PHD COURSE
ON
MODEL COMPANIONSHIP RESULTS
FOR
SET THEORY

22/6/2022

LECTURE 12

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$$C^{\omega}(\mathbb{R}) = \{ f : f \text{ is analytic } f : \mathbb{R} \to \mathbb{R} \}$$

$$< \; = \{ (f, g, [\{ x \in \mathbb{R} : f(x) < g(x) \}]_{\text{Meager}}) : f, g \in C^{\omega}(\mathbb{R}) \}$$

$$C = \{ (c_a, [\mathbb{R}]_{\text{Meager}}) : a \in \mathbb{R} \}$$

$$c_a : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto a$$

$$B = \frac{\text{Borel}}{\text{Meager}} \cong RO(\mathbb{R})$$

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$$[\exists x\; C(x) \wedge Id < x]$$

$$=$$

$$\bigvee_{g \in C^{\omega}(\mathbb{R})} [Id < g] \wedge [C(\boxed{x})]$$

$$=$$

$$\bigvee_{a \in \mathbb{R}} [Id < c_a]$$

$$=$$

$$\bigvee_{a \in \mathbb{R}} [(-\infty; a)]_{Meager}$$

$$=$$

$$[\mathbb{R}]_{Meager} = 1_B$$

if $G \supseteq \{ \neg [(-\infty; a)]_{\text{Meager}} : a \in \mathbb{R} \}$ ③

for any $f \in C^\omega(\mathbb{R})$

$[\![C(f)]\!] \wedge [\![Id < f]\!] = 0 \notin G$ if f is not constant

$[\![Id < f_a]\!] = (-\infty; a) \notin G$ if $f = c_a$

$\mathcal{M}/G \models [Id]_G < [f]_G$ iff $[\![Id < f]\!] \in G$

$\mathcal{M}/G \models C([f]_G)$ iff $[\![C(f)]\!] \in G$ iff f is c_a for some $a \in \mathbb{R}$

So for any $f \in C^\omega(\mathbb{R})$ 4

$$\mathcal{M}/_{\mathcal{G}} \not\models [Id]_{\mathcal{G}} < [f]_{\mathcal{G}} \wedge C([f]_{\mathcal{G}})$$

$$\Downarrow$$

$$\mathcal{M}/_{\mathcal{G}} \not\models \exists x \left([Id]_{\mathcal{G}} < x \wedge C(x) \right)$$

Def $\mathcal{M}$ is full if for all $\tau_1 \in \tau_n \in \mathcal{M}$ all $f(x_1 \dots x_n)$ formulae all $G \in St(B)$

$\mathcal{M}/G \models f([\tau_1]_G \dots [\tau_n]_G)$ iff $[\![f(\tau_1 \dots \tau_n)]\!] \in G$

(5)

equivalently if for $f(x_0 \dots x_n)$ and $\tau_1 \dots \tau_n \in \mathcal{M}$ there $\sigma_1 \dots \sigma_m$ s.t

$$[\![\exists x\, f(x, \tau_1 \dots \tau_n)]\!] = \bigvee_{\sigma \in \mathcal{M}} [\![f(\sigma, \tau_1 \dots \tau_n)]\!] = \bigvee_{i=1}^{m} [\![f(\sigma_i, \tau_1 \dots \tau_n)]\!]$$

<u>Def.</u>: A B-valued model $\mathcal{M}$ ~~with~~ has the K-mixing property if

for $\{\tau_a : a \in A\} \subseteq \mathcal{M}$ indexed by an antichain of size at most K

there is $\tau \in \mathcal{M}$ s.t. $[\![\tau = \tau_a]\!] \geq a$ $\forall a \in A$.

$\mathcal{M}$ has the mixing property if it has the $|B|$-mixing property.

Lemma If $\mathcal{M}$ has the mixing property $\mathcal{M}$ is full.

Pf: let $\varphi(x_0 \dots x_n)$ be a formula and $\tau_1 \dots \tau_n \in \mathcal{M}$

$$b = \bigvee_{\sigma \in \mathcal{M}} [\![\varphi(\sigma, \tau_1 \dots \tau_n)]\!] = [\![\exists x\, \varphi(x, \tau_1 \dots \tau_n)]\!]$$

$$D_\varphi = \{ a \leq b : \exists\, \sigma_a \; [\![\varphi(\sigma_a, \tau_1 \dots \tau_n)]\!] \geq a \}$$

let $A \subseteq D_\varphi$ be a maximal antichain, consider $\{\sigma_a : a \in A\}$

