PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

16/6/2022

LECTURE 10

Thm: Assume $(V, \in) \models ZFC + LC$ ①

(there are class many Woodin cardinals)

Let ~~the~~ G be V-generic for some $P \in V$

then $(V, \in_{\Delta_1}^V, NS^V, \omega_1^V) \equiv_1 (V[G], \in_{\Delta_1}^{V[G]}, NS^{V[G]}, \omega_1^{V[G]})$

Rmk: LC are needed for

$\omega_1 = \omega_1^L$ holds in L and fails in $L[G]$ for G

$\boxed{\text{atomic for } \in_{\Delta_1} \cup \{\omega_1\}}$

L-generic for $Coll(\omega, \omega_1^L)$

Def: NS is saturated if $\forall A \subseteq P(\omega_1)/NS$ antichain $|A| \leq \aleph_1$

Fact $P(\omega_2)/NS_{\omega_2}$ is saturated $\Rightarrow$ $P(\omega_2)/NS_{\omega_2}$ is complete

Pf: Let $X \subseteq P(\omega_2)/NS$ be any subset

let $A \subseteq \downarrow X = \{[S] : \exists [T] \in X \ [S] \leq [T]\}$ be a max. antichain. Then

$$\bigvee_{P(\omega_2)/NS} A = [\nabla A] = \bigvee X$$

Def: Let M be ctm of $ZFC + NS$ is saturated and G be M-generic for $\left(P(\omega_1)/NS\right)^M$ (3)

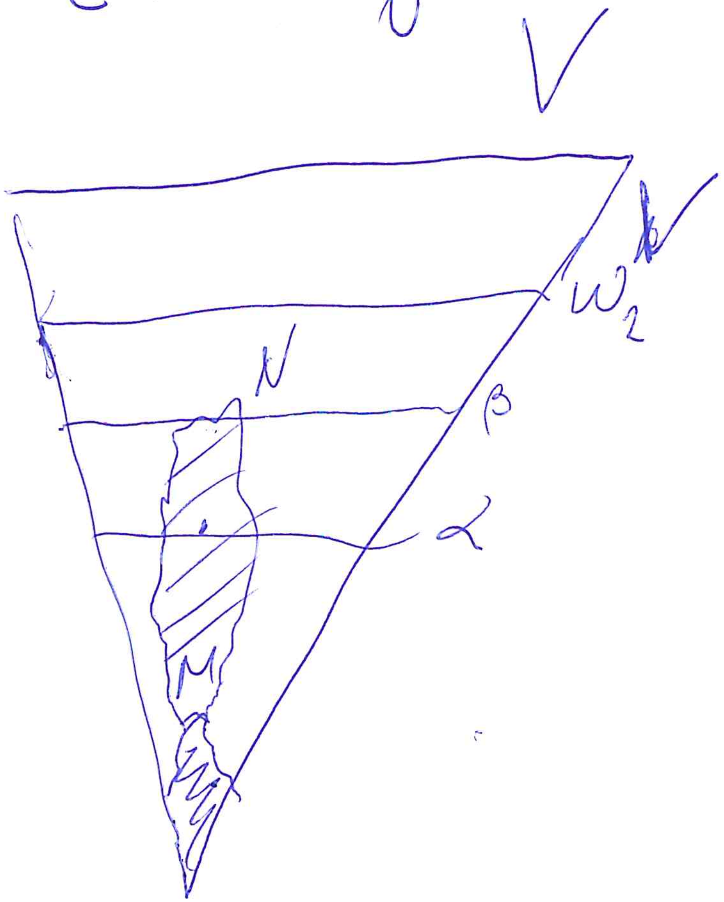
then we can define $Ult(M,G) = \{[f]_G : f : \omega_1^M \to M,\ f \in M\}$

with $[f]_G = \{h : \{\alpha \in \omega_1^M : h(\alpha) = f(\alpha)\} \in G\}$

$$j_G : M \to Ult(M,G)$$
$$a \mapsto [c_a]_G \qquad c_a : \omega_1 \to M$$
$$\alpha \mapsto a$$

