

INCOMPATIBLE BOUNDED CATEGORY FORCING AXIOMS

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ABSTRACT. We introduce bounded category forcing axioms for well-behaved classes Γ . These are strong forms of bounded forcing axioms which completely decide the theory of some initial segment of the universe $H_{\lambda_\Gamma^+}$ modulo forcing in Γ , for some cardinal λ_Γ naturally associated to Γ . These axioms naturally extend projective absoluteness for arbitrary set-forcing—in this situation $\lambda_\Gamma = \omega$ —to classes Γ with $\lambda_\Gamma > \omega$. Unlike projective absoluteness, these higher bounded category forcing axioms do not follow from large cardinal axioms, but can be forced under mild large cardinal assumptions on V . We also show the existence of many classes Γ with $\lambda_\Gamma = \omega_1$, and giving rise to pairwise incompatible theories for H_{ω_2} .

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1. INTRODUCTION

Forcing axioms are principles asserting the existence of sufficiently generic filters for all forcing notions in some reasonable class. In a general form they state, for a given class Γ of forcing notions and a given cardinal κ , that for every $\mathcal{P} \in \Gamma$ and every collection \mathcal{D} of $\leq \kappa$ -many dense subsets of \mathcal{P} there is a filter G of \mathcal{P} such that $G \cap D \neq \emptyset$ for each $D \in \mathcal{D}$.¹ We refer to this statement as $\text{FA}_\kappa(\Gamma)$. These axioms are successful in settling a wide range of problems undecidable on the basis of the commonly accepted axioms for set theory. This is not surprising given that, for reasonable classes Γ , forcing axioms can be often characterized as (or at least imply) *maximality principles* with respect to generic extensions via forcing notions coming from Γ .

For certain classes Σ of mathematical statements, such maximality principles can be formulated as principles of *generic absoluteness* of the universe with respect to the relevant forcing extensions, i.e., assertions stating that for every $\mathcal{P} \in \Gamma$, every \mathcal{P} -generic filter G over V , and every $\sigma \in \Sigma$, $V \models \sigma$ if and only if $V[G] \models \sigma$. By Shoenfield's absoluteness this principle of generic absoluteness holds, for example, if Σ is the class of all Σ_1 -sentences² with parameters in H_{ω_1} and Γ is the class Ω of all forcing notions.

Other well known examples of generic absoluteness principles are the bounded forcing axioms $\text{BFA}_\lambda(\Gamma)$: when λ is an infinite cardinal and Γ is a class of forcing axioms preserving the cardinality of all cardinals $\mu \leq \lambda$, the *bounded forcing axiom* $\text{BFA}_\lambda(\Gamma)$ states that $H_{\lambda^+} \prec_1 V^{\mathcal{P}}$ whenever $\mathcal{P} \in \Gamma$ (which is a more compact way to formulate the generic absoluteness principle for the family Σ given by Σ_1 -sentences with parameters in H_{λ^+} and the class of forcings Γ). An equivalent formulation of $\text{BFA}_\lambda(\Gamma)$ asserts that for all cbas $\mathbf{B} \in \Gamma$ and all families \mathcal{D} of $\leq \lambda$ -many maximal antichains of \mathbf{B} each of size at most λ there is a filter G on \mathbf{B} meeting all these antichains. This formulation makes it clear that the generic absoluteness principle $\text{BFA}_\lambda(\Gamma)$ follows from $\text{FA}_\lambda(\Gamma)$.

A further fundamental example of generic absoluteness principle is instantiated by choosing the family Σ of all projective sentences (i.e., all \in -sentences of the form $\sigma^{H_{\omega_1}}$ where σ is any \in -sentence with parameters in H_{ω_1}) and the class Ω of all forcing notions: it is a remarkable fact, due to Woodin, that the mere presence in V of sufficiently strong large cardinals—for example a proper class of Woodin cardinals, or a supercompact cardinal—outright implies this principle of generic absoluteness, which we will henceforth refer to as *Projective Absoluteness*³ (abbreviated by PA).

Generic absoluteness is a naturally attractive possible feature of the universe, at least for the set-theorist with realist inclinations, in that it manages to neutralize, to some extent, the effects of forcing on the universe (forcing being our prime method for proving independence over our base set theory).

¹This is equivalent to the axiom obtained by letting \mathcal{D} consist of maximal antichains of \mathcal{P} .

²Here by Σ_1 -sentence with parameters \vec{a} we mean an \in -formula $\exists x \psi(x, \vec{a})$ with $\psi(x, \vec{y})$ a Δ_0 -formula.

³See Theorem 2.4 for a more accurate formulation.

To continue our presentation we need to avail ourselves of the expressive power provided by second order set theory. To be specific: for us second order set theory denotes those *first order* axiomatizations of set theory in which we consider classes as legitimate elements of our universe of discourse (e.g. Gödel-Bernays or Morse-Kelley). For the remainder of this paper we will work in the extension of Morse-Kelley set theory (henceforth denoted by **MK**) with the axiom saying that the class Ord of all ordinals is Mahlo (i.e., every class of ordinals which is closed and unbounded contains an inaccessible cardinal).

In order to introduce the kind of generic absoluteness principles which will soon become our main focus, let us start by describing in more detail some key features of **PA** which can be better formulated in $\text{MK} + \text{Ord is Mahlo}$. First of all, in this set-up we can make sense of hypotheses which in terms of consistency strength lie way below the existence of Woodin cardinals and which nevertheless suffice to produce models of **PA**. In fact we can show that if $V \models \text{MK} + \text{Ord is Mahlo}$, then for every set-generic extension $V[G]$ of V there is a set-generic extension $V[G][H]$ of $V[G]$ satisfying **PA** (and $\text{MK} + \text{Ord is Mahlo}$). Of course now we do not obtain Projective Absoluteness in V but only prove that it is forcible (this is the content of Theorem 2.8). In order to even be able to state our starting hypothesis we need to transcend first order set theory (e.g. **ZFC**)—which of course was enough to express the existence of a proper class of Woodin cardinals—and work instead in our second order set theory.

The forcibility of **PA** over a model of $\text{MK} + \text{Ord is Mahlo}$ originates in the classical argument for deriving Projective Absoluteness from large cardinals. The main observation is that if Ord is Mahlo, then there are unboundedly (actually stationarily) many inaccessible cardinals δ with the property that whenever G is $\text{Coll}(\omega, < \delta)$ -generic over V and $H \supseteq G$ is $\text{Coll}(\omega, < \text{Ord})$ -generic over V

$$(H_{\omega_1}^{V[G]}, \in) \prec (V[H], \in).$$

Note again that the above is expressible as a first order property of the universe of sets V in parameter δ assuming **MK** while it is not so assuming just **ZFC** (note also that H is $V[G]$ generic for $\text{Coll}(\omega, < \text{Ord})$ as computed in $V[G]$). Using the fact that every set-forcing extension can be absorbed in a forcing extension via $\text{Coll}(\omega, < \kappa)$ for some high enough cardinal κ (and therefore it is also absorbed by $\text{Coll}(\omega, < \text{Ord})$), it follows that **PA** holds in $V[G]$ in the above situation. This rough proof-sketch (which will be developed with full details in Section 2.1) shows the following: If Ord is Mahlo, then Projective Absoluteness holds if and only if the identity on H_{ω_1} is an elementary embedding⁴, from this \in -structure into $V^{\text{Coll}(\omega, < \text{Ord})}$. This justifies defining Projective Absoluteness in **MK** to be precisely this axiom (which we do).

This paper can be naturally split into two parts. The first part of the paper is aimed at producing an analogue of **PA** for the theory of $H_{\lambda_\Gamma^+}$, for some uncountable cardinal λ_Γ , with respect to forcing notions in some given class Γ (where λ_Γ is the supremum of the class of cardinals preserved by all forcings in Γ).

PA states that $H_\kappa \prec V[G]$ with G V -generic for $\text{Coll}(\omega, < \text{Ord})$ and $\kappa = \omega^+$. Note that ω is the unique infinite cardinal preserved by all set-sized forcings and that $\text{Coll}(\omega, < \text{Ord})$ is universal for all set-sized forcings, i.e. any set-sized forcing embeds as a complete suborder of the class forcing $\text{Coll}(\omega, < \text{Ord})$.

⁴e.g. for any \in -formula $\phi(\vec{x})$, $(H_{\omega_1}, \in) \models \phi(\vec{a})$ if and only if $\Vdash_{\text{Coll}(\omega, < \text{Ord})} \phi(\vec{a})$.

Our guiding idea for obtaining analogues of PA is to single out for which classes Γ of set-sized forcings we can find:

- a cardinal λ_Γ such that all forcing in Γ preserve all cardinals up to and including λ_Γ and all other cardinals are destroyed by some forcing in Γ .
- a universal (class-sized) forcing \mathcal{P}_Γ absorbing all elements of Γ as complete suborders.

If these two conditions are met we can certainly formulate the variation of PA relative to Γ stating that $(H_{\lambda_\Gamma^+}, \epsilon) \prec (V[H], \epsilon)$ whenever H is V -generic for \mathcal{P}_Γ (note again that this is first order expressible in MK in parameter Γ).

We will devise a very general framework which establishes that the above can be obtained for a variety of classes of set sized forcings Γ by letting \mathcal{P}_Γ be the class pre-order (Γ^*, \leq_Γ) where Γ^* is the class of complete boolean algebras in Γ and $\mathbf{C} \leq_\Gamma \mathbf{B}$ if and only if there exists $i : \mathbf{B} \rightarrow \mathbf{C}$ complete (but possibly non-injective) homomorphism with a generic quotient in Γ (see Def. 2.21 for details).

- A first (and not surprising) key point is that all properties of Γ we are interested in will be forcing invariant. Hence the class of cbas in Γ suffices in order to investigate all such properties, and the class partial order (Γ^*, \leq_Γ) described above incorporates all the information we will need to know about Γ . Hence, from now on, and for the sake of economy, we will freely abuse notation and write Γ in place of Γ^* ; the context will clarify whether we intend for Γ the class of all its preorders or the narrower class consisting of its cbas.
- A second key point (which motivates our definition of the order \leq_Γ) is that if G is V -generic for \mathbf{C} , then $i^{-1}[G] = H$ is V -generic for \mathbf{B} (regardless of whether $i : \mathbf{B} \rightarrow \mathbf{C}$ is injective or not).
- A third key point is that under reasonable assumptions on Γ (e.g. if Γ is the class of proper, semiproper, or stationary set preserving forcings) we can prove that the first order formula defining in MK the set-sized forcings which are elements of Γ is satisfied by the class-sized forcing (Γ, \leq_Γ) itself! Furthermore under such assumptions we can prove that (Γ, \leq_Γ) is able to absorb all its elements $\mathbf{B} \in \Gamma$ (which are set-sized) as complete sub-forcings (possibly below some condition $\mathbf{C} \leq_\Gamma \mathbf{B}$); moreover, this absorption is particularly nice: e.g., if G is V -generic for \mathbf{B} , then $\Gamma^{V[G]}$ is exactly the quotient forcing $\Gamma \upharpoonright \mathbf{B}/G$ (however it will take us a long while to even be able to formulate this assertion properly).

Putting all these observations together we get by a simple reflection argument that for stationarily many inaccessible cardinals δ (the regular completion of the set-sized forcing) $\Gamma_\delta = \Gamma \cap V_\delta$ is itself a forcing in Γ which absorbs as complete subforcings all its elements, and also such that whenever H is V -generic for the class forcing $\Gamma \upharpoonright \Gamma_\delta$, $H \cap V_\delta = G$ is V -generic for Γ_δ and such that $H_\delta^{V[G]} = H_{\lambda_\Gamma^+}^{V[G]}$ is an elementary substructure of $V[H]$. Furthermore, a byproduct of our result will be that $\mathbf{BFA}_{\lambda_\Gamma}(\Gamma^{V[H]})$ holds in $V[H]$. This is the generalization to Γ of Projective Absoluteness we will be able to realize in $V[H]$.

For these reasons we will call BCFA(Γ) (the bounded category forcing axiom for Γ) the assertion

$$(H_{\lambda_\Gamma^+}, \epsilon) \prec (V^\Gamma, \epsilon).$$

This statement is of course first order expressible in **MK** in parameter Γ (or, as in the cases of interest to us, in the set sized parameters needed to define Γ).

Sections 3 and 4 are the bulk of the first part of this paper. In there we will prove $\text{BCFA}(\Gamma)$ consistent relative to $\text{MK} + \text{Ord is Mahlo}$ for what we will call the *absolutely well behaved classes* Γ . The second part of the paper (e.g. Section 5) shows that such absolutely well behaved classes are not exotic, as among them we will find the α -(semi)proper forcings for all countable indecomposable ordinals α , the ω^ω -bounding and proper forcings, the stationary set preserving forcings (assuming suitable large cardinal assumptions), and many others.

We note that the results of this paper expand on [27], which deals with generic absoluteness results for the ω_1 -Chang model and the class of complete boolean algebras preserving stationary subsets of ω_1 . In Section 1.2 of this introduction we give some more details on the context of [27] and how these results compare to those of [27].

Our results reverse our perspective on forcing: rather than a tool useful to prove undecidability results, it becomes a tool useful to prove theorems. Specifically, in order to show that $\text{BCFA}(\Gamma)$, together with the ambient set theory, implies $H_{\lambda_\Gamma^+} \models \sigma$ for a given sentence σ , it suffices to show that $\text{BCFA}(\Gamma)$, together with the ambient set theory, implies the existence of a forcing notion \mathbb{B} in Γ which forces both $\text{BCFA}(\Gamma)$ and $H_{\lambda_\Gamma^+} \models \sigma$, which is significantly simpler (we turned the universal problem of showing that something holds in *all* models of $\text{BCFA}(\Gamma)$ into a consistency problem).

Bounded category forcing axioms are to be seen as strong forms of bounded forcing axioms providing a picture of the universe as being saturated by *only* forcing coming from the relevant class Γ . For classes $\Gamma_0 \subseteq \Gamma_1$, even if it is of course true that $\text{BFA}_{\lambda_\Gamma}(\Gamma_1)$ implies $\text{BFA}_{\lambda_\Gamma}(\Gamma_0)$, there will typically⁵ be statements σ about $H_{\lambda_\Gamma^+}$ such that $\text{BFA}_{\lambda_\Gamma}(\Gamma_1)$ implies σ , whereas $\neg\sigma$ can be forced by a forcing in Γ_0 and, once forced, will be preserved by subsequent forcing in Γ_0 . This explains why σ will follow from $\text{BCFA}(\Gamma_1)$ whereas it will fail in the $\text{BCFA}(\Gamma_0)$ model.

One lesson to be learned from these incompatibility results is that natural forms of (bounded) forcing axioms do not, by themselves, favour a universalist conception of set theory. There is unavoidable branching at the level of these axioms; in particular, the set-theorist with a universalist mindset will need additional criteria—beyond the ‘naive’ view on maximality provided by looking at the containment relation between the classes Γ under consideration—to favour one of these axioms over the others. This should be compared with the fact that, as an empirical fact, consistent large cardinal axioms seem to be orderable under implication.⁶ A reasonable additional criterion available to the universalist when assessing bounded category forcing axiom—consistent with the view that these axioms are indeed strong forms of bounded forcing axioms—could be to focus on maximizing the class of Π_2 sentences over $H_{\lambda_\Gamma^+}$ implied by the axiom⁷. In this respect, when looking at classes Γ with $\lambda_\Gamma = \omega_1$,

⁵Our results in the second part of the paper provide some evidence that this will be the case for all choices of Γ_0 and Γ_1 .

⁶At least upwards directed, in the sense that for any two large cardinal axioms A_1 and A_2 there always seems to be a large cardinal axiom A_3 subsuming both A_1 and A_2 . This seems to apply both to large cardinal axioms over **ZFC** and to the family of large cardinal axioms in the wider **ZF** context.

⁷After all, the mathematical applicability of bounded forcing axioms is correlated to their Π_2 consequences for the theory of $H_{\lambda_\Gamma^+}$.

the class that fares the best is of course the class of all forcing notions preserving stationary subsets of ω_1 (we will denote this class by **SSP**).

1.1. Model companionship versus bounded category forcing axioms. This section correlates the results presented so far with the second author's work relating forcing axioms and generic absoluteness results to Robinson's notion of model companionship as applied to set theory. Its content is not needed to follow the remainder of this paper, but gives further motivation for our results.

Model companionship describes in a model-theoretic terminology applicable to an arbitrary first order theory T , the closure properties of algebraically closed fields and the way the elementary class given by such fields sits inside the elementary class given by arbitrary fields. Applied to the set-theoretic realm, the basic idea is that H_{ω_1} plays with respect to the generic multiverse (given by the generic extensions of V by set sized-forcings) the role \mathbb{C} does for arbitrary fields, while (assuming forcing axioms) H_{ω_2} plays this same role with respect to the generic multiverse given by forcings in an appropriate class (e.g. proper, **SSP**, etc).

A first key notion in this context is that of existentially closed structures in a signature τ . A τ -structure \mathcal{M} is existentially closed in a superstructure \mathcal{N} if $\mathcal{M} \prec_1 \mathcal{N}$. For a τ -structure \mathcal{M} with domain M , let us write (M, τ^M) as a shorthand for $(M, R^{\mathcal{M}} : R \in \tau)$.

Consider a signature τ_{ST} in which one adds a predicate symbol R_ϕ of arity n for any Δ_0 formula $\phi(x_1, \dots, x_n)$ and interprets in the models of set theory these predicate symbols R_ϕ as the extension of the formula ϕ . In this set-up, one of the key consequences of Shoenfield's absoluteness is that

$$(H_{\omega_1}^V, \tau_{\text{ST}}^V) \prec_1 (H_\kappa^{V[G]}, \tau_{\text{ST}}^{V[G]})$$

whenever G is a forcing extension of V and κ is an uncountable cardinal in $V[G]$; actually Shoenfield's absoluteness states that $H_{\omega_1}^V$ is existentially closed in all of its well-founded τ_{ST} -superstructures which model ZFC^- (i.e. ZFC minus the power-set axiom).

