PHD COURSE
ON
MODEL COMPANIONSHIP RESULTS
FOR
SET THEORY

LECTURE 13 23/6/2022

Cor. $(H_{\omega_2}, \in_{\Delta_2})$ is $\mathrm{Th}(V, \in_{\Delta_1})_{\cancel{\forall}}$-ec.

Pf. Let ~~δ be inacc. and~~ $\mathcal{M} \sqsupseteq (H_{\omega_1}, \in_{\Delta_2})$ be s.t. $\mathrm{Th}(\mathcal{M})_{\forall} = \mathrm{Th}(H_{\omega_1}, \in_{\Delta_1})_{\forall}$

We use the as yet unproved fact that

$$\left(H_{\omega_2}^{V^B}/G, \in_{\Delta_1}^{V^B}/G\right) \succ_1 (H_{\omega_1}, \in_{\Delta_2}) \quad \text{via } a \mapsto [\check{a}]_G$$

~~(1) this is what we need to re~~

Let $\delta > |\mathcal{M}|$ and let $G \in St(\mathrm{Coll}(\omega, <\delta))$ be s.t. $\left(H_{\check{\delta}}^{V^{\mathrm{Coll}(\omega,<\delta)}}/G, \in_{\Delta_1}^{B}/G\right)$ is saturated assume $\mathfrak{M} \models \exists x\, \varphi(x, a_1 \ldots a_n)$ with (2)

$a_1 \ldots a_n \in H_{\omega_1}$ let $c \in \mathfrak{M}$ be s.t. $\mathfrak{M} \models \varphi(c, a_1 \ldots a_n)$

then $p(x) = Th(V, \in_{\Delta_1})_{\forall} \cup \{ \psi(x, \vec{b}) : \vec{b} \in H_{\omega_2}^{<\omega}$, ψ q.f. for $\in_{\Delta_1}$, $\psi(c, \vec{b})$ holds in $\mathfrak{M} \}$

and the sentences of this type are realized in $\left(H_\delta^{V^{Coll(\omega,<\delta)}}/G, \in_{\Delta_1} \right)$, ~~while the~~ so

is cons

~~quantif~~ $p(x)$ is consistent when each $b \in H_{\omega_2}$ is assigned to $[\check{b}]_G$ in $H_{\check{\delta}}^{Coll(\omega,<\delta)}/G \Rightarrow p(x)$ is realized in $H_{\check{\delta}_1}^{Coll(\omega,<\delta)}/G$

3

Thm Assume there are class many inacc. Then for any B cba in V and $G \in St(B)$ $(H_{\omega_1}, \in_{\Delta_1}) \prec_1 (H_{\omega_1}^B, \in_{\Delta_1}^B)/G$ ($a \mapsto [\check{a}]_G$)

Pf: let $\delta > |B|$ (inacc.) be s.t. V_δ reflects

$[\![\exists x \varphi(x, \check{a}_1 \dots \check{a}_n)]\!]_B^{B^B} > 0_B$ ~~with~~

find τ s.t. $[\![\varphi(\tau, \check{a}_1 \dots \check{a}_n)]\!] > 0_B$ let δ be inacc. s.t. $\tau \in V_\delta$, then $(V_\delta, \in) \models [\![\varphi(\tau, \check{a}_1 \dots \check{a}_n)]\!] > 0_B$

Take $M \prec (V_\delta, \in)$ $|M| = \aleph_0$ $B, \tau, a_1 \dots a_n \in M$.

Let $\pi_M : M \to N$ be the transitive collapse. 4

Note that $\pi_M(a) = a$ for any $a \in H_{\omega_1} \cap M$

$\pi_M[a] = \{\pi_M(b) : b \in a \cap M\} = \{\pi_M(b) : b \in a\} = \cancel{\pi_M(a)}$

$\quad\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\pi_M(a) \qquad\qquad\qquad\qquad \{b : b \in a\} = a$

since

$M \models \exists f : \omega \to a$ surjection

and $\omega \subseteq M \Rightarrow f[\omega] \subseteq M$

$\check{a} \to \{\langle \check{b}, 1_B \rangle : b \in a\}$

$N \models [\![\varphi(\pi_M(\tau), \cancel{\check{a}_1}\ \cancel{\pi_M(\check{a}_1)}\ \check{a}_2, \ldots \check{a}_n)]\!]_{\pi_M(B)} > 0_{\pi_M(B)}$

assuming carrier of B ~~ϕ~~ is an Ord and $1_B = 0$, ~~[illegible]~~

So find G N-generic for $Q = M_\mu(B)$ (5)

s.t. $[\![\varphi(\check{\tau}, \check{a}_1 \dots \check{a}_n)]\!]^{N}_{Q} \in G$

$\Downarrow$

$N[G] \models \varphi(\tau_G, \check{a}_1 \dots a_n)$

$N[G]$ is countable and transitive $\Rightarrow$

$N[G] \in H_{\omega_1}$; H_{ω_1} is transitive $\Rightarrow$

$N[G] \subseteq H_{\omega_1}$

$(N[G], \in_{\Delta_1}) \sqsubseteq (H_{\omega_1}, \in_{\Delta_1}) \Rightarrow (H_{\omega_1}, \in_{\Delta_1}) \models \varphi(\tau_G, a_1 \dots a_n)$

$\boxtimes$

6

<u>Cor.</u> (Viale, Parente)

$(H_{\omega_2}, \in_{\Delta_2})$ is $Th(V, \in_{\Delta_1})$-ec.