Now by mixing find σ s.t. $[\![\sigma = \sigma_a]\!] \geq a \quad \forall a \in A.$

Note that $\bigvee A = \bigvee D_\varphi = [\![\exists x\, \varphi(x, \tau_1 \ldots \tau_n)]\!]$

$$[\![\varphi(\sigma, \tau_1 \ldots \tau_n)]\!] \geq [\![\varphi(\sigma_a, \tau_1 \ldots \tau_n)]\!] \wedge [\![\sigma = \sigma_a]\!] \geq a \wedge a = a$$

&

$$[\![\varphi(\sigma, \tau_1 \ldots \tau_n)]\!] = [\![\exists x\, \varphi(x, \tau_1 \ldots \tau_n)]\!] \qquad \forall a \in A$$

$\blacksquare$

Lemma V^B has the mixing property (9)
hence is full if B is a cba.

Cor. if $\dot{K} \in V^B$ is s.t.

$[\![\dot{K} \text{ is a regular cardinal}]\!]_B = 1_B$

let $\kappa = \sup\{\alpha : [\![\alpha = \dot{K}]\!] > 0_B\}$

$$H_{\dot{K}}^{V^B} = \{\tau \in V^B_{(\kappa^+ + |B|)^+} \cancel{\mathrm{rk}(\tau)} : [\![\mathrm{trcl}(\{\tau\}) < \dot{K}]\!] = 1_B\}$$

has the mixing property and is s.t.
whenever G is V-generic for B $H^{V[G]}_{\dot{K}_G} = \{\tau_G : \tau \in H^{V^B}_{\dot{K}}\}$

if $\tau_G \in H_{\check{\kappa}_G}^{V[G]}$ $b_0 [\![trcl(\{\tau\}) < \check{\kappa}]\!] \in G$ (10)

σ be the gluing of τ and ϕ along $\{b, \neg b\}$ then $[\![trcl(\{\sigma\}) < \check{\kappa}]\!] = 1_B$

$[\![\sigma = \tau]\!] \geq b \in G$,

$\Downarrow$

$\sigma_G = \tau_G$

$H_{\check{\kappa}}^{V^B}$ has the mixing property$_B$: if $\{\tau_a : a \in A\} \subseteq H_{\check{\kappa}}^{V^B} \subseteq \bigvee_{(\check{\kappa}+|\check{B}|)^+}$ if $\underbrace{rk(\tau) \leq \sup \{rk(\tau_a)\}_{a \in A}}$ if τ glues $\{\tau_a : a \in A\}$ (11)

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$H_{\dot{\kappa}}^{V^B}$ it has the mixing property

(modulo the fact that one correctly computes

$[\![\tau \in \sigma]\!]_B^{V^B}$ and $[\![\tau = \sigma]\!]$ for $\tau, \sigma \in H_{\dot{\kappa}}^{V^B}$

$$[\![\tau \in \sigma]\!]_B^{V^B} = \bigvee_{h \in dom(\sigma)} [\![\tau = h]\!] \wedge \sigma(h)$$

$$\Vert$$

$$[\![\tau \in \sigma]\!]_B^{H_{\dot{\kappa}}^{V^B}}$$

$$[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!]$$

$$[\![\sigma \subseteq \tau]\!] = \bigwedge_{h \in dom(\sigma)} (\neg \sigma(h)) \vee [\![h \in \tau]\!]$$

note that $\tau \in H_{\dot{\kappa}}^{V^B}$ and $\sigma \in dom(\tau)$

$$\Rightarrow [\![\cancel{\sigma \in} \text{ trcl}(\sigma) < \dot{\kappa}]\!] \geq \tau(\sigma) > 0_B$$

$$H_{\dot{\kappa}}^{V^B} = \left\{ \tau : [\![\text{trcl}(\tau)]\!] \leq 1_B \text{ and } \forall \sigma \in \text{trcl}(\tau) \cap V^B \; [\![\text{trcl}(\sigma)]\!] = 1_B \right\}$$

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~~$H_k^{V^B}$~~ **Fact** $H_{\dot{k}}^{V^B} \prec_1 V^B$ with respect to $\in_{\Delta_0}$

via (Id, Id)

if $\varphi(\tau_1 \dots \tau_n)$ is Σ_1 for $\in_{\Delta_0}$

and G is V-generic for B with

$[\![\varphi(\tau_1 \dots \tau_n)]\!]_B^{V^B} \in G$ and $\tau_1 \dots \tau_n \in H_{\dot{k}}^{V^B}$

$V[G] \models \varphi((\tau_1)_G \dots (\tau_n)_G)$ and $H_{\dot{k}_G}^{V[G]} \prec_1 V[G]$