Consider now bounded forcing axioms; these axioms state that $(H_{\omega_2}^V, \tau_{\text{ST}}^V) \prec_1 (H_\kappa^{V[G]}, \tau_{\text{ST}}^{V[G]})$ whenever G is V -generic for a forcing notion P in a given class of forcings Γ (stationary set preserving, proper, semiproper, etc) and $\kappa \geq \omega_2^{V[G]}$; once again, these axioms assert that $H_{\omega_2}^V$ is existentially closed in its τ_{ST} -superstructures which model ZFC^- and are obtained by *certain types of forcings*.

Let us explore briefly the notion of T -existentially closed structure and show how such structures are produced in model theory. Given a first order theory T for a signature τ , a τ -structure \mathcal{M} is T -e.c. if $\mathcal{M} \prec_1 \mathcal{N}$ whenever \mathcal{N} is a superstructure of \mathcal{M} realizing T_\forall (the universal fragment of T).

Note that neither \mathcal{M} nor \mathcal{N} need to be models of T , the only sure thing is that they model⁸ T_\forall . A key (and not so trivial) fact which will play an essential role in our arguments is that whenever \mathcal{M} is T -e.c., so is \mathcal{N} if $\mathcal{N} \prec_1 \mathcal{M}$. The standard example of a T -e.c. structure is an algebraically closed field, where T is the theory of fields in signature $\{+, \cdot, 0, 1\}$: if K is algebraically closed, for any $L \supseteq K$

⁸For example take T to be the theory of fields elementarily equivalent to \mathbb{Q} in signature $\{+, \cdot, 0, 1\}$, K an algebraically closed field, and $L \supseteq K$ a ring with no zero-divisors which is not a field. Then K is T -e.c., $K \prec_1 L$ are both models of T_\forall , but neither of them models T .

modeling⁹ T_V , any Σ_1 formula with parameters in K (i.e. statements of the form $\exists \vec{y} \left[\bigwedge_{i=1}^n p_i(\vec{y}) = 0 \wedge \bigwedge_{j=1}^m q_j(\vec{y}) \neq 0 \right]$ with p_i, q_j polynomials with coefficients in K) realized in L is already true in K .

How does one construct a T -e.c. structure? The simplest way is to start with a model \mathcal{M}_0 of T_V and construct using some book-keeping device a chain $(\mathcal{M}_\alpha : \alpha < \kappa)$ of models of T_V in which at each stage α one tries to make true in $\mathcal{M}_{\alpha+1}$ some existential formula with parameters in¹⁰ \mathcal{M}_α . Note that there is tension between the constraint given by $\mathcal{M}_{\alpha+1} \models T_V$ and the requirement that $\mathcal{M}_{\alpha+1}$ realizes some Σ_1 formula with parameters in \mathcal{M}_α (for example this formula cannot be the negation of some universal axiom of T).

A key point is that an increasing chain of models of T_V is still a model of T_V , hence the construction does not stop at limit levels; now if κ is large enough, at stage κ all existential formulae with parameters in some \mathcal{M}_α for $\alpha < \kappa$ have been realized (if possible) in some \mathcal{M}_β with $\beta < \kappa$, hence \mathcal{M}_κ is T -e.c.

Compare this procedure with the standard proof of the consistency of bounded forcing axioms: for example to establish the consistency of BPFA one does exactly the same, but now one starts from $\mathcal{M}_0 = (V, \in)$ and in passing from $\mathcal{M}_\alpha = (V[G_\alpha], \in)$ to $\mathcal{M}_{\alpha+1} = (V[G_{\alpha+1}], \in)$, one can use only forcings which are proper in $V[G_\alpha]$; moreover, to catch one's tail one must continue the iteration for a κ which is a reflecting cardinal. Also, in this case there is tension between realizing some new Σ_1 formula with parameters in $H_{\omega_2}^{V[G_\alpha]}$, and doing so by using a proper forcing (which preserves the Π_1 formula, for τ_{Σ_1} in parameter $S \subseteq \omega_1$, S is stationary).

Now one of the guiding ideas of this paper is: what happens if we continue this iterated construction along all the ordinals? If the ordinals are long enough, one should end up with a structure where all existential formulae which can be consistently made true by a proper forcing have been made true. Note, in addition, that $V[G]$ should also be a structure which resembles H_{ω_2} : on the one hand, every ordinal above ω_1^V will have been collapsed to ω_1^V at some point in the iteration; on the other hand, we should also expect that the regularity of Ord is preserved as all iterands are set-sized and, moreover, ω_1 is preserved as the iteration is proper. Hence, the Ord-length iteration is $<$ Ord-CC, ω_1 -preserving, and collapses all uncountable ordinals to size \aleph_1 , if things are properly organized.

In the above situation, we can look at whether there are stages α for which $H_{\omega_2}^{V[G_\alpha]}$ is an actual elementary substructure of $V[G]$. And if the ordinals are long enough (e.g., Ord is Mahlo), this will in fact be the case. We can pursue this approach not only for proper forcings but for a variety of classes Γ including the class of proper forcing notions, semiproper forcing notions, SSP, etc.

It turns out that if Γ is a sufficiently well-behaved class,¹¹ then the corresponding Ord-length iteration produces the same effects as the category forcing (Γ, \leq_Γ) . This gives a further motivation to define the bounded category forcing axiom for Γ ,

⁹Note that L may not even be a ring, it is just a structure with no-zero divisors, where $+, \cdot$ satisfy the commutativity, associativity, and distributivity laws, and 0 and 1 are the neutral elements of $+, \cdot$. L is a ring if it satisfies the Π_2 -sentence stating that $(L, +, 0)$ is a group.

¹⁰Much in the same way one builds the algebraic closure of a field L by passing from $L = L_0$ to L_1 , the field of fractions of $L_0[x]/p_0(x)$ (with p_0 an irreducible polynomial with coefficients in L), and then inductively from L_n to the field of fractions of $L_n[x]/p_n(x)$, with the p_n s given by some book-keeping device.

¹¹All of the above classes are.

$\text{BCFA}(\Gamma)$, as the assertion that $H_{\omega_2}^V$ is an elementary substructure of $V[H]$, for some generic extension H of V by (Γ, \leq_Γ) (as we did). Working in Gödel-Bernays or in Morse-Kelley set theory, this is a perfectly meaningful statement (while in ZFC it cannot even be stated).

These bounded category for axiom for Γ not only imply the corresponding bounded forcing axioms but also strengthen the resurrection axioms introduced by Johnstone and Hamkins [12] and their iterated version introduced by Audrito and the second author [6]. Both of these axioms were motivated by the comparison of the model theoretic notion of *resurrection* with the properties of generic multiverse; for those familiar with model theory, resurrection and T -existential closure are closely related concepts, hence we pursue here the same approach to the generic multiverse of [6, 12]. We are now in a position to briefly outline how the results of this paper compare to the notion of model completeness and model companionship.

A first order theory T is *model complete* if and only if $\mathcal{M} \prec \mathcal{N}$ whenever \mathcal{N} is a superstructure of \mathcal{M} and they are both models of T ; equivalently, if and only if any model of T is T_\forall -e.c. T is *the model companion of S* if T is model complete and if every model of T embeds into a model of S and vice versa; equivalently, if $T_\forall = S_\forall$ and any model of T is S -e.c.

A standard example is obtained by taking S to be the theory of rings with no zero divisors in signature $\{+, \cdot, 0, 1\}$ and T the theory of algebraically closed fields in the same signature.

Now let us look at the Γ -generic multiverse (for a sufficiently well-behaved class Γ):

$$\left\{ H_{\kappa}^{V[G]} : \kappa \geq \omega_2^{V[G]}, G \text{ is } V\text{-generic for a forcing in } \Gamma \right\}.$$

The above considerations show that the class

$$\left\{ H_{\omega_2}^{V[G]} : V[G] \models \text{BCFA}(\Gamma), G \text{ is } V\text{-generic for a forcing in } \Gamma \right\}$$

sits inside the Γ -generic multiverse much in the same way the elementary class of models of T sits inside the elementary class of models of S , whenever T is the model companion of S .

These considerations do *not* establish that the theory of H_{ω_2} is model complete if $V \models \text{BCFA}(\Gamma)$. However, much stronger connections between generic absoluteness, forcing axioms, and model companionship can be obtained if one works with richer signatures. For instance, [25] shows that the theory of $H_{\omega_1}^V$ is the model companion of the theory of V in the signature $\tau_{\text{UB}} = \tau_{\text{ST}} \cup \text{UB}^V$, where UB^V is the class of universally Baire sets, and each $B \in \text{UB}^V$ defines a predicate symbol for τ_{UB} . Using the result of the first author and Schindler establishing that Woodin's axiom (*) follows from MM^{++} [3], the second author [26] has shown that in models of MM^{++} where UB^\sharp is invariant across forcing extensions, the theory of H_{ω_2} is the model companion of the theory of V in signature $\tau_{\text{NS}_{\omega_1}, \text{UB}} = \tau_{\text{UB}} \cup \{\text{NS}_{\omega_1}\}$ (where NS_{ω_1} is the non-stationary ideal on ω_1 , and is the canonical interpretation of its associated unary predicate symbol in $\tau_{\text{NS}_{\omega_1}, \text{UB}}$). Note that assuming large cardinals, the τ_{UB} -theory of H_{ω_1} is generically invariant, while in [26] it is also shown that the universal fragment of the $\tau_{\text{NS}_{\omega_1}, \text{UB}}$ -theory of V is invariant across the generic multiverse. Also, the model completeness results of [26] entail that the $\tau_{\text{NS}_{\omega_1}, \text{UB}}$ -theory of H_{ω_2} is invariant across forcing extension of V which model MM^{++} . In particular, the results of the present paper are weaker than those obtained in [26] when predicated about the class Γ of

SSP-forcings, but as we will see below, they can also be asserted for a variety of classes Γ other than SSP (for example proper, semiproper, etc); finally the present paper generalizes and improves results appearing in [12, 27, 28, 7], where appropriate strengthenings of forcing axioms are introduced with the aim of achieving similar goals.

1.2. Unbounded category forcing axioms. It is worth¹² pointing out that bounded category forcing axioms are natural bounded forms of much stronger axioms, which we call *category forcing axioms*. Given an absolutely well-behaved class Γ of complete Boolean algebras, the category forcing axiom for Γ , $\text{CFA}(\Gamma)$, asserts that the class of certain pre-saturated towers of ideals whose regular open completion is in Γ and which have certain ‘rigidity’ properties, is dense in the category forcing¹³ (Γ, \leq_Γ) . The theory of category forcing axioms, developed by the authors of the present paper, generalizes the theory of the strong form of Martin’s Maximum MM^{+++} introduced in [27]. The axiom MM^{+++} is in fact $\text{CFA}(\text{SSP})$.

In the presence of sufficiently strong large cardinals¹⁴, one can obtain the following.

- (1) $\text{CFA}(\Gamma)$ can always be forced by a forcing in Γ , and in fact by the intersection of (Γ, \leq_Γ) with V_δ for some supercompact cardinal δ .
- (2) $\text{CFA}(\Gamma)$ implies a strong form of the forcing axiom $\text{FA}_{\lambda_\Gamma}(\Gamma)$.
- (3) If $\mathbb{B} \in \Gamma$, G is V -generic for \mathbb{B} , and $V[G] \models \text{CFA}(\Gamma)$, then $\mathcal{C}_{\lambda_\Gamma^+}^V$, the λ_Γ^+ -Chang model as computed in V , is elementarily equivalent to $\mathcal{C}_{\lambda_\Gamma^+}^{V[G]}$ (even considering the extended language with parameters from $H_{\lambda_\Gamma^+}$).

In (3) above, given an infinite cardinal λ , the λ -Chang model \mathcal{C}_λ is the \subseteq -minimal transitive model of ZF containing all ordinals and closed under λ -sequences. It can be construed as¹⁵ $\mathcal{C}_\lambda := L(\text{Ord}^\lambda)$. In particular, $L(\mathcal{P}(\lambda))$ is a definable inner model of \mathcal{C}_λ , and $H_{\lambda^+} \subseteq \mathcal{C}_\lambda$ is definable in \mathcal{C}_λ from λ . As is well-known, \mathcal{C}_λ need not satisfy AC. For instance, by a result of Kunen [15, Thm. 1.1.6 and Rmk. 1.1.28], if there are λ^+ -many measurable cardinals, then $\mathcal{C}_\lambda \models \neg\text{AC}$.

It is not difficult to see that if Γ is an absolutely well-behaved class, then $\text{CFA}(\Gamma)$ implies $\text{BCFA}(\Gamma)$ (again, assuming sufficiently strong large cardinals). In particular, all axioms $\text{CFA}(\Gamma)$ obtained from considering the classes Γ from the second part of the present paper are provably pairwise incompatible.

We should point out that, unlike the bounded category forcing axioms we will be studying in the present paper, the stronger category forcing axioms we are referring to here admit a definition in first order set theory. In fact, given that the relevant classes Γ admit a Δ_2 definition in ZFC, possibly with some parameter, it is easy to see that $\text{CFA}(\Gamma)$ can be expressed as a Π_3 sentence (in the same parameter) over models of ZFC.

¹²As already mentioned, also this section serves as a motivation of the results of this paper and is not needed to follow the remainder of the article.

¹³By ‘tower of ideals’ we mean the forcing notion resulting from naturally relativizing the definition of the stationary tower forcing to some given family of ideals of sets with suitable properties. Forcing with such a forcing gives rise to an elementary embedding $j : V \rightarrow M$, for a certain subclass M of the generic extension. In the current situation, j has critical point λ_Γ^+ .

¹⁴The relevant large cardinals—typically in the region of the existence of a proper class of supercompact cardinals together with one almost super-huge cardinal—are now much stronger than the ones we need for our analysis of *bounded* category forcing axioms.

¹⁵Where $L(\text{Ord}^\lambda) = \bigcup_{\alpha \in \text{Ord}} L(\text{Ord}^\lambda \cap V_\alpha) = \bigcup_{\alpha \in \text{Ord}} L[\text{Ord}^\lambda \cap V_\alpha]$.

We should clarify that, despite the more attractive aspects of category forcing axioms, compared to their bounded forms, we have opted to present in this paper only the theory of the latter. The reason for this is that the theory of their unbounded counterparts is much more involved and would require a longer article. This work will appear elsewhere.

1.3. Notational conventions on class forcing over models of MK. From now on we adopt the following conventions to denote models of MK:

Notation 1.1. (V, \mathcal{V}, \in) is the standard universe of sets and classes in the signature $\{\in, \text{Set}\}$ with Set a unary predicate defining the elements of V , i.e. such that

$$\forall x (\text{Set}(x) \leftrightarrow \exists y (x \in y))$$

holds in (V, \mathcal{V}, \in) .

We assume throughout that (V, \mathcal{V}, \in) models $\text{MK} + \text{Ord is Mahlo}$. We refer the reader to [1] for more details on class forcing over models of MK. The following information conveys all that will be needed here.

Whenever (P, \leq_P) is a class forcing (i.e. $P \in \mathcal{V}$ and $\leq_P \subseteq P \times P$ is a pre-order also in \mathcal{V}), and G is \mathcal{V} -generic for P (i.e. it meets all dense subsets of P in \mathcal{V}), it makes sense to define $\mathcal{V}[G]$ as the family of classes obtained by the evaluation map

$$\dot{C} \mapsto \dot{C}_G = \left\{ \tau_G : (\tau, p) \in \dot{C} : p \in G \right\}$$

as \dot{C} ranges over the elements of \mathcal{V} which are contained in $V^P \times P$ and

$$V^P = \{ \tau \in V : \tau \text{ is a set-sized } P\text{-name} \}$$

(the notion of set-sized P -name is recursively defined according to Kunen's standard definition as those $\tau \in V$ which are binary relations with domain a subset of V^P and range contained in P). $V[G]$ is actually $(V^P \times P)_G$.

We may occasionally write $V[G]$ with G V -generic for a class to denote the generic extension $(V[G], \mathcal{V}[G], \in)$ obtained by G over the MK-model (V, \mathcal{V}, \in) .

In general such class forcings may not even preserve replacement (as Ord could become countable, for example, if one forces with $\text{Coll}(\omega, \text{Ord})$). In this paper we will be interested in class forcings which preserve the regularity of Ord but may not preserve its inaccessibility; hence, for a generic G for such class forcings, $(V[G], \in)$ is a model of ZFC^- (i.e. ZFC without the power-set axiom), but possibly not a model of ZFC .

2. FORCING WITH FORCINGS

The aim of this section is to develop a general theory of category forcings i.e. class forcings (Γ, \leq_Γ) with Γ a definable class of forcings and $P \leq_\Gamma Q$ if whenever G is V -generic for P , in $V[G]$ there is H V -generic for Q such that $V[G]$ is a generic extension of $V[H]$ by a forcing in $\Gamma^{V[H]}$. It will become apparent that such an analysis can be systematically carried out if one focuses on the subclass of Γ given by the complete boolean algebras in Γ . First of all this is no loss of information since our analysis will identify properties of elements of Γ which are invariant with respect to boolean completions. For example “whenever G is V -generic for P , in $V[G]$ there is H V -generic for Q ” can be formulated in algebraic terms as “there is a (possibly non-injective) complete homomorphism $i : \text{RO}(Q) \rightarrow \text{RO}(P)$ ”. We hope that this

example makes transparent that, by focusing on cbas rather than posets, we will be able to leverage the algebraic structure of complete boolean algebra to greatly simplify the formulation of certain concepts, as well as many proofs. However most of the forcings in Γ we will consider are not naturally presented as cbas; for example one of our main result will be to show that $(\Gamma \cap V_\delta, \leq_\Gamma \cap V_\delta)$, which is not even separative, is in Γ for most inaccessible cardinals δ . So we will feel free to decide depending on the issue under examination whether to focus on Γ as a class of cbas or rather as a class of partial orders, keeping in mind that our results apply equally well in both set-ups.