Lemma if M, G are as above $(Ult(M,G), \in_G)$ is well founded modulo Mostowski collapse $Ult(M,G)$ is a transitive cm of $ZFC + NS_{\omega_1}$ is saturated

Def M is iterable if M is a ctm of ZFC + NS is saturated and $\forall \alpha < \omega_1^V$ there is ~~$\beta \geq \alpha$ and~~ N ctm of ZFC s.t. $M = (V_{M \cap \omega_1})^N$



Lemma (Lemma 1.6 of Larson's chapter of Handbook of set theory)

Assume M is iterable and $\gamma \leq \omega_1^V + 1$

$\mathcal{J} = \{ j_{\alpha\beta} : M_\alpha \to M_\beta : \alpha \leq \beta < \gamma \}$ is an iteration of M then

M_α is well founded for any $\alpha < \gamma$.

Def: $J = \{ j_{\alpha\beta} : M_\alpha \to M_\beta : \alpha \leq \beta < \gamma \}$ $\gamma \leq \omega_1$ 6

is an iteration of M_0 if:

(i): $j_{\alpha\alpha+1} : M_\alpha \to M_{\alpha+1}$ is $j_{G} : M_\alpha \to Ult(M_\alpha, G_\alpha)$

with G_α M_α-generic for

$\left(P(\omega_1) / NS \right)^{M_\alpha}$

(ii) $j_{0\beta} : M_0 \to M_\beta$ is isomorphic to the direct limit of $\{ j_{0\alpha} : M_0 \to M_\alpha : \alpha < \beta \}$ for β-limit

(iii) for every $\alpha \leq \beta \leq \gamma$ $j_{\alpha\gamma} = j_{\beta\gamma} \circ j_{\alpha\beta}$ and $j_{\alpha\alpha} = Id$

(iv) each M_β is transitive.

Def: Given $\{J_{\alpha\beta} : \alpha \leq \beta < \gamma\}$ system of commuting embeddings which gives an iteration of M_0 (7)

the direct limit is given by ~~pairs~~

$$\{[(a,\alpha)] : a \in M_\alpha], \alpha < \gamma\}$$

$$[(a,\alpha)] = \{(b,\beta) : J_{\alpha\beta}(a) = b \text{ or } J_{\beta\alpha}(b) = a\}$$

$[(a,\alpha)] \; E \; [(b,\beta)]$ if $\alpha \leq \beta$ and $J_{\alpha\beta}(a) \in b$

$\alpha > \beta$ and $a \in J_{\beta\alpha}(b)$

Fact if G is M-generic for $\left(P(\omega_1)/NS\right)^M$ and $M \models ZFC + NS$ is sat.

$\operatorname{crit}(j_G) = \omega_1^M$

Pf: $j_G(\alpha) = [c_\alpha]_G$ for ~~$\alpha < \omega$~~ any α

for any $\alpha \in \omega_1^M$

~~$[h]_G \in [c_{\omega_1^M}]$ define~~ $[c_\alpha]_G \in [Id_{\omega_1}]_G \in [c_{\omega_1^M}]_G$

$Id_{\omega_1}(\alpha) = \alpha \in \omega_1^M = c_{\omega_1^M}(\alpha)$

~~if for α~~ if $[h]_G \in [Id_{\omega_1}]_G$ then

$\exists \beta \in \omega_1^M$ s.t. $[h]_G = [c_\beta]$ $S = \{\alpha \in \omega_1^M : h(\alpha) \in Id(\alpha) = \alpha\} \in G \Rightarrow$

$\cancel{M \models h\restriction S : \omega_1^M \to \omega_1^M}$

$M \models h\restriction S : S \to \omega_1^M$ is regressive

by pressing down we get that

$S = \nabla_{\alpha<\omega_1^M} S_\alpha$

$\nabla_{\alpha<\omega_1} S_\alpha = \{\beta : \exists \alpha<\beta\ \beta\in S_\alpha\} =$

$S_\alpha = \{\beta\in S : h(\beta) = \alpha\}$

$= \{\beta : \exists \alpha<\beta : h(\beta)=\alpha \text{ and } \beta\in S\} =$

$= \{\beta : h(\beta)\in\beta\} = S$

$[S] = \bigvee_{P(\omega_1)^M/NS} \{[S_\alpha] : \alpha<\omega_1\}$ since G is M-generic and $[S]\in G$ $\exists\alpha\ [S_\alpha]\in G$

