_2}, \omega_1\} \cup \text{(certain UB-sets)} \supseteq \tau \supseteq \in_{\Delta_0} \cup \{\omega_2\}$

and $T \supseteq ZFC_\tau + LC + \omega_2$ is the first unc. cardinal

$\neg CH \in SCH(T)$

Pf: Let $(V, \in) \models$ ~~$ZFC + LC$~~ T

Let G be V-generic for $Add(\omega, \omega_2)$

$V[G] \models ZFC_\tau + LC + \neg CH$ and $(V[G], \tau^{V[G]}) \equiv_\Delta (V, \tau^V)$

Π_2 for τ

i.e $\neg CH$ is strongly $T_{\forall\vee\exists}$-consistent 23

hence $\neg CH \in SCH(T)$ ■

<u>Def</u>: $C \subseteq \omega_1$ is club if

(i) $\forall \alpha \in \omega_1 \; \exists \beta \in C \;\; \beta > \alpha$

(ii) $\forall X \subseteq C$ st $\left(\sup(X) \in \omega_1 \Rightarrow \sup(X) \in C\right)$

$S \subseteq \omega_2$ is stat. if

$\forall C \subseteq \omega_2$ club $C \cap S \neq \emptyset$.

~~Fact~~

$NS_{\omega_1} = \{ X \subseteq \omega_1 \ ; \ \exists C \text{ club } C \cap X = \emptyset \}$

Fact NS_{ω_1} is an ideal.

Pf: if $X \in NS_{\omega_1}$ and $Y \subseteq X$ $Y \in NS_{\omega_1}$

if $X_1, X_2 \in NS_{\omega_1}$ as witnessed by C_1, C_2

then $X_1 \cup X_2 \in NS_{\omega_1}$ // by $C_1 \cap C_2$

$C_1 \cap C_2$ is unbounded ; take $\alpha_0 \in \omega_1$ take

$\alpha_{2n} \in C_2$ $\alpha_{2n} > \alpha_{2n-1}$

$\alpha_{2n+1} \in C_1$ $\alpha_{2n+1} > \alpha_{2n}$

$\alpha = \sup_n \alpha_n$ is in $C_1 \cap C_2$

$\vee$

α_0 $\quad \alpha = \sup_m \underset{\substack{\cap \\ C_2}}{\alpha_{2m}} = \sup_\ell \underset{\substack{\cap \\ C_1}}{\alpha_{2\ell+1}}$

Similar argument shows

NS_{ω_1} is a σ-ideal

<u>Lemma</u> (Fodor's Lemma): if $f: S \to \omega_1$ is a choice function for $S \notin NS_{\omega_1}$

$\exists T \subseteq S$ $\quad T \notin NS_{\omega_1}$ s.t. $f \upharpoonright T$ is constant.