Notation 2.1. Given a cba $\mathbb{B} \in V$ and a family A of elements of¹⁶ $V^\mathbb{B}$,

$$A^\circ = \{\langle \tau, 1_\mathbb{B} \rangle : \tau \in A\}$$

For a partial order P and $p \in P$, $P \upharpoonright p = \{q \in P : q \leq p\}$.

Definition 2.2. Let \mathbb{B} be an infinite complete boolean algebra and $\dot{\kappa} \in V^\mathbb{B}$ be such that

$$\llbracket \dot{\kappa} \text{ is a regular cardinal} \rrbracket_\mathbb{B} = 1_\mathbb{B}.$$

We define

$$H_\dot{\kappa}^\mathbb{B} = \{\tau : \tau \in V^\mathbb{B}, \llbracket \text{trcl}(\tau) \text{ is hereditarily of size less than } \dot{\kappa} \rrbracket_\mathbb{B} = 1_\mathbb{B}\}$$

$(H_\dot{\kappa}^\mathbb{B})^\circ \in \mathcal{V}^\mathbb{B}$ is a canonical name for $H_{\dot{\kappa}_G}^{V[G]}$ whenever G is V -generic for \mathbb{B} . More precisely:

Assume G is V -generic for \mathbb{B} . Then

$$(H_\dot{\kappa}^\mathbb{B})^\circ_G = \{\tau_G : \tau \in H_\dot{\kappa}^\mathbb{B}\} = H_{\dot{\kappa}_G}^{V[G]}.$$

On the face of its definition $H_\dot{\kappa}^\mathbb{B}$ is a proper class. To avoid unnecessary complications we can use Scott's trick and consider just its elements whose rank is bounded by some sufficiently large fixed α . As we will see, in many cases of interest it suffices to take $\alpha = |\mathbb{B}|$.

Definition 2.3. Given a complete boolean algebra \mathbb{B} , we denote by $\dot{\omega}_1$ a canonically chosen member of $V^\mathbb{B}$ such that

$$\llbracket \dot{\omega}_1 \text{ is the first uncountable cardinal} \rrbracket_\mathbb{B} = 1_\mathbb{B}.$$

There is of course a dependence on \mathbb{B} in the above definition of $\dot{\omega}_1$. However, we will not need to make this dependence explicit in the notation.

2.1. Projective absoluteness. We want to sketch the reason why Solovay's model for set theory obtained by forcing with $\text{Coll}(\omega, < \delta)$ gives the "correct" theory of projective sets. While doing so we outline how these results could be generalized to larger fragment of the universe sets. The following is a weakening of Woodin's generic absoluteness results for the ω -Chang model:

Theorem 2.4. [15, Cor. 3.1.7] (**Woodin**). *Assume V is a ZFC model in which there are class-many Woodin cardinals. Let $\phi(x_1, \dots, x_n)$ be any formula for the signature $\{\in\}$ and $a_1, \dots, a_n \in H_{\dot{\omega}_1}$. TFAE:*

- (1) $H_{\dot{\omega}_1}^V \models \phi(a_1, \dots, a_n)$.
- (2) $V \models \llbracket H_{\dot{\omega}_1} \models \phi(\check{a}_1, \dots, \check{a}_n) \rrbracket_\mathbb{B} = 1_\mathbb{B}$ for some cba \mathbb{B} .

¹⁶The (in our opinion very smart) A° notational trick is due to Asaf Karagila.

(3) $V \models \llbracket H_{\check{\omega}_1} \models \phi(\check{a}_1, \dots, \check{a}_n) \rrbracket_{\mathbb{B}} = 1_{\mathbb{B}}$ for all cbas \mathbb{B} .

In particular, the second order theory of the natural numbers is provably invariant under set-sized forcing.

We will prove a weak form of the above theorem (considerably weakening the large cardinal assumptions); this will be a first approximation to the type of axiom we will introduce to get generic absoluteness for the theory of H_{λ^+} with respect to arbitrary infinite cardinals λ .

Notation 2.5. For δ an ordinal

- $p \in \text{Coll}(\omega, <\delta)$ if and only if $p : \omega \times \delta \rightarrow \delta$ is such that $\text{dom}(p)$ is finite and $p(n, \alpha) \in \alpha$ for all $\alpha \in \text{dom}(p)$.
- \mathbb{B}_δ denotes the boolean completion of the forcing notion $\text{Coll}(\omega, <\delta)$.
- In models (V, \mathcal{V}) of MK, $\text{Coll}(\omega, <\text{Ord})$, $\mathbb{B}_{\text{Ord}} \in \mathcal{V}$ are the class forcings obtained replacing δ with Ord .

\mathbb{B}_{Ord} is a well defined element of \mathcal{V} because $\text{Coll}(\omega, <\text{Ord})$ is $<\text{Ord}$ -CC, hence the suprema needed to define \mathbb{B}_{Ord} are suprema of set-sized antichains, and therefore \mathbb{B}_{Ord} is a proper class.

Generically $\text{Coll}(\omega, <\delta)$ adds bijections between ω and all ordinals less than δ , while making δ the first uncountable cardinal; $\text{Coll}(\omega, <\text{Ord})$ therefore makes all sets existing in the generic extension hereditarily countable.

Solovay realized that these forcings produce natural models of the theory of projective sets (equivalently of the first order theory of (H_{ω_1}, \in)).

The following are well-known properties of \mathbb{B}_δ (which hold also for $\delta = \text{Ord}$ by a straightforward generalization of the proofs).

Theorem 2.6. *Let (V, \mathcal{V}) be a model of MK. Assume δ is an inaccessible cardinal or $\delta = \text{Ord}$. Then:*

\mathbb{B}_δ is $<\delta$ -CC: [13, Theorem 15.17(iii)] Hence $H_{\check{\omega}_1}^{\mathbb{B}_\delta} = V_\delta \cap V^{\mathbb{B}_\delta}$.

Universality: [13, Lemma 26.9] \mathbb{B}_δ contains as a complete subalgebra an isomorphic copy of any cba of size smaller than δ .

Factor Lemma: [13, Cor. 26.11] For all cbas $\mathbb{B} \in V_\delta$,

$$\mathbb{B} * \text{Coll}(\check{\omega}, <\check{\delta}) \cong \text{Coll}(\omega, <\delta).$$

Logical Completeness: \mathbb{B}_δ is homogeneous [13, Thm 26.12].

Hence

$$T_\delta = \left\{ \phi(a_1, \dots, a_n) : a_1, \dots, a_n \in H_{\omega_1}, \right. \\ \left. \llbracket H_{\check{\omega}_1} \models \phi(\check{a}_1, \dots, \check{a}_n) \rrbracket_{\mathbb{B}_\delta} = 1_{\mathbb{B}_\delta} \right\}$$

is a complete first order theory.

Definition 2.7. PA (Projective Absoluteness)

$$H_{\omega_1} \models T_{\text{Ord}}.$$

PA states that for all formulae $\phi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in H_{\omega_1}$ we have that

$$H_{\omega_1} \models \phi(a_1, \dots, a_n) \text{ if and only if } \llbracket \phi(\check{a}_1, \dots, \check{a}_n) \rrbracket_{\mathbb{B}_{\text{Ord}}} = 1_{\mathbb{B}_{\text{Ord}}}.$$

A complete boolean algebra $\mathbb{C} \in V$ satisfies PA if and only if (viewing it as a complete subalgebra of \mathbb{B}_{Ord}) for all $\tau_1, \dots, \tau_n \in H_{\check{\omega}_1}^{\mathbb{C}}$ and formulae $\phi(x_1, \dots, x_n)$

without class quantifiers, we have that

$$\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}_{\text{Ord}}} = \llbracket (H_{\omega_1}^{\mathbf{C}})^\circ \models \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{C}}.$$

Theorem 2.8. *Assume*

$$(V, \mathcal{V}) \models \text{MK} + \text{Ord is Mahlo.}$$

(i.e. every club subclass of the ordinals in \mathcal{V} contains a regular cardinal).

Let $(\Omega, \rightarrow^\Omega)$ be the category given by the class of cbas as objects and the class of complete homomorphisms between them as arrows.¹⁷

Set $\mathbf{C} \leq_\Omega \mathbf{B}$ if there is $i : \mathbf{B} \rightarrow \mathbf{C}$ in \rightarrow^Ω .

Then

$$D_{\text{PA}} = \{\mathbf{B} \in \Omega : \mathbf{B} \text{ satisfies PA}\}$$

is dense in the class partial order (Ω, \leq_Ω) .

In particular PA is a consistent axiom.

Proof. Since $\mathbf{B} \geq_\Omega \mathbf{B}_\delta$ for any inaccessible $\delta > |\mathbf{B}|$, it suffices to show that there are stationarily many δ such that \mathbf{B}_δ satisfies PA.

We need to put together three separate observations on the properties of \mathbf{B}_γ :

- (1) For any ordinal γ

$$\text{Coll}(\omega, < \gamma) \sqsubseteq \text{Coll}(\omega, < \text{Ord}),$$

(i.e. the inclusion map is a complete embedding of partial orders).

- (2) Assume δ is an inaccessible cardinal. Since \mathbf{B}_δ is $< \delta$ -CC and has size δ , it is not hard to check that

$$(H_{\omega_1}^\delta)_G^\circ = \{\tau_G : \tau \in V_\delta \cap V_{\mathbf{B}}, \llbracket \text{trcl}(\tau) \text{ is countable} \rrbracket_{\mathbf{B}_\delta} = 1_{\mathbf{B}}\} = H_{\omega_1}^{V[G]} = V_\delta[G]$$

whenever G is V -generic for \mathbf{B}_δ .

Observe also that $\text{Coll}(\omega, < \delta)$ is a definable class forcing in the ZFC-model (V_δ, \in) , which the MK-model $(V_\delta, V_{\delta+1}, \in)$ proves to be $< \delta$ -CC.

By the forcing theorem¹⁸ applied in V_δ to the definable class forcing $\text{Coll}(\omega, < \delta)$, we get that for all V -generic filters G for \mathbf{B}_δ , all formulae $\phi(x_1, \dots, x_n)$, and all $\tau_1, \dots, \tau_n \in V_\delta \cap V_{\mathbf{B}}$,

$$(H_{\omega_1}^\delta)_G^\circ = V_\delta[G] \models \phi(\tau_1, \dots, \tau_n)$$

if and only if for some $p \in G \cap \text{Coll}(\omega, < \delta)$,

$$V_\delta \models p \Vdash \phi(\tau_1, \dots, \tau_n).$$

- (3) Since $\langle V, \mathcal{V} \rangle$ is a model of MK, the class

$$C = \{\gamma : (V_\gamma, \in) \prec (V, \in)\}$$

is a club subset of Ord.

Now for each $\gamma \in C$, each formula $\phi(x_1, \dots, x_n)$ without class quantifiers, and each $\tau_1, \dots, \tau_n \in V_\gamma \cap V^{\mathbf{B}}$,

$$V_\gamma \models p \Vdash_{\text{Coll}(\omega, < \gamma)} \phi(\tau_1, \dots, \tau_n)$$

¹⁷Note that the homomorphism may not be injective. The key point is that the preimage $i^{-1}[G]$ by a complete (possibly non-injective) homomorphism $i : \mathbf{B} \rightarrow \mathbf{C}$ of a V -generic filter G for \mathbf{C} is a V -generic filter for \mathbf{B} .

¹⁸The forcing theorem applies in V_δ to the class forcing (relative to V_δ) $\text{Coll}(\omega, < \delta)$ because $\text{Coll}(\omega, < \delta)$ is definably $< \delta$ -CC, in the sense that any definable antichain over (V_δ, \in) belongs to V_δ .

if and only if

$$V \models p \Vdash_{\text{Coll}(\omega, < \text{Ord})} \phi(\tau_1, \dots, \tau_n).$$

This is the case since the classes involved in the definition of

$$p \Vdash_{\text{Coll}(\omega, < \text{Ord})} \phi(\tau_1, \dots, \tau_n)$$

are all definable in V using parameters in V_γ .

Since Ord is Mahlo in $\langle V, \mathcal{V} \rangle$, there are stationarily many inaccessible cardinals δ in C , and for any inaccessible $\delta \in C$ the three properties outlined above for \mathbf{B}_δ hold simultaneously.

The following claim suffices to complete the proof of the theorem.

Claim 1. *Assume $\delta \in C$ is inaccessible. Then for all formulae $\phi(x_1, \dots, x_n)$ without class quantifiers and $\tau_1, \dots, \tau_n \in H_{\dot{\omega}_1}^{\mathbf{B}_\delta} = V_\delta \cap V^{\mathbf{B}_\delta}$,*

$$\llbracket H_{\dot{\omega}_1} \models \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}_\delta} = \llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}_{\text{Ord}}}.$$

Proof. Since $\delta \in C$ is inaccessible, ϕ does not have class quantifiers, and $\text{Coll}(\omega, < \delta) \sqsubseteq \text{Coll}(\omega, < \text{Ord})$, we have that:

$$\begin{aligned} & \llbracket H_{\dot{\omega}_1} \models \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}_\delta} = \\ &= \bigvee_{\text{Coll}(\omega, < \delta)} \{ p \in \text{Coll}(\omega, < \delta) : \langle V_\delta, \in \rangle \models p \Vdash_{\text{Coll}(\omega, < \delta)} \phi(\tau_1, \dots, \tau_n) \} = \\ &= \bigvee_{\text{Coll}(\omega, < \delta)} \{ p \in \text{Coll}(\omega, < \text{Ord}) : \langle V, \in \rangle \models p \Vdash_{\text{Coll}(\omega, < \text{Ord})} \phi(\tau_1, \dots, \tau_n) \} = \\ &= \bigvee_{\text{Coll}(\omega, < \text{Ord})} \{ p \in \text{Coll}(\omega, < \text{Ord}) : \langle V, \in \rangle \models p \Vdash_{\text{Coll}(\omega, < \text{Ord})} \phi(\tau_1, \dots, \tau_n) \} = \\ &= \llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}_{\text{Ord}}}. \end{aligned}$$

The first and last equalities hold by definition (for the first observe that $\dot{H}_{\dot{\omega}_1}^\delta = V_\delta \cap V^{\mathbf{B}_\delta}$ is a canonical \mathbf{B}_δ -name for $H_{\dot{\omega}_1}^{V[G]}$ for G V -generic for \mathbf{B}_δ); the second equality holds because $\langle V_\delta, \in \rangle \prec \langle V, \in \rangle$; the third equality holds because $\text{Coll}(\omega, < \delta) \sqsubseteq \text{Coll}(\omega, < \text{Ord})$. \square

The theorem is proved. \square

Remark 2.9. If $\gamma \in C$ is not inaccessible, then $\text{Coll}(\omega, < \gamma)$ is not $< \gamma$ -cc and collapses γ to become countable. Hence $H_{\dot{\omega}_1}^{\mathbf{B}_\gamma} \neq V_\gamma \cap V^{\mathbf{B}_\gamma}$. This gives that PA may not be satisfied by \mathbf{B}_γ in this case, since the boolean value of $\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}_{\text{Ord}}}$ may require some $p \notin \text{Coll}(\omega, < \gamma)$ to be computed if there is some τ_i which is not in V_γ .

Corollary 2.10. *Assume $\langle V, \mathcal{V} \rangle \models \text{MK}$ and $i : \mathbf{B} \rightarrow \mathbf{C}$ is a complete homomorphism. Assume \mathbf{B} and \mathbf{C} both satisfy PA . Then for any G V -generic for \mathbf{C} , letting $H = i^{-1}[G]$, we get that*

$$H_{\dot{\omega}_1}^{\mathbf{B}}[H] = H_{\dot{\omega}_1}^{V[H]} \prec H_{\dot{\omega}_1}^{V[G]} = H_{\dot{\omega}_1}^{\mathbf{C}}[G].$$

Proof. The Corollary is an immediate consequence of the following basic model-theoretic observation:

Fact 2.11. *Assume $\mathcal{M}_0, \mathcal{M}_1$ and \mathcal{N} are L -structures such that the following diagram is realized:*

$$\begin{array}{ccc} \mathcal{M}_0 & \xrightarrow{\Sigma_\omega} & \mathcal{N} \\ & \searrow & \nearrow \\ & \mathcal{M}_1 & \end{array}$$

Then

$$\mathcal{M}_0 \prec \mathcal{M}_1.$$

Notice that if K is V -generic for \mathbf{B}_{Ord} , $G \in V[K]$ is V -generic for \mathbf{C} , and $H \in V[G]$ is V -generic for \mathbf{B} , we obtain the above configuration:

$$\begin{array}{ccc} H_{\omega_1}^{V[H]} & \xrightarrow{\Sigma_\omega} & H_{\omega_1}^{V[K]} \\ & \searrow & \nearrow \\ & H_{\omega_1}^{V[G]} & \end{array}$$

□

Corollary 2.12. *Assume*

$$(V, \mathcal{V}) \models \text{MK} + \text{PA} + \text{Ord is Mahlo.}$$

TFAE:

- (1) $H_{\omega_1} \models \phi(a_1, \dots, a_n)$.
- (2) $\llbracket H_{\omega_1}^{\mathbf{B}} \models \phi(\check{a}_1, \dots, \check{a}_n) \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$ for some cba \mathbf{B} which satisfies PA.
- (3) $\llbracket H_{\omega_1}^{\mathbf{B}} \models \phi(\check{a}_1, \dots, \check{a}_n) \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$ for all cbas \mathbf{B} which satisfy PA.