$\Rightarrow S_\alpha = \{\beta\in S_1 : h(\beta)=\alpha\} \subseteq \{\alpha\in\omega_1^M : h(\alpha)\in f d_{\omega_1^M}(\alpha)\}$

Fact if $\{j_{\alpha\beta} : M_\alpha \to M_\beta : \alpha \le \beta \le \omega_1^N\} \in N$

is an iteration of M_0 $\quad j_{0\omega_1^N}(\omega_1^{M_0}) = \omega_1^N$

Proof because $\{\omega_1^{M_\alpha} : \alpha < \omega_1^N\} \in N$

and is strictly increasing in type ω_1^N

and contained in ω_1^N hence

$$\sup_{\alpha<\omega_1^N} \omega_1^{M_\alpha} = \omega_1^N$$

$$\rotatebox{90}{$\le$}$$

$\omega_1^{M_{\omega_1^N}}$ $\quad \mathrm{crit}(j_{\alpha\omega_1^N}) = \omega_1^{M_\alpha}$ $\quad N \models \omega_1^N$ is a count ord. $\lightning$

if $\omega_1^{M_{\omega_1^N}} > \omega_1^N$ then $M_{\omega_1} \models \omega_1^N$ is a countable ordinal

Def: $J = \{ J_{\alpha\beta} : M_\alpha \to M_\beta : \alpha \leq \beta \leq \omega_2^N \} \in N$

is NS-correct for N (N a tm of ZFC)

if $J_{0\omega_2^N}(NS^{M_0}) = NS^N \cap M_{\omega_1^N}$

$\cap$

$NS^{M_{\omega_2^N}}$

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if $S \in M_{\omega_1^N}$ and $N \models S$ is stat.

$M_{\omega_1} \subseteq N$ is a trans. substruct

$M_{\omega_1} \models S$ is stat

but possible $S \in M_{\omega_2^N}$ $M_{\omega_1^N} \models S$ is stat. but $\exists C \in N$ $N \models S \cap C = \emptyset$
$N \models C$ is club.

Lemma (Lemma 2.8 in Larson's chapter of HST) Assume $N \models ZFC + \cancel{\ldots}$ is transitive and $N \models M$ is iterable.

Then there is $\{ \mathcal{J}_{\alpha\beta} : \alpha \le \beta \le \omega_1^N \} \in N$ NS-correct iteration of $M = M_0$.

Hence

$$\left(M_{\omega_2^N}, \in_{\Delta_2}^{M_{\omega_1^N}}, NS^{M_{\omega_2^N}}, \omega_2^{M_{\omega_2^N}} \right) \equiv$$

$$\equiv \left(N, \in_{\Delta_1}^N, NS^N, \omega_1^N \right).$$

~~Let~~ Thm: on preservation of Π_2 truth value of Π_2-sentences.

Pf: Let φ be a Π_2-sentence for $\in_{\Delta_2} \cup \{NS, \omega_1\}$ true in V.

Assume: G is V-generic for some $P \in V$ and $\cancel{\delta >} |P|$ is Woodin and $V[G] \models \neg\varphi$ note that $H_{\omega_2}^{V[G]} \models \neg\varphi$ as well hence so does $V_\alpha[G]$ for $\alpha > |P|$ inaccessible in V.

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Thm (Woodin) Assume δ is a Woodin cardinal.
Then there ~~is~~ $T_\delta^{\omega_1}$ forcing notion s.t.
if G is V-generic for $T_\delta^{\omega_1}$, in $V[G]$
~~there $\exists$~~ it can be defined

$$j_G : V \to \mathrm{Ult}(V, G) \quad \text{s.t.}$$

(i) $\mathrm{crit}(j_G) = \omega_1^V$

(ii) $j_G(\omega_2^V) = \delta = \omega_1^{V[G]} = \omega_2^{\mathrm{Ult}(V,G)}$

(iii) $\underbrace{\mathrm{Ult}(V,G)^{<\delta} \subseteq \mathrm{Ult}(V,G)}$

(iv) $V_\alpha \in \mathrm{Ult}(V,G)$ for all $\alpha < \delta$.