Hence $\cancel{H_{\dot{\kappa}}^{V^B}} = H_{\dot{\kappa}_G}^{V[G]} \models \varphi((\tau_1)_G \cdots (\tau_n)_G) \Rightarrow$ (13)

$$[\![\varphi(\tau_1 \cdots \tau_n)]\!]_B^{H_{\dot{\kappa}}^{V^B}} \in G \qquad \{\tau_G : \tau \in H_{\dot{\kappa}}^{V^B}\}$$

$$[\![\varphi(\tau_1 \cdots \tau_n)]\!]_B^{H_{\dot{\kappa}}^{V^B}} = [\![\varphi(\tau_1 \cdots \tau_n)]\!]_B^{V^B} \quad \blacksquare$$

$\cancel{\text{If}}$ if G is V-generic for B

$$(V[G], \in) \cong \left(V^B/G, \in^B/G\right) \text{ and } \left(H_{\dot{\kappa}_G}^{V[G]}, \in\right) \cong \left(H_{\dot{\kappa}}^{V^B}/G, \ldots\right)$$

$$\tau_G \longleftarrow [\tau]_G = \{\sigma \in V^B : [\![\tau = \sigma]\!] \in G\}$$

$\{\sigma_G : \overset{\shortparallel}{\tau(\sigma)} \in G, \sigma \in dom(\tau)\}$

Generic multiverse is

$$\left\{ H_{\dot\kappa}^{V^B}/_G \;:\; G \in St(B) \quad [\![\dot\kappa \text{ is a cardinal}]\!]_B^{V^B} = 1_B \right\}$$

Morphisms: given $\iota : B \to C$ complete homomorphism of cbas, we let

$$\hat\iota : V^B \longrightarrow V^C$$

$$\tau \longmapsto \hat\iota(\tau) = \{ \langle \hat\iota(\sigma) : \iota(\tau(\sigma)) \rangle : \sigma \in dom(\tau) \}$$

if $\dot\kappa \in V^B$ and $[\![\dot\kappa \text{ is a cardinal}]\!]_B^{V^B} = 1_B$

$\dot\delta \in V^C$ and $[\![\dot\delta \ \ldots\ldots \ ]\!]_C^{V^C} = 1_C$

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Let $H \in St(C)$ and $\iota : B \to C$ be a complete ingective hom.

we let $G = \iota^{-1}[H] \in St(B)$

$$\hat{\iota}/H : V^{B}/G \longrightarrow V^{C}/H$$

$$[\tau]_G \longmapsto [\hat{\iota}(\tau)]_H$$

and if $[\![\hat{\iota}(\check{K}) < \dot{\delta}]\!] \in H$ then

$\hat{\iota}/H \restriction H^{V^B}_{\check{K}}/G$ maps into $H^{V^C}_{\dot{\delta}}/H$

(Viale, Parente) (26)

Thm. Assume $(V,\in) \models \exists$ class many inaccessible

then ~~there are~~ every set sized model $\mathcal{M}$

of $Th(V, \in_{\Delta_1})$ ~~$\forall A$~~ is an $\in_{\Delta_1}$-substructure

of $\left(H_{\delta}^{Coll(\omega,<\delta)} / G, \in_{\Delta_1} / G \right) = \mathcal{M}_{\delta}$

with δ inaccessible s.t. $|\mathcal{M}| < \delta$

since $G \in St(RO(Coll(\omega,<\delta)))$ can be

chosen so that ~~the above~~ $\mathcal{M}_{\delta}$ is saturated

Suppose $\qquad \mathcal{M} \sqsupseteq (H_{\omega_1}, \in_{\Delta_0})$

$$\mathcal{M} \models Th(V, \in_{\Delta_0})_{\forall \vee \exists}$$

$$\mathcal{M} \models \exists x\, \varphi(x) \cancel{a_1 \dots a_n)}$$

$$\mathcal{M} \sqsubseteq \left(H_\delta^{Coll(\omega,<\delta)} / G, \ \in_{\Delta_0} / G \right)$$

$$\Pi$$

$$\exists x\, \varphi(x)$$

<u>Thm</u> Let $\mathbb{P}$ be any forcing and $\kappa > 2^{|\mathbb{P}|}$ be regular then

$$RO(\mathbb{P}) \hookrightarrow RO(\mathrm{Coll}(\omega, <\kappa)).$$

Since $\mathbb{P} * \dot{\mathrm{Coll}}(\omega, <\kappa) \cong \mathrm{Coll}(\omega, <\kappa)$