Woodin's generic absoluteness results for projective sets provide significant strengthenings of the conclusion of the theorem above in the presence of stronger hypotheses. The following is a weakening of [15, Theorem 3.1.2] which has the same flavor of what we have been showing so far:

Theorem 2.13 (Woodin). *Assume δ is a Woodin cardinal which is a limit of Woodin cardinals and $\mathbf{B} \in V_\delta$ is a cba.*

Then

$$H_{\omega_1}^{\mathbf{B}} \prec H_{\omega_1}^{\mathbf{B}_\delta}.$$

Now observe that the fact whether or not δ is Woodin is detected at stage $V_{\delta+1}$:

$$V \models \delta \text{ is a Woodin cardinal} \iff V_{\delta+1} \models \delta \text{ is a Woodin cardinal.}$$

In particular 'Ord is Woodin' is expressible in any model of MK.

Corollary 2.14. *Assume*

$$(V, \mathcal{V}) \models \text{MK} + \text{Ord is Woodin} + \text{there are class many Woodin cardinals.}$$

Then **every** \mathbf{B} in Ω satisfies PA.

Hence TFAE:

- (1) $H_{\omega_1} \models \phi(a_1, \dots, a_n)$.
- (2) $\llbracket H_{\omega_1}^{\mathbf{B}} \models \phi(\check{a}_1, \dots, \check{a}_n) \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$ for some cba \mathbf{B} .
- (3) $\llbracket H_{\omega_1}^{\mathbf{B}} \models \phi(\check{a}_1, \dots, \check{a}_n) \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$ for all cbas \mathbf{B} .

We observe the following:

Remark 2.15. The key steps in the proof of the density of

$$D_{\text{PA}} = \{ \mathbf{B} : H_{\omega_1}^{\mathbf{B}} \prec V^{\text{Coll}(\omega, < \text{Ord})} \}$$

for the preorder given by the class category $(\Omega, \rightarrow^\Omega)$ in a model of

$$\text{MK} + \text{Ord} \text{ is Mahlo}$$

rely on the following crucial properties:

- $\text{Coll}(\omega, < \delta)$ preserves the regularity of δ for any regular cardinal δ and makes δ the first uncountable cardinal;
- $\text{Coll}(\omega, < \delta) \sqsubseteq \text{Coll}(\omega, < \text{Ord})$ for all regular cardinals δ ;
- any cba $\mathbf{B} \in V_\delta$ embeds as a complete suborder in $\text{Coll}(\omega, < \delta)$ for any inaccessible δ ;

We want to replicate the above proof pattern for arbitrary classes of forcings Γ closed under two-step iterations. The guiding idea will be that projective absoluteness is just $\text{BCFA}(\Gamma)$ for $\Gamma = \Omega$, the class of all forcing notions. We will show that for a variety of definable classes of forcings Γ there is a (uniformly) definable cardinal λ_Γ associated to Γ such that

$$D_\Gamma = \{ \mathbf{B} \in \Gamma : H_{\lambda_\Gamma^+}^{\mathbf{B}} \prec V^{\Gamma|\mathbf{B}} \}$$

is dense in the class forcing induced by the category $(\Gamma, \rightarrow^\Gamma)$.

Our analysis of projective absoluteness in the previous section shows that $\lambda_\Omega = \omega$.

We will focus in section 5 on the analysis of classes Γ for which $\lambda_\Gamma = \omega_1$. However, the machinery we will present below is modular and shows that for any class of forcings Γ satisfying certain reasonable properties, λ_Γ is well defined and D_Γ is dense in the class forcing induced by the category $(\Gamma, \rightarrow^\Gamma)$.

First of all we need to introduce the terminology and notation required to formulate precisely what is meant by the class forcing induced by the category $(\Gamma, \rightarrow^\Gamma)$, by $\text{BCFA}(\Gamma)$, etc.

Notation 2.16. Let $i : \mathbf{B} \rightarrow \mathbf{C}$ be a complete homomorphism of cbas.

$$\ker(i) = \bigvee \{ b \in \mathbf{B} : i(b) = 0_{\mathbf{C}} \},$$

$$\text{coker}(i) = \neg \ker(i),$$

i is a regular embedding if it is injective and complete.

For any $b \in \mathbf{B}$

$$k_b : \mathbf{B} \rightarrow \mathbf{B} \upharpoonright b$$

$$c \mapsto c \wedge b.$$

$$\dot{G}_{\mathbf{B}} = \{ \langle \check{b}, b \rangle : b \in \mathbf{B} \}$$

is the canonical \mathbf{B} -name for the V -generic filter.

Remark 2.17. All complete homomorphisms $i : \mathbf{B} \rightarrow \mathbf{C}$ are of the form $i_0 \circ k_b$ with $i_0 : \mathbf{B} \upharpoonright b \rightarrow \mathbf{C}$ a regular embedding and $b = \text{coker}(i)$.

Notation 2.18. Let $i : \mathbf{B} \rightarrow \mathbf{C}$ a complete homomorphism and $\dot{\kappa} \in \mathcal{V}^{\mathbf{B}}$, $\dot{\delta} \in \mathcal{V}^{\mathbf{C}}$ be such that

$$\llbracket \dot{\kappa} \text{ is a regular uncountable cardinal} \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$$

and

$$\llbracket \dot{\delta} \text{ is a regular uncountable cardinal} \rrbracket_{\mathbf{C}} = 1_{\mathbf{C}},$$

where

$$\llbracket \hat{i}(\dot{\kappa}) \leq \dot{\delta} \rrbracket = 1_{\mathbf{C}}.$$

We say that

$$H_{\dot{\kappa}}^{\mathbf{B}} \prec H_{\dot{\delta}}^{\mathbf{C}}$$

if and only if for all $\tau_1, \dots, \tau_n \in H_{\dot{\kappa}}^{\mathbf{B}}$ and formulae $\phi(x_1, \dots, x_n)$ without class quantifiers, we have that

$$i(\llbracket H_{\dot{\kappa}} \models \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}) = \llbracket H_{\dot{\delta}} \models \phi(\hat{i}(\tau_1), \dots, \hat{i}(\tau_n)) \rrbracket_{\mathbf{C}}.$$

Here,

$$\begin{aligned} \hat{i} : \mathcal{V}^{\mathbf{B}} &\rightarrow \mathcal{V}^{\mathbf{C}} \\ \tau &\mapsto \left\{ \langle \hat{i}(\sigma), i(b) \rangle : \langle \sigma, b \rangle \in \tau \right\} \end{aligned}$$

Our analysis of projective absoluteness covers the situation in which

$$i : \mathbf{B} = \text{RO}(\text{Coll}(\omega, < \delta)) \rightarrow \mathbf{C} = \text{RO}(\text{Coll}(\omega, < \gamma))$$

is the inclusion, $\delta < \gamma$ are both in C (hence they both force PA), and $\dot{\kappa} = \dot{\omega}_1^{\mathbf{B}}$ and $\dot{\delta} = \dot{\omega}_1^{\mathbf{C}}$ are the canonical names for the first uncountable cardinal, for \mathbf{B} and \mathbf{C} , respectively.

Notation 2.19. Given the standard model (V, \mathcal{V}) of MK, a *definable* class Γ denotes an element of \mathcal{V} defined by a first order formula $\phi_{\Gamma}(x, a_{\Gamma})$ with $a_{\Gamma} \in V$ and $\phi_{\Gamma}(x, y)$ a formula without class quantifiers.

Note that we allow the free variable x to also take values in \mathcal{V} , since one of our objective will be to infer that $\phi_{\Gamma}(\Gamma, a)$ holds in (V, \mathcal{V}, \in) for a wide range of classes Γ .

Definition 2.20. Let Γ be a definable class of pre-orders.

- $(\Gamma, \rightarrow^{\Gamma})$ is the category whose objects are cbas in Γ , and whose arrows are the Γ -correct homomorphisms (not necessarily injective).
- For cbas \mathbf{B}, \mathbf{C} , we write
 - $\mathbf{B} \geq_{\Gamma} \mathbf{C}$ if there is a Γ -correct $i : \mathbf{B} \rightarrow \mathbf{C}$, and
 - $\mathbf{B} \geq_{\Gamma}^* \mathbf{C}$ if there is a Γ -correct $i : \mathbf{B} \rightarrow \mathbf{C}$ which is also injective (i.e. i is regular).

For any ordinal δ , $\Gamma_{\delta} = \Gamma \cap V_{\delta}$.

Definition 2.21. Let Γ be a definable class of pre-orders.

- Γ is *stable under forcing* if for any $P \in \Gamma$, any pre-order Q such that $\text{RO}(P) \cong \text{RO}(Q)$ is also in Γ .
- A complete (not necessarily injective) homomorphism of cbas $i : \mathbf{B} \rightarrow \mathbf{C}$ is *Γ -correct* if

$$\llbracket \phi_{\Gamma}(\mathbf{C}/_{i[\dot{G}_{\mathbf{B}}]}, \check{a}_{\Gamma}) \rrbracket_{\mathbf{B}} = \llbracket \mathbf{C}/_{i[\dot{G}_{\mathbf{B}}]} \in \dot{\Gamma} \rrbracket_{\mathbf{B}} \geq \text{coker}(i).^{19}$$

- Γ is *closed under two-step iterations* if $\mathbf{C} \in \Gamma$ whenever $\mathbf{B} \in \Gamma$ and $i : \mathbf{B} \rightarrow \mathbf{C}$ is Γ -correct.

¹⁹This set-up provides an algebraic expression of the condition that whenever G is V -generic for \mathbf{C} with $i(\text{coker}(i)) \in G$ and $H = i^{-1}[G]$, $\mathbf{C}/_{i[H]}$ is a cba in $V[H]$ which is in $\Gamma^{V[H]}$.

- $(\Gamma, \rightarrow^\Gamma)$ is the category whose objects are cbas in Γ , and whose arrows are the Γ -correct homomorphisms (not necessarily injective).
- For cbas \mathbf{B}, \mathbf{C} , we write
 - $\mathbf{B} \geq_\Gamma \mathbf{C}$ if there is a Γ -correct $i : \mathbf{B} \rightarrow \mathbf{C}$, and
 - $\mathbf{B} \geq_\Gamma^* \mathbf{C}$ if there is a Γ -correct $i : \mathbf{B} \rightarrow \mathbf{C}$ which is also injective (i.e. i is regular).

For any ordinal δ , $\Gamma_\delta = \Gamma \cap V_\delta$.

Remark 2.22. Stability under forcing in Γ asserts that Γ is closed with respect to the equivalence relation on preorders given by $P \sim Q$ iff $\text{RO}(P) \cong \text{RO}(Q)$. All the properties of Γ we are interested in are invariant under this equivalence relation, and each equivalence class according to \sim has a canonical representative given by the unique (modulo isomorphism) cba \mathbf{B} belonging to it. In particular, we can restrict our analysis of Γ concentrating just on its subclass given by the cbas in it. It will be however sometimes convenient to allow as elements of Γ also partial orders $P \in V$ which are not cbas but are such that $\phi_\Gamma(P, a_\gamma)$ holds: specifically, we will need to infer that for well-behaved classes Γ , $\Gamma_\delta \in \Gamma$ for many inaccessible cardinals δ ; ($\Gamma_\delta, \leq_\Gamma$) is a pre-order but not a cba, and it is much simpler to describe the combinatorial properties of Γ_δ than those of its boolean completion.

- If $(\Gamma, \rightarrow^\Gamma)$ has lower bounds for its finite subsets, then \leq_Γ is a trivial forcing notion since all conditions are compatible.
- For any cba \mathbf{B} there is a regular embedding $i : \mathbf{B} \rightarrow \text{RO}(\text{Coll}(\omega, \delta))$ for any large enough δ ; consequently the class $(\Omega, \rightarrow^\Omega)$ of all cbas and all complete homomorphisms between them has lower bounds for its finite subsets, hence \leq_Ω^* and \leq_Ω are trivial forcing notions.

Definition 2.23. A partial order P is in SSP if for any stationary $S \subseteq \omega_1$,

$$P \Vdash \check{S} \text{ is stationary.}$$

Fact 2.24. $(\text{SSP}, \leq_\Omega)$ (hence also $(\text{SSP}, \leq_{\text{SSP}})$) is an atomless partial order.

Proof. Assume

- P is Namba forcing on \aleph_2 ,
- Q is $\text{Coll}(\omega_1, \omega_2)$.

Then $\text{RO}(P)$ and $\text{RO}(Q)$ are incompatible conditions in $(\text{SSP}, \leq_\Omega)$ (and therefore also in $(\text{SSP}, \leq_{\text{SSP}})$): Assume $\mathbf{R} \leq_{\text{SSP}} \text{RO}(P), \text{RO}(Q)$, and H is V -generic for \mathbf{R} . Then

- $\omega_1^{V[H]} = \omega_1$,
- there are $G, K \in V[H]$ V -generic filters for P and Q respectively (since $\mathbf{R} \leq_\Omega \text{RO}(P), \text{RO}(Q)$).

G gives in $V[H]$ a sequence cofinal in ω_2^V of type ω .

K gives in $V[H]$ a sequence cofinal in ω_2^V of type ω_1^V .

Contradiction with the preservation of ω_1 in $V[H]$ (which holds since $\mathbf{R} \in \text{SSP}$).

This argument can be repeated in $V^{\mathbf{B}}$ for any $\mathbf{B} \in \text{SSP}$. \square

Remark 2.25. Similar arguments show that \leq_Γ defines an atomless partial order for a variety of $\Gamma \subseteq \text{SSP}$ (for example for Γ being the class of proper posets, or the class of semiproper posets).

2.2. BCFA(Γ) and how to get it. First of all we define the cardinal λ_Γ we attach to a given class Γ of forcing notions. λ_Γ could actually be Ord. However, in all cases of interest λ_Γ will in fact exist as a cardinal. It will be needed to formulate BCFA(Γ) properly (it will

also be a key parameter in the proper formulation of the iteration theorem we will later require to hold for forcings in Γ).

Definition 2.26. Given a definable class of forcings Γ , λ_Γ is the supremum of all cardinals $\eta \in V$ such that all forcings in Γ preserve η .

Remark 2.27. $\lambda_\Omega = \omega$, $\lambda_{\text{SSP}} = \omega_1$, and the same holds for all $\Gamma \subseteq \text{SSP}$ such that every countably closed forcing is in Γ (e.g. the class of proper forcings and the class of semiproper forcings). It is easy to see that λ_Γ is either Ord or the maximum of the set of cardinals of which it is a supremum. We will be interested just in the case in which λ_Γ is an uncountable regular cardinal (and the reader can safely assume throughout the paper that $\lambda_\Gamma = \omega_1$).

Definition 2.28. Let (V, \mathcal{V}) be the standard model of MK and $\Gamma \in \mathcal{V}$ be a definable class of forcings closed under two-step iterations and such that $\lambda_\Gamma \in \text{Ord}$. We say that the *Bounded Category Forcing Axiom* for Γ , $\text{BCFA}(\Gamma)$, holds if

$$H_{\lambda_\Gamma^+} \prec V^\Gamma,$$

and

$$D_\Gamma = \left\{ \mathbb{B} \in \Gamma : H_{\lambda_\Gamma^+}^{\mathbb{B}} \prec V^{\Gamma \restriction \mathbb{B}} \right\}$$

is dense in (Γ, \leq_Γ) .²⁰

As we will see, the density of D_Γ can be proved right away in MK for many classes of forcings Γ , hence $\text{BCFA}(\Gamma)$ holds once we force with some element of D_Γ .

Assume we have a definable class of forcings $(\Gamma, \rightarrow_\Gamma)$ with the following properties.²¹

- *Pretameness:* $V^\Gamma \models \text{Ord} = \lambda_\Gamma^+$.
- *Factor Lemma:* Γ is closed under two-step iterations; therefore²² $\mathbb{B} * \dot{\Gamma} \cong \Gamma \restriction \mathbb{B}$ holds for all cbas $\mathbb{B} \in \Gamma$;
- *Self-similarity:* For stationarily many inaccessible cardinals α ,
 - $\Gamma \cap V_\alpha = \Gamma_\alpha$ is such that $(\Gamma_\alpha, \leq_\Gamma) \in \Gamma$, and (as in the case $\alpha = \text{Ord}$) $(\Gamma_\alpha, \leq_\Gamma)$ preserves the regularity of α making it the successor of λ_Γ , and
 - the map $\mathbb{B} \mapsto \Gamma_\alpha \restriction \mathbb{B}$ regularly embeds Γ_α into $\Gamma \restriction \Gamma_\alpha$.
- *Universality:* For stationarily many inaccessible cardinals α , every $\mathbb{B} \in \Gamma_\alpha$ regularly embeds into $\Gamma_\alpha \restriction \mathbb{C}$ for some $\mathbb{C} \leq_\Gamma \mathbb{B}$ in Γ_α .

Then we would be able to replicate the same proof pattern we used to establish the consistency of projective absoluteness replacing $\text{Coll}(\omega, < \text{Ord})$ by Γ and $\text{Coll}(\omega, < \alpha)$ by Γ_α to infer that

$$D_\Gamma = \left\{ \mathbb{B} \in \Gamma : H_{\lambda_\Gamma^+}^{\mathbb{B}} \prec V^\Gamma \right\}$$

is dense in (Γ, \leq_Γ) , as witnessed by the forcings $\Gamma_\alpha \restriction \mathbb{C}$ for α inaccessible with $\Gamma_\alpha \in \Gamma$, $V_\alpha \prec V$, and $\mathbb{C} \in \Gamma_\alpha$.

We also want to observe some other basic facts concerning the above conditions.

²⁰ $H_{\lambda_\Gamma^+} \prec V^\Gamma$ means that $H_{\lambda_\Gamma^+}^V \prec V[G]$ for every Γ -generic filter G over V , whereas $H_{\lambda_\Gamma^+}^{\mathbb{B}} \prec V^{\Gamma \restriction \mathbb{B}}$ means that whenever G is \mathcal{V} -generic for $\Gamma \restriction \mathbb{B}$, in $V[G]$ there is H , V -generic for \mathbb{B} , such that

$$H_{\lambda_\Gamma^+}^{V[H]} \prec V[G].$$

²¹While parsing through these items, the reader should keep in mind the case Γ being SSP or the class of (semi)proper forcings with $\lambda_\Gamma = \omega_1$.