Let now H be V-generic for $\mathbb{T}_\delta^{\omega_1}$ (15)

since $\mathbb{P}$ is a countable forcing in $V[H]$

and ~~the~~ ~~V_α~~ the dense in $\mathbb{P}$ sets of V are
$\alpha > |\mathbb{P}|$, $\alpha > \delta$ Woodin in V.

all in V_α which is countable in $Ult(V,H)$

in $Ult(V,H)$ we can find G s.t.

$V_\alpha[G] \models \neg\varphi$ (choose G and $p \in G$ s.t. $V_\alpha \models p \Vdash_{\mathbb{P}} \neg\varphi$).

+ δ_0 is Woodin

~~$V_\alpha[G]$~~ is ~~ ~~ $\mathcal{I}$tera

<u>Thm</u> (Shelah): Assume δ is Woodin

then $\exists\ \mathbb{P}_{NS}$ s.t. if H is V-generic

for $\mathbb{P}_{NS}$ ~~…~~ $V[H] \models NS$ is saturated

and $(V, \in_{\Delta_1}, \underbrace{NS, \omega_1) \sqsubseteq (V[H]}, \in_{\Delta_1}, NS, \omega_1)$

Now in $V_\alpha[G] \models \delta_0$ is Woodin we force

with $\mathbb{P}_{NS}^{V_\alpha[G]}$ add let ~~H~~ $K \in Ult(V, H)$ be

$V_\alpha[G]$-generic for $\mathbb{P}_{NS}^{V[G]}$ and let ~~M~~ $= V_\alpha[G][K] \models NS$ is sat.

$\shortparallel$ $\qquad\qquad\qquad\qquad$ $\shortparallel$ $\qquad$ $\cap$

$\neg\varphi$ $\qquad\qquad\qquad\qquad$ $\neg\varphi$ $\qquad$ $V[G][K]$

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So in $\mathrm{Ult}(V,H)$ we can find

NS-correct iteration of $\underset{\models \neg\varphi}{M} = V_\alpha[G][K]$

$\mathrm{Ult}(V,H) \models \neg\varphi$

$\delta = \omega_1^{\mathrm{Ult}(V,H)}$

let $\{ J_{\alpha\beta} : \alpha \leq \beta \leq \delta \} \in \mathrm{Ult}(V,H)$ be

NS-correct iteration of M

then $\underset{\models\neg\varphi}{\left(M_\delta, \in_{\Delta_1}^{M_\delta}, NS^{M_\delta}, \omega_1^{M_\delta}\right)} \sqsubseteq \left(\mathrm{Ult}(V,H), \in_{\Delta_1}^{\mathrm{Ult}(V,H)}, NS^{\mathrm{Ult}(V,H)}, \omega_1^{M_\delta}\right)$

$\lightning$

if $V \models \varphi$ for φ Π_2

and G is V-generic for some $P \in V$

$V[G] \models \varphi$

Assume now $V \models \psi$ with ψ Σ_1.

and G is V-generic for some $P \in V$

then $V[G] \models \psi$.

So let H be $V[G]$-generic for $(\mathbb{T}^{\omega_1}_{\delta})^{V[G]}$ (19)

with $\delta > \alpha > \delta_0$, ~~...~~

$\neg\psi \dashv \left(\mathrm{Ult}(V[G], H)\right) \models V_\alpha$ is iterable, NS is sat $+ \psi$

for α inaccessible $\delta > \alpha > |P|$

let $\mathcal{J} = \{ I_{\gamma\beta} : \gamma \leq \beta \leq \delta \}$ NS-correct iteration of V_α in $\mathrm{Ult}(V[G], H)$ $\boxtimes$

then $I_{0\delta}(V_\alpha) = M_\delta$, $\neg\psi$ is s.t.

$\psi \dashv (M_\delta, \in_{\Delta_1}^{M_\delta}, NS^{M_\delta}, \omega_1^{M_\delta}) \sqsubseteq (\mathrm{Ult}(V[G], H), \in_{\Delta_1}, NS, \omega_1)$ Σ_1