Using

$Cod_\omega : \cancel{W}FE_\omega \to H_{\omega_2}$

$R \mapsto a$ if $a = \pi_R(0)$

sends Σ^1_{n+1}-set in a Σ_n-set

we get that $Th(H_{\omega_2}, \in_{\Delta_2})$ is not model complete (there are Σ_2-sets which are not definable as Π_2-sets)

UNIV. BAIRE SETS

Def: Let (X,τ) be a compact Polish space (e.g. (X,τ) is 2^ω with product topology)

$A \subseteq X$ is universally Baire if for any $f: Y \to X$ with (Y,σ) Hausdorff compact and f continuous $f^{-1}[A]$ has the Baire property in Y

recall $Z \subseteq Y$ is nowhere dense if $Y \setminus Z$ contains a dense open set

Z is meager if it is contained in a countable union of nowhere dense sets.

$Z \subseteq Y$ has BP if $Z \triangle \mathrm{Reg}(Z) = \cancel{\mathrm{Int}\,\mathrm{Cl}}$ (8)
is meager

$\mathrm{Reg}(Z) = \mathrm{Int}(\mathrm{Cl}(Z))$

Counterexample to univ. Baireness

Let $C \subseteq [0;1]$ be the Cantor set.
then C has Lebesgue measure 0.
Now in C take $P \subseteq C$ s.t. P is
not Leb. measurable according to Leb. measure in C
note that $\mathrm{Id}: C \hookrightarrow [0;1]$ is continuous
and P is Leb. meas in $[0;1]$

Find $C \subseteq [0;1]$ meager given by ~~intersectio~~ union of countably many nwd sets with $|C| = 2^{\aleph_0}$. (9)

A: ~~iff~~ C is meager in $[0;1]$

Let $P \subseteq C$ be path. for BP

then P is meager in $[0;1]$ but

~~iff~~ $Id^{-1}[P] = P$ is pathological in C.

<u>Fact</u> Borel sets are UB
~~and fo~~ UB-sets are a σ-algebra.

~~<u>Fact</u> Analytic sets~~

<u>Thm</u> Assume there are class many Woodin, then projective sets are UB, actually any $X \subseteq 2^\omega$ s.t. X is definable in $L(\mathrm{Ord}^\omega)$ is UB.

<u>Thm</u> Assume LC as above then UB-sets are closed under projections.

<u>Thm</u> (Assume LC) UB-sets are determined.

<u>Thm</u> LC The following families $\mathcal{A}_i$ of sets of reals are contained in UB and are s.t. $(H_{\omega_2}, \in_{\Delta_0}, A : A \in \mathcal{A}_i)$ is model complete:

- $\mathcal{A}_0 = \{X : X \subseteq P(\omega)^n : X$ is definable in $L(\mathrm{Ord}^\omega)\}$
- $\mathcal{A}_1 = \{X : X$ is projective$\}$
- $\mathcal{A}_2 = \{X : X \subseteq P(\omega)^n$ is definable in $L(\mathbb{R})\}$

; note that WFE is Π^1_1 hence UB

$$\Big(\underbrace{WFE, \cancel{\omega} \, E, \{(Cod^{k})^{-1}[A] : A \in \mathcal{A}_1}_{\mathcal{A}_1^*}\}\Big) \cong \Big(H_{\omega_1}, \in, \{A : A \in \mathcal{A}_1\}\Big) \; (1)$$

$$\Downarrow$$

$$(WFE, \in_{\Delta_1}, \mathcal{A}_1^*) \cong (H_{\omega_2}, \in_{\Delta_1}, \mathcal{A}_1, \ldots$$

$$Cod : WFE \longrightarrow H_{\omega_2}$$

$$A \subseteq (2^\omega)^k$$

$$\big((Cod)^{k}\big)^{-1}[A] = \Big\{(R_1 \ldots R_k) : \langle Cod(R_1), \ldots Cod(R_k)\rangle \in A\Big\}$$

~~WF~~

13

Cor. if ~~$A \neq$~~ $\{A \subseteq \mathrm{Form}_\in \times 2$ is

~~φ~~ s.t. $\in_A = \cancel{\{R_\psi : \psi\ \Delta_1\text{-property}\}}\ \in_{\Delta_1} \cup$

$\{R_\psi : \cancel{\text{st}}\ \cancel{\varphi}\ \cancel{\&}\ ZFC + LC \vdash \forall \vec{x}\ \psi(\vec{x}) \to \bigwedge_{i=1}^{n} (x_i \in 2^\omega) \wedge$

all quantifiers in ψ range over $L(\mathrm{Ord}^\omega)$ ($\cancel{\text{or}}$ $\cancel{\text{st}}$ $L(\mathbb{R})$, $H_{\omega_1} \ldots$)

$(ZFC + LC + T_{\in,A})$ has as AMC

$\{\psi : \psi$ is Π_2 for $\in_A$ and $(H_{\omega_1}^{M}, \in_A^{M}) \models \psi$ for any $M \models ZFC + LC + T_{\in,A}\}$

Thm.: Assume there are class many Woodin and G is V-generic for some $P \in V$

$$\cancel{V}\left(L(\mathrm{Ord}^{\omega})^{V}, \in\right) \equiv \left(L(\mathrm{Ord}^{\omega})^{V[G]}, \in\right)$$

Thm. Assume there are ~~class~~ many stat. many inacc. ~~...~~ cardinals and we work in MK.

Then there ~~st~~ are stat many inacc. δ s.t. $\left(\underbrace{H_{\omega_2}^{\mathrm{Coll}(\omega,<\delta)}}, \in_{\Delta_1}, \dots\right)$ are all models of the same T.

and this theory ~~with~~ extends the $KH(Th(V, \in_{\Delta_1}, \dots))$.