²²As we will see below, it will be rather delicate to define $\dot{\Gamma}$ and to infer that $\mathbb{B} * \dot{\Gamma}$ and $\Gamma \restriction \mathbb{B}$ define equivalent class forcings in V ; this despite the intuition that closure of Γ under two-step iterations should grant it.

Fact 2.29. *Assume Γ is pretame, satisfies the Factor Lemma, the Self-similarity condition, and the Universality condition. Then whenever K is \mathcal{V} -generic for Γ and $C \in K$, in $V[K]$ there is a V -generic filter for C .*

Proof. This is easily granted if $C = \Gamma_\alpha \in K$ ($K \cap V_\alpha$ is V -generic for Γ_α by **self-similarity**). Now assume $B \geq_\Gamma \Gamma_\alpha \upharpoonright C$ for some $\Gamma_\alpha \upharpoonright C \in K$. Let $i : B \rightarrow \Gamma_\alpha \upharpoonright C$ witness $B \geq_\Gamma \Gamma_\alpha \upharpoonright C$ (i exists by **universality**). Then $i^{-1}[K] \in V[K]$ is V -generic for B . \square

The following observation is a first indication that these properties are closely related to generic absoluteness results and to forcing axioms.

Fact 2.30. *Assume Γ is pretame, satisfies the Factor Lemma, the Self-similarity condition, and the Universality condition. Let $B \in D_\Gamma$, and let H be V -generic for B . Then*

$$H_{\lambda_\Gamma^+}^{V[H]} \prec_{\Sigma_1} V[H]^C$$

for all $C \in \Gamma^{V[H]}$.

Proof. Notice that if $C \leq_\Gamma B$, K is \mathcal{V} -generic for Γ , with $H \in V[K]$ V -generic for C , and $G \in V[H]$ is V -generic for B , we obtain the configuration:

$$\begin{array}{ccc} H_{\lambda_\Gamma}^{V[G]} & \xrightarrow{\Sigma_\omega} & V[K] \\ & \searrow & \nearrow \\ & V[H] & \end{array}$$

This gives that any Σ_1 -property with parameters in $H_{\lambda_\Gamma}^{V[G]}$ true in $V[H]$ remains true in $V[K]$ and thus reflects to $H_{\lambda_\Gamma}^{V[G]}$. \square

In particular, the elements of D_Γ are forcing strong versions of the corresponding bounded forcing axiom for Γ .

We now give rigorous definitions and outline how to infer the above properties of $(\Gamma, \rightarrow^\Gamma)$ for a wide family of classes of forcings which includes the classes of proper, semiproper, and SSP forcings.

3. FORCING WITH FORCINGS: DEFINITIONS

3.1. The Factor Lemma for Γ . In Subsection 3.4 we will define the notion of absolutely well-behaved class. One of the things we will prove about such classes is the following.

Lemma 3.1 (The Factor Lemma for Γ). *Let $\langle V, \mathcal{V} \rangle \models \text{MK}$, and let $\Gamma \in \mathcal{V}$ be a definable absolutely well-behaved class of forcings. Let $B \in \Gamma$ and*

$$\begin{aligned} \Gamma_B &= \left\{ \dot{C} \in \mathbf{V}^B : \left[\left[\phi_\Gamma(\dot{C}, \check{a}_\Gamma) \right] \right]_B = 1_B \right\}, \\ \rightarrow_B^\Gamma &= \left\{ \dot{k} \in \mathbf{V}^B : \left[\left[\dot{k} : \dot{C} \rightarrow \dot{D} \text{ is } \Gamma\text{-correct} \right] \right]_B = 1_B \right\}. \end{aligned}$$

Then whenever G is V -generic for B , we have that $(\Gamma_B)_G^\circ = \Gamma^{V[G]}$ and $(\rightarrow_B^\Gamma)_G^\circ = (\rightarrow^\Gamma)^{V[G]}$. Moreover, for any $C \leq_\Gamma B$ fix in V $i_C : B \rightarrow C$ witnessing this. Then the map:

$$\begin{aligned} \Theta_B : \Gamma \upharpoonright B &\rightarrow B * \Gamma_B^\circ \\ C &\mapsto B * (C / i_C[\dot{G}_B]) \end{aligned}$$

defines a dense embedding of $(\Gamma \upharpoonright B, \leq_\Gamma)$ into the class partial order $B * \Gamma_B^\circ$ with order given by $(d, \dot{D}) \leq (c, \dot{C})$ if and only if $d \leq_B c$ and

$$\left[\left[\exists \dot{k} : \dot{C} \rightarrow \dot{D} \text{ } \Gamma\text{-correct} \right] \right]_B \geq d.$$

We defer the proof to Section 4.3. We don't need the Factor Lemma for any of the proofs occurring in Sections 3.2, 3.3, 4.1, 4.2. On the other hand, this lemma is needed in the proof of Theorem 3.14, which will be an easy corollary of all the results obtained in the above sections combined with the Factor Lemma, and it is also needed in the derivation of Corollary 4.10.

Despite the apparent self-evidence of the Factor Lemma, its proof requires a careful formulation and unfolding of the relation existing between elements of $\Gamma^{V[G]}$ and $\rightarrow^{\Gamma^{V[G]}}$ and their corresponding \mathbf{B} -names $\Gamma_{\mathbf{B}}, \rightarrow_{\mathbf{B}}^{\Gamma}$. To appreciate the difficulties one may encounter when proving this lemma, observe that the following set of equalities holds true²³ for all V -generic filters G for \mathbf{B} :

$$\begin{aligned} (\Gamma_{\mathbf{B}})_{\mathbf{B}}^{\circ} &= \left\{ \dot{C}_G : \dot{C} \in \Gamma_{\mathbf{B}} \right\} = \\ &= \left\{ \dot{C}_G : \left[\left[\dot{C} \in (\Gamma_{\mathbf{B}})^{\circ} \right]_{\mathbf{B}} = 1_{\mathbf{B}} \right] \right\} = \\ &= \left\{ \dot{C}_G : \left[\left[\dot{C} \in (\Gamma_{\mathbf{B}})^{\circ} \right]_{\mathbf{B}} \in G \right] \right\} = \\ &= \left\{ \mathbf{C}/i_{\dot{C}[G]} : \mathbf{C} = \text{RO}(\mathbf{B} * \dot{C}) \text{ and } \left[\left[\phi_{\Gamma}(\dot{C}, \check{a}_{\Gamma}) \right]_{\mathbf{B}} = 1_{\mathbf{B}} \right] \right\} = \\ &= \left\{ \mathbf{C}/i_{[G]} : \mathbf{C} \in \Gamma \upharpoonright \mathbf{B}, i : \mathbf{B} \rightarrow \mathbf{C} \text{ is } \Gamma\text{-correct} \right\}. \end{aligned}$$

Nonetheless, the equality between the terms in the second and third lines is a bit delicate to prove, and requires a careful reformulation of Cohen's forcing theorem.

It will be even more delicate to prove the corresponding set of equalities for the canonical \mathbf{B} -name $(\rightarrow_{\mathbf{B}}^{\Gamma})^{\circ}$ for $\rightarrow^{\Gamma^{V[G]}}$, and to infer the other desired properties of the \mathbf{B} -names $(\Gamma_{\mathbf{B}})^{\circ}, (\rightarrow_{\mathbf{B}}^{\Gamma})^{\circ}$.

3.2. Γ -iterability. There are two further key properties of Γ we need to outline in order to infer the nice properties for Γ needed to establish the consistency of $\text{BCFA}(\Gamma)$. Formulated in categorical terms, we need to have that Γ is closed under set sized products and that many Γ -valued diagrams have a colimit. Formulated in the forcing terminology, we need the closure of Γ under lottery sums, and an iteration theorem for Γ .

Let's first focus on the definition of the iterability property for forcings in Γ :

Definition 3.2. Assume Γ is a definable class of pre-orders closed under two-step iterations, and λ_{Γ} is a regular uncountable cardinal.²⁴

- Γ has the *Baumgartner property* if whenever δ is inaccessible and

$$\mathcal{F} = \{i_{\alpha\beta} : \mathbf{B}_{\alpha} \rightarrow \mathbf{B}_{\beta} : \alpha \leq \beta < \delta\} \subseteq \rightarrow^{\Gamma} \cap V_{\delta}$$

is an iteration system with $\mathbf{B}_{\alpha} = \varinjlim \mathcal{F} \upharpoonright \alpha \in \Gamma$ for stationarily many $\alpha < \delta$, then $\varinjlim \mathcal{F} \in \Gamma$.

- $\overline{\Gamma}$ is *iterable* if Γ has the Baumgartner property and Player *II* has a winning strategy $\Sigma(\Gamma)$ in the game $\mathcal{G}(\Gamma)$ of length Ord between players *I* and *II* defined as follows:
 - players *I* and *II* alternate playing Γ -correct *injective* homomorphism $i_{\alpha, \alpha+1} : \mathbf{B}_{\alpha} \rightarrow \mathbf{B}_{\alpha+1}$;
 - player *I* plays at odd stages, player *II* at even stages (0 and all limit ordinals are even);
 - at stage 0, *II* plays the identity on the trivial cba 2.

²³Given a \mathbf{B} -name $\dot{C} \in V^{\mathbf{B}}$ for a forcing notion, $i_{\dot{C}} : \mathbf{B} \rightarrow \text{RO}(\mathbf{B} * \dot{C}), b \mapsto \langle b, 1_{\dot{C}} \rangle$ denotes the canonical embedding of \mathbf{B} in the boolean completion of the two-step iteration $\mathbf{B} * \dot{C}$.

²⁴The reader may keep in mind $\lambda_{\Gamma} = \omega_1$ and Γ being the class of semiproper forcings while parsing through the definition.

- at limit stages η , II must play²⁵ a $\mathbb{B}_\eta \in \Gamma$ which admits for each $\alpha < \eta$ a Γ -correct $i_{\alpha\eta} : \mathbb{B}_\alpha \rightarrow \mathbb{B}_\eta$ such that $i_{\beta,\eta} \circ i_{\alpha,\beta} = i_{\beta,\eta}$ for all $\alpha \leq \beta < \eta$; moreover II must play $\varinjlim\{\mathbb{B}_\alpha : \alpha < \eta\}$ at stage η if:
 - * either $\text{cof}(\eta) = \lambda_\Gamma$,
 - * or η is inaccessible and $\{i_{\alpha\beta} : \mathbb{B}_\alpha \rightarrow \mathbb{B}_\beta : \alpha \leq \beta < \eta\} \subseteq \Gamma \cap V_\eta$;
- II wins $\mathcal{G}(\Gamma)$ if she can play at all stages.

Remark 3.3.

- If Γ has the Baumgartner property, $\delta > \lambda_\Gamma$ is inaccessible, and

$$\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \rightarrow \mathbb{B}_\beta : \alpha \leq \beta < \delta\} \subseteq \rightarrow^\Gamma \cap V_\delta$$

is a play of $\mathcal{G}(\Gamma)$, $\varinjlim \mathcal{F} \in \Gamma$ is $<\delta$ -CC, since:

- All \mathbb{B}_α are $<\delta$ -CC having size less than δ , and for all $\alpha < \delta$ of cofinality λ_Γ , \mathbb{B}_α is the direct limit of $\mathcal{F} \upharpoonright \alpha$, and therefore Baumgartner's theorem 6.10 applies.
- $\varinjlim \mathcal{F} \in \Gamma$ by the Baumgartner property.

In particular, if Γ is closed under two-step iterations but is not iterable, and Σ is a strategy for player II , a play

$$\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \rightarrow \mathbb{B}_\beta : \alpha \leq \beta < \delta\}$$

of the game $\mathcal{G}(\Gamma)$ which II cannot win using Σ is such that either

- $\text{cof}(\delta) < \delta$, or
- $\text{cf}(\delta) = \lambda_\Gamma$ and $\varinjlim(\mathcal{F}) \notin \Gamma$, or
- $\text{cf}(\delta) \neq \lambda_\Gamma$ and there is no lower bound in Γ to the cbas played in the game before stage δ , or
- some \mathbb{B}_α has size bigger than δ .

Moreover II can always play at successor stages of $\mathcal{G}(\Gamma)$.

- The class Γ of proper forcings is iterable: II plays the identity at all non-limit stages, the full limit at limit stages of countable cofinality, and the direct limit at limit stages of uncountable cofinality.
- With slightly more refined strategies one can prove that also the class of semiproper forcings is iterable, and that so is the class of stationary set preserving forcings assuming the existence of class many supercompact cardinals. We will address this issue with more care in Section 5.

3.3. Universality of (Γ, \leq_Γ) and Γ -rigidity. What conditions grant that (Γ, \leq_Γ) absorbs as a complete suborder any set sized $P \in \Gamma$?

The optimal case is that there is a complete embedding

$$i_B : B \rightarrow \Gamma \upharpoonright B$$

for a dense set of $B \in \Gamma$.

If this is the case, take $Q \in \Gamma$, find $B \leq_\Gamma Q$ in the above dense set and $i : Q \rightarrow B$ in \rightarrow^Γ witnessing this.

Then $i_B \circ i : Q \rightarrow \Gamma \upharpoonright B$ will witness that $\Gamma \upharpoonright B$ absorbs Q as well.

Now, given $B \in \Gamma$, we have a natural candidate for a complete embedding $i_B : B \rightarrow \Gamma \upharpoonright B$:

$$\begin{aligned} i_B : B &\rightarrow \Gamma \upharpoonright B \\ b &\mapsto B \upharpoonright b. \end{aligned}$$

²⁵ $\varinjlim\{\mathbb{B}_\alpha : \alpha < \eta\}$ is the direct limit of the iteration system given by the maps $i_{\gamma\beta} : \mathbb{B}_\gamma \rightarrow \mathbb{B}_\beta$ which are built along the play of $\mathcal{G}(\Gamma)$ (see appendix 6 for the definition of direct and inverse limit of an iteration system).

- i_B is order preserving:

If $b_1 \leq b_0$, the map

$$\begin{aligned} i_{b_1} : B \upharpoonright b_0 &\rightarrow B \upharpoonright b_1 \\ c &\mapsto c \wedge b_1 \end{aligned}$$

is Γ -correct and witnesses that $B \upharpoonright b_0 \geq_\Gamma B \upharpoonright b_1$.

- i_B preserves sups:

If $\{a_i : i \in I\} \subset B^+$ is a maximal antichain in B , the product algebra

$$\prod_{i \in I} (B \upharpoonright a_i)$$

(the lottery sum of $\{B \upharpoonright a_i : i \in I\}$) is isomorphic to B , the top element of $\Gamma \upharpoonright B$ in $(\Gamma \upharpoonright B, \leq_\Gamma)$.

- **PROBLEM:** Does this map preserve incompatibility? In general NO!

Assume B is homogeneous (for example B is the boolean completion of Cohen's forcing $2^{<\omega}$). Assume s, t are incompatible conditions in B ; by homogeneity, $B \upharpoonright s$ is isomorphic to $B \upharpoonright t$; therefore the incompatible $s, t \in B$ are mapped to the compatible conditions $B \upharpoonright s, B \upharpoonright t$ in (Γ, \leq_Γ) .

To overcome this problem we need to find densely many highly inhomogeneous $B \in \Gamma$.

Suppose for the moment that the map

$$\begin{aligned} i_B : B &\rightarrow \Gamma \upharpoonright B \\ b &\mapsto B \upharpoonright b. \end{aligned}$$

defines a complete embedding (i.e. preserves the incompatibility relation), and pick a \mathcal{V} -generic H for Γ such that $B \in H$. Then $G = i_B^{-1}[H]$ is V -generic for B .

Suppose now that

$$\{B : i_B \text{ defines a complete embedding}\}$$

is dense in (Γ, \leq_Γ) . Assume H is \mathcal{V} -generic for Γ . Pick $Q \in H$. By density there is $B \in H$ refining Q and such that i_B defines a complete embedding of B into $\Gamma \upharpoonright B$. Let $i : Q \rightarrow B$ witness $B \leq_\Gamma Q$. Then $K = (i_B \circ i)^{-1}[H]$ is V -generic for Q .

In particular we get the following:

Fact 3.4. *Assume*

$$E_\Gamma = \{B \in \Gamma : i_B : b \mapsto B \upharpoonright b \text{ defines a complete embedding of } B \text{ into } \Gamma \upharpoonright B\}$$

is dense in (Γ, \leq_Γ) .

Then any \mathcal{V} -generic filter H for Γ adds a V -generic filter for any $Q \in H$.

How do we get to the density of E_Γ ? The key step is to reformulate properly the condition that i_B is a complete embedding.

Definition 3.5. Let Γ be a definable class of cbas closed under two-step iterations, and let $B \in \Gamma$. $B \in \Gamma$ is Γ -rigid if for $i_0, i_1 : B \rightarrow Q$ in \rightarrow^Γ we have that $i_0 = i_1$.

Remark 3.6. Γ -rigid cbas B are absorbed by $(\Gamma \upharpoonright B, \leq_\Gamma)$ using the map $i_B : b \mapsto B \upharpoonright b$. See the Lemma below.

Lemma 3.7. *The following are equivalent characterizations of Γ -rigidity for an algebra $B \in \Gamma$:*

- (1) *for all $b_0, b_1 \in B$ such that $b_0 \wedge_B b_1 = 0_B$, $B \upharpoonright b_0$ is incompatible with $B \upharpoonright b_1$ in (Γ, \leq_Γ) .*
- (2) *For every $C \leq_\Gamma B$ and every V -generic filter H for C , there is just one Γ -correct V -generic filter $G \in V[H]$ for B .*
- (3) *For all $C \leq_\Gamma B$ in Γ there is only one Γ -correct homomorphism $i : B \rightarrow C$.*

Proof. We prove these equivalences by contraposition as follows:

2 implies 1: Assume 1 fails as witnessed by $i_j : \mathbb{B} \upharpoonright b_j \rightarrow \mathbb{Q}$ for $j = 0, 1$ with b_0 incompatible with b_1 in \mathbb{B} . Pick H , a V -generic filter for \mathbb{Q} . Then $G_j = i_j^{-1}[H] \in V[H]$ (for $j = 0, 1$) are distinct and Γ -correct V -generic filters for \mathbb{B} in $V[H]$, since $b_j \in G_j \setminus G_{1-j}$.

1 implies 3: Assume 3 fails for \mathbb{B} as witnessed by $i_0 \neq i_1 : \mathbb{B} \rightarrow \mathbb{C}$. Let b be such that $i_0(b) \neq i_1(b)$. W.l.o.g. we can suppose that $r = i_0(b) \wedge i_1(\neg b) > 0_{\mathbb{C}}$. Then $j_0 : \mathbb{B} \upharpoonright b \rightarrow \mathbb{C} \upharpoonright r$ and $j_1 : \mathbb{B} \upharpoonright \neg b \rightarrow \mathbb{C} \upharpoonright r$ given by $j_k(a) = i_k(a) \wedge r$ for $k = 0, 1$ and a in the appropriate domain witness that $\mathbb{B} \upharpoonright \neg b$ and $\mathbb{B} \upharpoonright b$ are compatible in (Γ, \leq_{Γ}) , i.e. that 1 fails.

3 implies 2: Assume 2 fails for \mathbb{B} as witnessed by some $\mathbb{C} \leq_{\Gamma} \mathbb{B}$, a V -generic filter H for \mathbb{C} , and Γ -correct V -generic filters $G_1 \neq G_2 \in V[H]$ for \mathbb{B} . Let $\dot{G}_1, \dot{G}_2 \in V^{\mathbb{C}}$ be such that $(\dot{G}_1)_H = G_1 \neq (\dot{G}_2)_H = G_2$ are Γ -correct V -generic filters for \mathbb{B} in $V[H]$ for both $j = 1, 2$. Find $q \in G$ forcing that $b \in \dot{G}_1 \setminus \dot{G}_2$ for some fixed $b \in \mathbb{B}$. Then for some $r \in H$ refining q , we have that both homomorphisms $i_j = i_{\dot{G}_j, r} : \mathbb{B} \rightarrow \mathbb{C} \upharpoonright r$ defined by $a \mapsto \left[\check{a} \in \dot{G}_j \right]_{\mathbb{C}} \wedge r$ are Γ -correct. However $i_1(b) = r = i_2(\neg b)$, and hence $i_1 \neq i_2$ witness that 3 fails for \mathbb{B} and $\mathbb{C} \upharpoonright r$. □

Fact 3.8. *The class of Γ -rigid cbas is closed under set-sized products (i.e. lottery sums), and the restriction operation $\mathbb{B} \mapsto \mathbb{B} \upharpoonright b$ for $b \in \mathbb{B}^+$.*

Proof. We leave to the reader to check that for all $b \in \mathbb{B}^+$, $\mathbb{B} \upharpoonright b$ is Γ -rigid if so is \mathbb{B} .

Assume now that $\{\mathbb{B}_i : i \in I\}$ is a family of Γ -rigid cbas.

Towards a contradiction, assume $k_j : (\prod_{i \in I} \mathbb{B}_i) \rightarrow \mathbb{C}$ are distinct and Γ -correct for $j = 0, 1$. If $k_0 \upharpoonright \mathbb{B}_i = k_1 \upharpoonright \mathbb{B}_i$ for all $i \in I$, we get that $k_0 = k_1$, which is a contradiction. Hence for some $i \in I$, $k_0 \upharpoonright \mathbb{B}_i \neq k_1 \upharpoonright \mathbb{B}_i$.

Then for some $a \in \mathbb{B}_i$, $k_0(a) \neq k_1(a)$, therefore $k_0(a) \Delta k_1(a) > 0_{\mathbb{C}}$. So either $k_0(a) \wedge k_1(\neg a) = c > 0_{\mathbb{C}}$ or $k_1(a) \wedge k_0(\neg a) = c > 0_{\mathbb{C}}$. In either cases we get that \mathbb{B}_i is not Γ -rigid as witnessed by the distinct Γ -correct maps (for $i = 0, 1$) $k_i^* : b \mapsto k_i(b) \wedge c$ with domain \mathbb{B}_i and range $\mathbb{C} \upharpoonright c$. □

Lemma 3.9. *Let Γ be closed under set-sized products (i.e. set-sized lottery sums) and such that the Γ -rigid forcings are dense in (Γ, \leq_{Γ}) . Assume D is a dense open class of (Γ, \leq_{Γ}) .*

Then for all $\mathbb{B} \in \Gamma$, there is a Γ -rigid $\mathbb{C} \leq_{\Gamma}^ \mathbb{B}$ and a maximal antichain A of \mathbb{C} such that $k_{\mathbb{C}}[A] \subseteq D$.*

Proof. Given $\mathbb{B} \in \Gamma$, find a Γ -rigid $\mathbb{C}_0 \leq_{\Gamma} \mathbb{B}$ with $\mathbb{C}_0 \in D$, with $k_0 : \mathbb{B} \rightarrow \mathbb{C}_0$ a witness that $\mathbb{C}_0 \leq_{\Gamma} \mathbb{B}$ and

$$b_0 = \text{coker}(k_0).$$

Then

$$\begin{aligned} k_0 \upharpoonright b_0 : \mathbb{B} \upharpoonright b_0 &\rightarrow \mathbb{C}_0 \\ b &\mapsto k_0(b) \end{aligned}$$

is injective and Γ -correct and witnesses $\mathbb{B} \upharpoonright b_0 \geq_{\Gamma}^* \mathbb{C}_0$.

Now find $\mathbb{C}_1 \leq_{\Gamma} \mathbb{B} \upharpoonright \neg b_0$ Γ -rigid and in D with $k_1 : \mathbb{B} \upharpoonright \neg b_0 \rightarrow \mathbb{C}_1$ a witness that $\mathbb{C}_0 \leq_{\Gamma} \mathbb{B}$ and

$$b_1 = \text{coker}(k_1) \leq \neg_{\mathbb{B}} b_0.$$

Then $k_1 \upharpoonright b_1 : \mathbb{B} \upharpoonright b_1 \rightarrow \mathbb{C}_1$ is injective.

Continuing this way we construct by induction a maximal antichain $E = \{b_\alpha : \alpha < \gamma\}$ of \mathbf{B} such that for all $\alpha < \gamma$ there is $b_\alpha \leq \neg \bigvee_{\beta < \alpha} b_\beta$, $C_\alpha \in D$, and $k_\alpha : \mathbf{B} \upharpoonright b_\alpha \rightarrow C_\alpha$ witnessing $C_\alpha \leq_\Gamma^* \mathbf{B} \upharpoonright b_\alpha$.

Let

$$k : \mathbf{B} \rightarrow \prod_{\alpha < \gamma} C_\alpha$$

$$b \mapsto \langle k_\alpha(b \wedge b_\alpha) : \alpha < \gamma \rangle.$$

Then $\mathbf{C} \leq_\Gamma^* \mathbf{B}$ as witnessed by k , and $A = \{c_\alpha = k(b_\alpha) : \alpha < \gamma\} = k[E]$ is a maximal antichain of \mathbf{C} such that for all $\alpha < \gamma$:

- $k(b_\alpha) = \langle 0_{C_0}, \dots, 0_{C_\eta}, \dots, 1_{C_\alpha}, 0_{C_{\alpha+1}}, \dots, 0_{C_\xi}, \dots \rangle \in A$;
- $C_\alpha \cong \mathbf{C} \upharpoonright k(b_\alpha) = k_{\mathbf{C}}(c_\alpha) \in D$.

Finally notice that \mathbf{C} is Γ -rigid, being the lottery sum of Γ -rigid forcings. \square

It can also be shown that Γ -rigidity is preserved by passing to generic quotients; i.e.

Assume $\mathbf{C} \in V$ is Γ -rigid, G is V -generic for \mathbf{B} , and $i : \mathbf{B} \rightarrow \mathbf{C}$ is a Γ -correct homomorphism. Then $\mathbf{C}/i[G]$ is $\Gamma^{V[G]}$ -rigid in $V[G]$.

But this fact (which we don't need) has a very convoluted proof (it essentially amounts to a different proof of the Factor Lemma), so we omit it.

3.4. Well-behaved classes Γ . We can now give the key definitions and state the main theorem which will be repeatedly used in Section 5.

Definition 3.10. Let (V, \mathcal{V}) be the standard model of MK. Given a definable class of cbas $\Gamma \in \mathcal{V}$ closed under two-step iterations:

- (Γ, \leq_Γ) is *strategically $< \text{Ord}$ -closed* if λ_Γ is a regular uncountable cardinal, and Γ is iterable with an iteration strategy $\Sigma(\Gamma)$ definable in the ZFC-model (V, \in) .
- Γ is *closed under lottery sums* if any set-sized product of cbas in Γ is in Γ .
- Γ is *closed under isomorphisms* if $\mathbf{C} \in \Gamma$ whenever $\mathbf{B} \in \Gamma$ and $\mathbf{C} \cong \mathbf{B}$.
- Γ is *closed under restrictions and complete subalgebras* if for every $\mathbf{B} \in \Gamma$:
 - for every $b \in \mathbf{B}$, the map

$$k_b : \mathbf{B} \rightarrow \mathbf{B} \upharpoonright b$$

$$c \mapsto c \wedge b$$

- is Γ -correct, and
- any complete subalgebra of \mathbf{B} is in Γ .

The following is the key definition of the paper.

Definition 3.11. Assume (V, \mathcal{V}) is a model of MK+ *Ord is Mahlo*.

A definable class Γ is *well-behaved in V* if:

- (1) λ_Γ is a regular uncountable cardinal.
- (2) Γ is closed under isomorphisms, two-step iterations, lottery sums, restrictions, and complete subalgebras.
- (3) For all $\mathbf{B} \in \Gamma$ and G V -generic for \mathbf{B} , $V[G]$ models that $\Gamma^{V[G]}$ is strategically $< \text{Ord}$ -closed.
- (4) Γ contains as elements all $< \lambda_\Gamma$ -closed forcings.
- (5) For all inaccessible cardinals $\delta > |a_\Gamma|$ and for all $\mathbf{B} \in V_\delta$,

$$V_\delta \models \phi_\Gamma(\mathbf{B}, a_\Gamma) \quad \text{if and only if} \quad V \models \phi_\Gamma(\mathbf{B}, a_\Gamma).$$

- (6) The Γ -rigid cbas are dense in (Γ, \leq_Γ) .

Γ is *absolutely well-behaved* if $\Gamma^{V[G]}$ is well-behaved in $V[G]$ for any V -generic filter G for some $\mathbf{B} \in \Gamma$.

Remark 3.12. In all classes Γ we will consider, clause 5 in Definition 3.11 will be proved by showing that Γ can be defined both by a Σ_2 formula and by a Π_2 formula, possibly with parameters.

Remark 3.13.

- We will show that SSP, proper, semiproper and many other well-known classes of forcings contained in SSP are absolutely well-behaved (in some cases assuming large cardinals in V).

The key point is that being well-behaved for all these Γ is provable in ZFC (+ large cardinals). The only property of well-behavedness not covered elsewhere in the literature for these classes of forcings is the density of Γ -rigid forcings.

- The class Γ given by CCC forcings is not well-behaved; for example it is not closed under lottery sums (easy), and it does not have Γ -rigid elements (less straightforward).

The following is one of the main results of this paper.

Theorem 3.14. *Assume $\langle V, \mathcal{V} \rangle$ satisfies*

- MK,
- λ_Γ is a regular uncountable cardinal, and
- Γ is absolutely well-behaved.

Then

V^Γ models that Ord is the successor of λ_Γ .

Moreover, for all inaccessible δ such that $V_\delta \prec V$:

- (1) $\Gamma_\delta \in \Gamma$ is Γ -rigid, and is such that for all $B \in \Gamma_\delta$ there is $C \in \Gamma_\delta$ such that $\Gamma_\delta \upharpoonright C \leq_\Gamma B$.

Hence the class of Γ -rigid forcings is pre-dense as witnessed by the forcings Γ_δ as δ ranges on the inaccessible cardinals with $V_\delta \prec V$.

- (2) Γ_δ preserves the regularity of δ and forces it to become the successor of λ_Γ .
- (3) If $\lambda_\Gamma = \omega_1$, Γ_δ forces BFA(Γ), and if δ is supercompact, it forces also FA $_{\omega_1}$ (Γ).
- (4) For all G V -generic for Γ with $\Gamma_\delta \in V[G]$, letting $G_\delta = G \cap V_\delta$ we have that $H_{(\lambda_\Gamma)^+}^{V[G_\delta]} = V_\delta[G_\delta] \prec V[G] = H_{(\lambda_\Gamma)^+}^{V[G]}$.

Corollary 3.15. *Assume $\langle V, \mathcal{V} \rangle$ satisfies*

- MK,
- Ord is Mahlo,
- λ_Γ is a regular uncountable cardinal, and
- Γ is absolutely well-behaved.

Then

$$D_\Gamma = \left\{ B \in \Gamma : H_{\lambda_\Gamma^+}^B \prec H_{\lambda_\Gamma^+}^{\Gamma \upharpoonright B} \right\}$$

is dense as witnessed by $\Gamma_\delta \upharpoonright C$ as δ ranges among the inaccessible cardinals γ with $(V_\gamma, \in) \prec (V, \in)$ and C among the elements of Γ_δ .

In particular, for every $B \in \Gamma$ there is some $C \leq_\Gamma B$ such that C forces BCFA(Γ).

The next section is devoted to the proof of Theorem 3.14. We will pay special attention to giving detailed proofs of 3.14(1) and 3.14(2). The reader familiar with these proofs will be able to fill in the details needed to prove the remaining assertions of Theorem 3.14.

The key result is 3.14(1).

Notation 3.16. Given a category forcing (Γ, \leq_Γ) with Γ a definable class of complete boolean algebras and \leq_Γ the order induced on Γ by the Γ -correct homomorphisms between elements of Γ , we denote the incompatibility relation with respect to \leq_Γ by \perp_Γ , and the subclass of Γ given by its Γ -rigid elements by Rig^Γ .

4. FORCING WITH FORCINGS: PROOFS

 4.1. Why $\Gamma_\delta \in \Gamma$?

Notation 4.1. Given a Γ -rigid forcing \mathbb{B} :

- $k_{\mathbb{B}} : b \mapsto \mathbb{B} \upharpoonright b$ is the canonical embedding of \mathbb{B} into $\Gamma \upharpoonright \mathbb{B}$.
- if $\mathbb{C} \leq_{\Gamma} \mathbb{B}$, $k_{\mathbb{B}\mathbb{C}}$ denotes the unique Γ -correct homomorphism of \mathbb{B} into \mathbb{C} .

Theorem 4.2. *Assume Γ is absolutely well-behaved, and δ is inaccessible and such that $V_\delta \prec V$. Then $\Gamma_\delta \in \Gamma$ preserves the regularity of δ making it the successor of λ_Γ .*

Notice that Theorem 4.2 proves Theorem 3.14(1) (except for the assertion that for all $\mathbb{B} \in \Gamma_\delta$ there is $\mathbb{C} \in \Gamma_\delta$ such that $\Gamma_\delta \upharpoonright \mathbb{C} \leq_{\Gamma} \mathbb{B}$) and Theorem 3.14(2).

Proof. Let P_Γ^δ be the set of $\mathcal{F} = \{i_{\alpha,\beta} : \mathbb{B}_\alpha \rightarrow \mathbb{B}_\beta : \alpha \leq \beta < \eta\}$ subsets of $\rightarrow^\Gamma \cap V_\delta$ such that:

- $\mathcal{F} \in V_\delta$ is a partial play of $\mathcal{G}_\delta(\Gamma)$ according to $\Sigma_\delta(\Gamma)$;
- all the moves of I in \mathcal{F} are Γ -rigid forcings;
- $\mathcal{G} \leq \mathcal{F}$ if \mathcal{G} is an end-extension of \mathcal{F} .

Since Γ is iterable, it is immediate to check that P_Γ^δ is $<\delta$ -closed, hence in Γ , since $\delta \geq \lambda_\Gamma$ and Γ is well-behaved (by clause 3.11(4)).

Moreover, let $\mathcal{G} = \{\mathbb{B}_\alpha : \alpha < \delta\}$ be (the union of) a V -generic G filter for P_Γ^δ . Then \mathcal{G} is an iteration system in $V[G]$, and it is clear that $V[G] = V[\mathcal{G}]$.

Since P_Γ^δ is $<\delta$ -closed, $V_\delta^{V[G]} = V_\delta \prec V$. In particular δ is inaccessible in $V[G]$, hence

$$\Gamma_\delta^{V[G]} = \Gamma^{V[G]} \cap V_\delta^{V[G]} = \Gamma \cap V_\delta$$

where the equalities hold because of clause 3.11(5) in the definition of well-behaved class applied in V and in $V[G]$.

Now $\Gamma^{V[G]}$ is iterable in $V[\mathcal{G}]$ (by clause 3.11(3)), hence it has the Baumgartner property in $V[G]$. Observe that for all $\xi < \delta$ of cofinality λ_Γ , \mathbb{B}_ξ is the direct limit of $\mathcal{G} \upharpoonright \xi$, and the set of such ξ is stationary in δ in $V[G]$ since P_Γ^δ is $<\delta$ -closed.

By the Baumgartner property of $\Gamma^{V[G]}$ and Theorem 6.10 applied in $V[G]$, letting $\mathbb{B}_\mathcal{G} \in V[\mathcal{G}]$ be the direct limit of the iteration system \mathcal{G} ,

$$V[\mathcal{G}] \models \mathbb{B}_\mathcal{G} \text{ is in } \Gamma^{V[G]} \text{ and is } <\delta\text{-CC.}$$

Therefore $V \models P_\Gamma^\delta * \mathbb{B}_\mathcal{G}$ is in Γ (by clause 3.11(2)) and preserves the regularity of δ , being a two-step iteration of a $<\delta$ -closed forcing with a $<\delta$ -CC forcing.

Claim 2. *Let H be $V[\mathcal{G}]$ -generic for $\mathbb{B}_\mathcal{G}$. Then*

$$\{\mathbb{B}_\alpha \upharpoonright f(\alpha) : f \in H, \alpha < \delta\}$$

is V -generic for Γ_δ .

By the Claim we get that $\Gamma_\delta \sqsubseteq P_\Gamma^\delta * \dot{\mathbb{B}}$, hence Γ_δ is in Γ by clause 3.11(2).

Therefore it suffices to prove the Claim to conclude that Γ_δ is in Γ and preserves the regularity of δ .

Proof. We leave to the reader to check that

$$\{\mathbb{B}_\alpha \upharpoonright f(\alpha) : f \in H, \alpha < \delta\}$$

is a filter on Γ_δ . We need to prove that this filter is V -generic.

The key to the proof is the following:

Subclaim 1. For any D dense open subset of Γ_δ , the set of partial plays $\mathcal{F} = \{\mathbf{B}_\eta : \eta < \alpha\} \in P_\Gamma^\delta$ such that there is $\xi < \alpha$ and A maximal antichain of \mathbf{B}_ξ with

$$\{\mathbf{B}_\xi \upharpoonright a : a \in A\} = k_{\mathbf{B}_\xi}[A] \subseteq D$$

is dense open in P_Γ^δ .

Assume the subclaim holds. Then for D dense open subset of Γ_δ , there is $\mathcal{F} = \{\mathbf{B}_\eta : \eta < \alpha\} \in \mathcal{G}$, $\xi < \alpha$ and A maximal antichain of \mathbf{B}_ξ such that

$$\{\mathbf{B}_\xi \upharpoonright a : a \in A\} \subseteq D.$$

Now

$$\{f(\xi) : f \in H\}$$

is $V[G]$ -generic for \mathbf{B}_ξ , and A is still a maximal antichain of \mathbf{B}_ξ in $V[G]$. Therefore for some $a \in A$ and $f \in H$, $f(\xi) \leq a$. Hence $\mathbf{B}_\xi \upharpoonright a \in D \cap H$, proving the claim.

We prove the subclaim:

Proof. Assume $\{\mathbf{B}_\xi : \xi < \alpha\} \in P_\Gamma^\delta$ and D is dense open in Γ_δ .

Then $\{\mathbf{B}_\xi : \xi < \alpha\}$ is a play according to $\Sigma(\Gamma)$. Notice that there is some freedom to decide what \mathbf{B}_ξ is only for odd ξ and for \mathbf{B}_0 , because the even stages are decided by the winning strategy $\Sigma(\Gamma)$ for player *II*. W.l.o.g. (by prolonging $\{\mathbf{B}_\xi : \xi < \alpha\}$ if necessary) we may assume that α is odd so that it is *I*'s turn to play. This gives that $\alpha = \beta + 1$. Then (by Lemma 3.9) there is $\mathbf{B}_{\beta+1} \leq_\Gamma^* \mathbf{B}_\beta$ which is Γ -rigid and such that some $A \subseteq \mathbf{B}_{\beta+1}$ is a maximal antichain with $k_{\mathbf{B}_{\beta+1}}[A] \subseteq D$. By definition of P_Γ^δ , $\{\mathbf{B}_\xi : \xi \leq \beta\} \cup \{\mathbf{B}_{\beta+1}\} \in P_\Gamma^\delta$ and

$$\{B_{\mathbf{B}_{\beta+1}} \upharpoonright a : a \in A\} = k_{\mathbf{B}_{\beta+1}}[A] \subseteq D. \quad \square$$

The Claim is proved. □

To conclude the proof of the Theorem, we are left with proving the following.

Claim 3. Γ_δ makes δ the successor of λ_Γ .

Proof. Since λ_Γ is preserved by all forcings in Γ , we get that λ_Γ is preserved by Γ_δ . We also know that δ is a regular cardinal of $V[G]$ whenever G is V -generic for Γ_δ .

We must show that δ is the successor of λ_Γ in $V[G]$.

For any ordinal $\alpha \geq \lambda_\Gamma$ and $\mathbf{B} \in \Gamma$, V models that $\mathbf{B} * \text{Coll}(\lambda_\Gamma, \alpha) \in \Gamma$ (since Γ contains all λ_Γ -closed forcings and is closed under two-step iterations); we easily get (since δ is inaccessible) that $\mathbf{B} * \text{Coll}(\lambda_\Gamma, \alpha) \in \Gamma_\delta$ for all $\lambda_\Gamma \leq \alpha < \delta$.

In particular, for any $\lambda_\Gamma \leq \alpha < \delta$, the set D_α of $\mathbf{C} \in \Gamma_\delta$ which collapse α to have size λ_Γ is dense open in Γ_δ .

Since $V_\delta \prec V$, also the set $\text{Rig}^\Gamma \cap D_\alpha$ is dense in Γ_δ for all $\alpha < \delta$. By Remark 3.6 (and using $V_\delta \prec V$) $b \mapsto \mathbf{B} \upharpoonright b$ is a complete embedding²⁶ of \mathbf{B} into $\Gamma_\delta \upharpoonright \mathbf{B}$ for any $\mathbf{B} \in \text{Rig}^\Gamma$. In particular, if G is V -generic for Γ_δ , in $V[G]$ there is a V -generic filter H for some $\mathbf{B} \in D_\alpha$ for any $\alpha < \delta$; any such generic filter H adds a surjection of λ_Γ onto α existing in $V[G]$. We are done. □

The theorem is proved. □

²⁶We do not as yet say that it is also a Γ -correct embedding; this is indeed the case but to infer it we need the Factor Lemma, whose proof is not as yet granted.

4.2. Γ -freezeability versus Γ -rigidity. It will be convenient in order to establish that a certain class is well-behaved to prove that it satisfies a clause weaker than 3.11(6): the Γ -freezing property.

Definition 4.3. Let Γ be a definable class of cbas closed under two-step iterations, and $\mathbb{B} \in \Gamma$. A Γ -correct $k : \mathbb{B} \rightarrow \mathbb{C}$ is Γ -freezing if for all $i_0, i_1 : \mathbb{C} \rightarrow \mathbb{Q}$ in \rightarrow^Γ we have that $i_0 \circ k = i_1 \circ k$ (i.e. if the map $b \mapsto \mathbb{C} \upharpoonright k(b)$ is incompatibility preserving for \leq_Γ).

We can give the following characterizations of Γ -freezeability, the proof of which is along the same lines of the proof of Lemma 3.7 and is left to the reader.

Lemma 4.4. *Let $k : \mathbb{B} \rightarrow \mathbb{Q}$ be a Γ -correct homomorphism. The following are equivalent:*

- (1) *For all $b_0, b_1 \in \mathbb{B}$ such that $b_0 \wedge_{\mathbb{B}} b_1 = 0_{\mathbb{B}}$ we have that $\mathbb{Q} \upharpoonright k(b_0)$ is incompatible with $\mathbb{Q} \upharpoonright k(b_1)$ in (Γ, \leq_Γ) .*
- (2) *For every $\mathbb{R} \leq_\Gamma \mathbb{Q}$ and every V -generic filter H for \mathbb{R} , there is just one Γ -correct V -generic filter $G \in V[H]$ for \mathbb{B} such that $G = k^{-1}[K]$ for all Γ -correct V -generic filters $K \in V[H]$ for \mathbb{Q} .*
- (3) *For all $\mathbb{R} \leq_\Gamma \mathbb{Q}$ in Γ and $i_0, i_1 : \mathbb{Q} \rightarrow \mathbb{R}$ witnessing that $\mathbb{R} \leq_\Gamma \mathbb{Q}$ we have that $i_0 \circ k = i_1 \circ k$.*

Theorem 4.5. *Assume Γ is a definable class of forcings satisfying clauses 3.11(1), 3.11(2), 3.11(3), 3.11(4), 3.11(5) of Def. 3.11. Assume further that for all $\mathbb{B} \in \Gamma$ there is $i_{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{C}$ injective and Γ -freezing \mathbb{B} . Then the class of Γ -rigid partial orders is dense in (Γ, \leq_Γ^*) .*

Proof. Fix $\mathbb{B}_0 \in \Gamma$ and $\Sigma_{\lambda_\Gamma}(\Gamma)$ be a winning strategy for player II in $\mathcal{G}_{\lambda_\Gamma}(\Gamma)$.

Define

$$\mathcal{F} = \{k_{\alpha\beta} : \mathbb{B}_\alpha \rightarrow \mathbb{B}_\beta : \alpha \leq \beta < \lambda_\Gamma\}$$

by recursion on λ_Γ as follows:

- at stage 0, II plays $k_0 : \mathbb{2} \rightarrow \mathbb{B}_0$;
- at odd stages α , I plays $k_{\alpha, \alpha+1} : \mathbb{B}_\alpha \rightarrow \mathbb{B}_{\alpha+1}$ Γ -correct and injective which freezes \mathbb{B}_α ;
- at even stages α , II plays according to Σ_{λ_Γ} .

By iterability of Γ , $\mathbb{B}_{\lambda_\Gamma}$ is the direct limit of \mathcal{F} and belongs to Γ . Clearly $k_{0\lambda_\Gamma} : \mathbb{B}_0 \rightarrow \mathbb{B}_{\lambda_\Gamma}$ witnesses $\mathbb{B}_{\lambda_\Gamma} \leq_\Gamma^* \mathbb{B}_0$. It suffices to prove the following:

Claim 4. $\mathbb{B}_{\lambda_\Gamma}$ is Γ -rigid.

Proof. Assume $\mathbb{B}_{\lambda_\Gamma} \upharpoonright f$ is compatible with $\mathbb{B}_{\lambda_\Gamma} \upharpoonright g$ in (Γ, \leq_Γ) for some threads f, g incompatible in $\mathbb{B}_{\lambda_\Gamma}$.

Let $\mathbb{R} \leq_\Gamma \mathbb{B}_{\lambda_\Gamma} \upharpoonright f, \mathbb{B}_{\lambda_\Gamma} \upharpoonright g$.

Since $\mathbb{B}_{\lambda_\Gamma}$ is a direct limit, f, g are threads with support bounded by some $\beta < \lambda_\Gamma$. Hence: $f(\alpha)$ and $g(\alpha)$ are incompatible²⁷ in \mathbb{B}_α for all $\lambda_\Gamma \geq \alpha \geq \beta$.

Now $k_{\beta\beta+2}$ freezes \mathbb{B}_β , hence $\mathbb{B}_{\beta+2} \upharpoonright k_{\beta\beta+2}(f(\beta))$ is incompatible with $\mathbb{B}_{\beta+2} \upharpoonright k_{\beta\beta+2}(g(\beta))$.

But $k_{\beta\beta+2}(f(\beta)) = f(\beta+2)$ and $k_{\beta\beta+2}(g(\beta)) = g(\beta+2)$, since both threads f, g have support at most β . Hence

$$\mathbb{B}_{\beta+2} \upharpoonright g(\beta+2) \perp_\Gamma \mathbb{B}_{\beta+2} \upharpoonright f(\beta+2).$$

We reached a contradiction:

- On the one hand, for all $\alpha < \lambda_\Gamma$:

$$\mathbb{B}_\alpha \upharpoonright f(\alpha) \geq_\Gamma \mathbb{B}_{\lambda_\Gamma} \upharpoonright f \geq_\Gamma \mathbb{R},$$

and

$$\mathbb{B}_\alpha \upharpoonright g(\alpha) \geq_\Gamma \mathbb{B}_{\lambda_\Gamma} \upharpoonright g \geq_\Gamma \mathbb{R}.$$

²⁷Here we use crucially use that $\mathbb{B}_{\lambda_\Gamma}$ is a direct limit! For inverse limits it is well possible that two incompatible threads f, g are such that $f(\alpha)$ and $g(\alpha)$ are compatible in \mathbb{B}_α for all α .

- On the other hand,

$$\mathbf{B}_{\beta+2} \upharpoonright g(\beta+2) \perp_{\Gamma} \mathbf{B}_{\beta+2} \upharpoonright f(\beta+2).$$

□

The theorem is proved. □

Notice the following:

Fact 4.6. *Assume $\Gamma \subseteq \Delta$ are definable classes of forcings. Then $\leq_{\Gamma} \subseteq \leq_{\Delta}$ and $\perp_{\Delta} \subseteq \perp_{\Gamma}$. Hence, if $i : \mathbf{B} \rightarrow \mathbf{C}$ is Γ -correct and Δ -freezes \mathbf{B} , we also have that i is Δ -correct and Γ -freezes \mathbf{B} .*

This fact will be repeatedly used to show that various classes of forcings Δ have the Δ -freezeability property providing for some $\Gamma \subseteq \Delta$ an $i : \mathbf{B} \rightarrow \mathbf{C}$ which is Γ -correct and Δ -freezes \mathbf{B} . As we will see, all our freezeability results proceed by proving the existence, given $\mathbf{B} \in \Gamma$, of a \mathbf{B} -name $\dot{\mathbf{Q}}$ for a forcing in Γ such that $\mathbf{C} = \mathbf{B} * \dot{\mathbf{Q}}$ codes the generic filter $\dot{G}_{\mathbf{B}}$ for \mathbf{B} as a subset $A_{\dot{G}_{\mathbf{B}}}$ of ω_1 in some absolute manner, in the sense that in every outer model M of $V^{\mathbf{C}}$ preserving stationary subsets of ω_1 , $A_{\dot{G}_{\mathbf{B}}}$ is the unique subset of $\mathcal{P}(\omega_1)$ satisfying some given property. It will thus follow that \mathbf{C} SSP-freezes \mathbf{B} , which will be an instance of the above since we will always have $\text{SSP} \supseteq \Gamma$ for the Γ of interest to us.

4.3. Proof of the Factor Lemma for a well-behaved Γ .

Notation 4.7. Given a well-behaved Γ , for each $\mathbf{R} \in \text{Rig}^{\Gamma}$ let

$$k_{\mathbf{R}} : \mathbf{R} \rightarrow \Gamma \upharpoonright \mathbf{R}$$

be given by $r \mapsto \mathbf{R} \upharpoonright r$. Then $k_{\mathbf{R}}$ is an order and incompatibility preserving embedding of \mathbf{R} in the class forcing $\Gamma \upharpoonright \mathbf{R}$ which maps maximal antichains to maximal antichains. Moreover, for every $\mathbf{B} \geq_{\Gamma} \mathbf{C}$ with $\mathbf{B} \in \text{Rig}^{\Gamma}$, let

$$i_{\mathbf{B}, \mathbf{C}} : \mathbf{B} \rightarrow \mathbf{C}$$

denote the unique Γ -correct homomorphism from \mathbf{B} into \mathbf{C} .

By the results of the previous sections, Rig^{Γ} is a dense subclass of Γ and is a separative partial order. Hence, in order to simplify our calculations slightly, we focus on Rig^{Γ} rather than on Γ when analyzing this class forcing.

Definition 4.8. Given $\mathbf{B}_0 \in \Gamma$, fix $k_0 : \mathbf{B}_0 \rightarrow \mathbf{B}$ Γ -freezing \mathbf{B}_0 and such that $\mathbf{B} \in \text{Rig}^{\Gamma}$. Let $i_{\mathbf{C}} = i_{\mathbf{B}, \mathbf{C}} \circ k_0$ and

$$\begin{aligned} k &= k_{\mathbf{B}} \circ k_0 : \mathbf{B}_0 \rightarrow \Gamma \upharpoonright \mathbf{B} \\ b &\mapsto \mathbf{B} \upharpoonright k_0(b) \end{aligned}$$

Given G a V -generic filter for \mathbf{B}_0 , define in $V[G]$ the class quotient forcing

$$P_{\mathbf{B}_0} = ((\text{Rig}^{\Gamma} \upharpoonright \mathbf{B})^V /_{k[G], \leq_{\Gamma} /_{k[G]})$$

as follows:

$$\mathbf{C} \in P_{\mathbf{B}_0}$$

if and only if $\mathbf{C} \in (\text{Rig}^{\Gamma} \upharpoonright \mathbf{B})^V$ and, letting J be the dual ideal of G , we have that $1_{\mathbf{C}} \notin i_{\mathbf{C}}[J]$ (or, equivalently, if and only if $\text{coker}(i_{\mathbf{C}}) \in G$).

We let

$$\mathbf{C} \leq_{\Gamma /_{k[G]}} \mathbf{R}$$

if $\mathbf{C} \leq_{\Gamma} \mathbf{R}$ holds in V .

Theorem 4.9. *Suppose Γ is absolutely well-behaved. Let $\mathbf{B}_0 \in \Gamma$, and let $k_0 : \mathbf{B}_0 \rightarrow \mathbf{B}$ be a Γ -freezing homomorphism for \mathbf{B}_0 with $\mathbf{B} \in \text{Rig}^\Gamma$. Set $k = k_{\mathbf{B}} \circ k_0$ and $i_{\mathbf{C}} = i_{\mathbf{B}, \mathbf{C}} \circ k_0$ for all $\mathbf{C} \leq_\Gamma \mathbf{B}$ in Γ .*

$$\begin{array}{ccc}
 & & k \\
 & \curvearrowright & \\
 V & & \\
 \mathbf{B}_0 & \xrightarrow{k_0} & \mathbf{B} \xrightarrow{k_{\mathbf{B}}} \Gamma \upharpoonright \mathbf{B}
 \end{array}$$

Let G be V -generic for \mathbf{B}_0 . Then:

(1) The class forcing

$$P_{\mathbf{B}_0} = ((\text{Rig}^\Gamma \upharpoonright \mathbf{B})^V /_{k[G]}, \leq_\Gamma /_{k[G]})$$

is in $V[G]$ forcing equivalent to the class forcing

$$Q_{\mathbf{B}} = (\Gamma^{V[G]} \upharpoonright (\mathbf{B}/_{k_0[G]}), \leq_{\Gamma^{V[G]}})$$

via the map

$$\begin{aligned}
 i^* : P_{\mathbf{B}_0} &\rightarrow Q_{\mathbf{B}} \\
 \mathbf{C} &\mapsto \mathbf{C}/_{i_{\mathbf{C}}[G]}.
 \end{aligned}$$

$$\begin{array}{ccccc}
 V[G] & & & & \\
 \mathbf{B}_0/G & \xrightarrow{k_0} & \mathbf{B}/_{k[G]} & \xrightarrow{k_{\mathbf{B}}} & (\Gamma \upharpoonright \mathbf{B})/_{k[G]} \\
 \downarrow \cong & & \searrow & \cong \nearrow & \\
 2 & & \Gamma^{V[G]} \upharpoonright (\mathbf{B}/_{k_0[G]}) & & \mathbf{C} \\
 & & \downarrow & & \nearrow \\
 & & \mathbf{C}/_{i_{\mathbf{C}}[G]} & &
 \end{array}$$

(2) Moreover let $\delta > |\mathbf{B}|$ be inaccessible and such that $(V_\delta, \in) \prec (V, \in)$. Then:

(a) $\Gamma_\delta^{V[G]} \upharpoonright (\mathbf{B}/_{k_0[G]})$ is forcing equivalent in $V[G]$ to $(\Gamma_\delta \upharpoonright \mathbf{B})^V /_{k[G]}$ via the same map.

(b) V models that $k_{\mathbf{B}} : \mathbf{B} \rightarrow \Gamma_\delta \upharpoonright \mathbf{B}$ is Γ -correct.

Notice that this theorem proves the missing part of Theorem 3.14(1), i.e. the assertion:

For all $\mathbf{B} \in \Gamma_\delta$ there is $\mathbf{C} \in \Gamma_\delta$ such that $\Gamma_\delta \upharpoonright \mathbf{C} \leq_\Gamma \mathbf{B}$.

In particular this theorem and Theorem 4.2 give a completely self-contained and detailed proof of Theorem 3.14(1) and Theorem 3.14(2).

Proof. Part 2a of the theorem follows immediately from its part 1 relativizing every assumption in part 1 to $V_{\delta+1}$. To prove part 2b, first observe that if $\mathbf{B} = \mathbf{B}_0$, k_0 is necessarily the identity map, G is V -generic for \mathbf{B} , and this gives that $(\mathbf{B}/_{k_0[G]})$ is the trivial complete boolean algebra $2 = \{0, 1\}$, i.e.:

$$\Gamma_\delta^{V[G]} \upharpoonright (\mathbf{B}/_{k_0[G]}) = \Gamma_\delta^{V[G]}.$$

Now

- δ is inaccessible in $V[G]$;
- $V_\delta \prec V$ grants that $V_\delta[G] \prec V[G]$, since $G \in V_\delta[G]$;

hence the set of $\Gamma^{V[G]}$ -rigid forcings is dense in $\Gamma^{V[G]} \cap V_\delta[G]$, since it is dense in $\Gamma^{V[G]}$, being $\Gamma^{V[G]}$ well-behaved in $V[G]$. By Theorem 4.2 applied in $V[G]$, $(\Gamma_\delta)^{V[G]} \in \Gamma^{V[G]}$.

By part 2a (applied in $V_{\delta+1}[G]$) for the case $\mathbf{B}_0 = \mathbf{B}$ (so that $k = k_{\mathbf{B}}$), we get that $(\Gamma_\delta \upharpoonright \mathbf{B})/_{k_{\mathbf{B}}[G]} \cong (\Gamma_\delta)^{V[G]}$ holds in $V[G]$ for all G V -generic for \mathbf{B} . This concludes the proof of 2b in case $\mathbf{B} = \mathbf{B}_0$. The desired conclusion 2b for an arbitrary $\mathbf{B}_0 \in \Gamma_\delta$ follows using the fact that the set of $\mathbf{B} \leq_\Gamma \mathbf{B}_0$ in Rig^Γ is dense in Γ_δ and applying 2b to all such \mathbf{B} .

We are left with proving part 1: Following the notation introduced in 4.8, we let $i_{\mathbf{R}}$ denote the Γ -correct homomorphism $i_{\mathbf{B},\mathbf{R}} \circ k_0$ for any $\mathbf{R} \leq_\Gamma \mathbf{B}$, and we let k denote the map $k_{\mathbf{B}} \circ k_0 : \mathbf{B}_0 \rightarrow \Gamma \upharpoonright \mathbf{B}$ given by $b \mapsto \mathbf{B} \upharpoonright k_0(b)$.

Let G be V -generic for \mathbf{B}_0 and J denote its dual prime ideal. We first observe that in $V[G]$,

$$\downarrow k[J] = \{\mathbf{R} \in \Gamma^V : \exists q \in J \mathbf{R} \leq_\Gamma^V \mathbf{B} \upharpoonright k_0(q)\}.$$

We show that in $V[G]$ the map i^* is total, order and incompatibility preserving, and with a dense target. This suffices to prove this part of the theorem.

i^* is total and with a dense target: By Theorem 6.2, any $\mathbf{Q} \in Q_{\mathbf{B}}$ is isomorphic to $\mathbf{C}/_{i_{\mathbf{C}}[G]}$ for some $\mathbf{C} \in (\Gamma \upharpoonright \mathbf{B})^V$ such that $1_{\mathbf{C}} \notin \downarrow i_{\mathbf{C}}[J]$, since \mathbf{Q} is a non-trivial complete boolean algebra in $V[G]$. Let in V $\mathbf{R} \in \text{Rig}^\Gamma$ refine \mathbf{C} in the \leq_Γ^* -order.

We claim that $\mathbf{R}/_{i_{\mathbf{R}}[G]}$ refines \mathbf{Q} in $Q_{\mathbf{B}}$.

Assume towards a contradiction that $\mathbf{R}/_{i_{\mathbf{R}}[G]} \notin Q_{\mathbf{B}}$. Then we would get that $1_{\mathbf{R}} \in i_{\mathbf{R}}[J]$. Therefore for any Γ -correct *injective* $u : \mathbf{C} \rightarrow \mathbf{R}$ witnessing that $\mathbf{R} \leq_\Gamma^* \mathbf{C}$, we would have that $i_{\mathbf{C}}[J] = u^{-1}[i_{\mathbf{R}}[J]]$. This gives that $1_{\mathbf{C}} \in i_{\mathbf{C}}[J]$, and contradicts our assumption that $1_{\mathbf{C}} \notin \downarrow i_{\mathbf{C}}[J]$.

Therefore $1_{\mathbf{R}} \notin i_{\mathbf{R}}[J]$, and

$$u/J : \mathbf{C}/_{i_{\mathbf{C}}[G]} \rightarrow \mathbf{R}/_{i_{\mathbf{R}}[G]}$$

witnesses that $i^*(\mathbf{R})$ refines \mathbf{Q} in $Q_{\mathbf{B}}$. Hence i^* has a dense image.

Moreover for any $\mathbf{R} \in P_{\mathbf{B}_0}$, $1_{\mathbf{R}} \notin i_{\mathbf{R}}[J]$, hence $\mathbf{R}/_{i_{\mathbf{R}}[G]}$ is a non-trivial complete boolean algebra in $\Gamma^{V[G]}$. Thus i^* is also well defined on all of $(\text{Rig}^\Gamma \upharpoonright \mathbf{B})^V/k[G]$.

i^* is order and compatibility preserving: Let $i_{\mathbf{Q}_0\mathbf{Q}} : \mathbf{Q}_0 \rightarrow \mathbf{Q}$ be a Γ -correct complete homomorphism in V with $\mathbf{Q}_0, \mathbf{Q} \in P_{\mathbf{B}_0}$ witnessing that $\mathbf{Q} \leq_\Gamma/k[G] \mathbf{Q}_0$. This occurs only if $1_{\mathbf{Q}} \notin i_{\mathbf{Q}}[J]$. By Lemma 6.5, $i_{\mathbf{Q}_0\mathbf{Q}}/J : \mathbf{Q}_0/i_{\mathbf{Q}_0}[J] \rightarrow \mathbf{Q}/i_{\mathbf{Q}}[J]$ is $\Gamma^{V[G]}$ -correct and witnesses that $\mathbf{Q}_0/i_{\mathbf{Q}_0}[J] \geq_\Gamma \mathbf{Q}/i_{\mathbf{Q}}[J]$ holds in $V[G]$. This shows that i^* is order preserving and maps non-trivial conditions to non-trivial conditions. In particular we can also conclude that i^* maps compatible conditions to compatible conditions.

i^* preserves the incompatibility relation: We prove this by contraposition. Assume $j_h : \mathbf{Q}_h/i_{\mathbf{Q}_h}[G] \cong \mathbf{R}_h \rightarrow \mathbf{Q}$ for $h = 0, 1$ witness that $\mathbf{Q}_0/i_{\mathbf{Q}_0}[G]$ and $\mathbf{Q}_1/i_{\mathbf{Q}_1}[G]$ are compatible in $(\Gamma)^{V[G]}$. We can assume that $\mathbf{Q} \cong \mathbf{C}/_{i_{\mathbf{C}}[G]}$.

By Proposition 6.6 applied for both $h = 0, 1$ to $\mathbf{B}, i_{\mathbf{Q}_h}, j_h$ we have that $j_h = l_h/G$ for some Γ -correct homomorphism $l_h : \mathbf{Q}_h \rightarrow \mathbf{C}_h$ in V such that:

- $l_h \circ i_{\mathbf{Q}_h} = i_{\mathbf{C}_h}$ for both $h = 0, 1$,
- $\mathbf{C}_1/i_{\mathbf{C}_1}[G] \cong \mathbf{Q} \cong \mathbf{C}_0/i_{\mathbf{C}_0}[G]$ in $V[G]$,
- $0_{\mathbf{C}_h} \notin i_{\mathbf{C}_h}[G]$ for both $h = 0, 1$.

By Proposition 6.4, we can find $s_j \notin i_{\mathbf{C}_j}[J]$ such that $\mathbf{C}_1 \upharpoonright s_1$ and $\mathbf{C}_0 \upharpoonright s_0$ are isomorphic. Without loss of generality we can suppose that $\mathbf{C}_h \upharpoonright s_h = \mathbf{C} \in \Gamma$. This gives that (modulo the refinement via s_h) $l_h \circ i_{\mathbf{Q}_h} = i_{\mathbf{C}}$ for both $h = 0, 1$, since both $l_h \circ i_{\mathbf{Q}_h}$ factor through k_0 which is Γ -freezing \mathbf{B}_0 .

In particular each l_h witnesses in V that $\mathbf{Q}_h \geq_\Gamma \mathbf{C}$ and are both such that $1_{\mathbf{C}} \notin i_{\mathbf{C}}[J]$. Find in V $\mathbf{R} \leq_\Gamma^* \mathbf{C}$ with $\mathbf{R} \in \text{Rig}^\Gamma$. Then $i_{\mathbf{R}}[J] = u \circ i_{\mathbf{C}}[J]$ for some (any) Γ -correct *injective* $u : \mathbf{C} \rightarrow \mathbf{R}$. Hence $1_{\mathbf{R}} \notin i_{\mathbf{R}}[J]$, else $1_{\mathbf{C}} \in u^{-1}[i_{\mathbf{R}}[J]] = i_{\mathbf{C}}[J]$.

This grants that \mathbf{R} is a non-trivial condition in $(\mathbf{Rig}^\Gamma \upharpoonright \mathbf{B})^V /_{k[G]}$ refining \mathbf{Q}_h for both $h = 0, 1$.

The proof of the theorem is completed. \square

We also obtain the following completeness result as a corollary of Theorems 3.14 and 4.9.

Corollary 4.10. *Assume $\langle V, \mathcal{V} \rangle$ satisfies*

- MK,
- λ_Γ is a regular uncountable cardinal, and
- Γ is absolutely well-behaved.

Suppose $\mathbf{B}, \mathbf{C} \in \Gamma$ are such that $V^\mathbf{B} \models \text{BCFA}(\Gamma)$ and $V^\mathbf{C} \models \text{BCFA}(\Gamma)$. If G and H are generic filters over V for, respectively, \mathbf{B} and \mathbf{C} , then $H_{\lambda_\Gamma^+}^{V[G]}$ and $H_{\lambda_\Gamma^+}^{V[H]}$ have the same theory.

5. ABSOLUTELY WELL-BEHAVED CLASSES

We organize this part of the paper as follows:

- We start giving the necessary definitions in 5.1.
- We state our main results in 5.2. Specifically we assert that there are uncountably many absolutely well-behaved definable classes of forcing notions with $\lambda_\Gamma = \omega_1$, whose bounded category forcing axioms yield pairwise incompatible theories for H_{ω_2} (this is incompatibility in first order logic).
- In 5.3 we give the proofs, specifically:
 - In 5.3.1 we isolate four types of freezing posets which will be used to establish the freezeability property.
 - In 5.3.2 we present the iteration lemmas that will be used to establish the iterability property (all of which were already known).
 - In 5.3.3 we give the proof that there are \aleph_1 -many definable classes Γ of forcing notions which are absolutely well-behaved with $\lambda_\Gamma = \omega_1$.
 - In 5.3.4 we prove that bounded category forcing axioms for the uncountably many absolutely well-behaved classes we produced in 5.3.3 yield pairwise incompatible theories for H_{ω_2} .

5.1. Forcing classes. We will now define the main classes of forcing notions considered in this paper. Most of these classes are completely standard, but we nevertheless include their definition here for the benefit of some readers.

Definition 5.1. A poset has the *countable chain condition* (is *c.c.c.*, for short) if and only if it has no uncountable antichains.

Given an ordinal ρ , we will call a sequence $(X_i)_{i \leq \rho}$ a *continuous chain* (or a *continuous ρ -chain*, if we want to bring in the length) if

- $(X_k)_{k \leq i} \in X_{i+1}$ whenever $i + 1 \leq \rho$ and
- $X_i = \bigcup_{k < i} X_k$ for every nonzero limit ordinal $i \leq \rho$.

An ordinal ρ is said to be *indecomposable* if $\rho = \omega^\tau$ for some ordinal τ .²⁸ Equivalently, ρ is indecomposable if $\text{ot}(\rho \setminus \eta) = \rho$ for every $\eta < \rho$. 1 is of course the first indecomposable ordinal.

Definition 5.2. Given a countable indecomposable ordinal ρ , a poset \mathcal{P} is *ρ -proper* if and only if there is a cardinal θ such that $\mathcal{P} \in H_\theta$ and there is a club $D \subseteq [H_\theta]^{\aleph_0}$ with the property that for every continuous chain $(N_i)_{i \leq \rho}$ of countable elementary submodels

²⁸Here, and elsewhere in the remainder of the paper, ω^τ denotes ordinal exponentiation.

of H_θ containing \mathcal{P} and every $p \in N_0 \cap \mathcal{P}$, there is an extension q of p such that q is (N_i, \mathcal{P}) -generic for all $i \leq \rho$, i.e., for every $i \leq \rho$ and every dense subset D of \mathcal{P} , $D \in N_i$, $q \Vdash_{\mathcal{P}} D \cap \dot{G} \cap N_i \neq \emptyset$.

Remark 5.3. \mathcal{P} is ρ -proper if and only if for every cardinal θ such that $\mathcal{P} \in H_\theta$ there is such a club $D \subseteq [H_\theta]^{\aleph_0}$ as in the above definition.

ρ -PR denotes the class of ρ -proper posets. We write $<\omega_1$ -PR to denote the class of those posets that are in ρ -PR for every indecomposable $\rho < \omega_1$. We say that \mathcal{P} is *proper* if it is 1-proper, and denote 1-PR also by PR.

The following is a simple but crucial observation:

Fact 5.4. *For any countable indecomposable ordinal ρ , ‘RO(P) is ρ -proper’ is both a Σ_2 property in parameters ρ and ω_1 and a Π_2 property in the same parameters, and the same holds for ‘RO(P) is $<\omega_1$ -proper’.*

Proof. Let θ be large enough such that $\text{RO}(P) \in H_\lambda$ for some $\lambda < \theta$. Then ‘RO(P) is ρ -proper’ holds in V if and only if it holds in any (some) transitive set $X \supseteq H_\theta$. Hence, $\text{RO}(P)$ is ρ -proper if and only if there is some regular cardinal θ and some transitive $X \supseteq H_\theta$ such that $(X, \in) \models$ ‘RO(P) is ρ -proper’.

Now:

- The formulae $(X, \in) \models \text{RO}(P)$ is ρ -proper and X is transitive are Δ_1 in the parameters X, P, ρ, ω_1 .
- The formula θ is a regular cardinal is Π_1 in parameter θ .
- The formula $X \supseteq H_\theta$ is Π_1 in parameters X, θ since it can be stated as

$$\forall w (|\text{trcl}(w)| < \theta \rightarrow w \in X),$$

where $\text{trcl}(w)$ is the Δ_1 -definable operation assigning to the set w its transitive closure.

It is now easy to check that $\text{RO}(P)$ is ρ -proper is a Σ_2 property in parameters ρ and ω_1 . We leave it to the reader to check that it is also Π_2 in the same parameters. \square

Being ρ -proper, for a forcing \mathcal{P} , is equivalent to \mathcal{P} preserving a certain combinatorial property: Given a set X , we say that $S \subseteq {}^\rho([X]^{\aleph_0})$ is ρ -stationary if for every club $D \subseteq [X]^{\aleph_0}$ there is a continuous ρ -chain σ of members of D such that $\sigma \in S$.

Recalling the standard characterization of properness, the following is not difficult to see.

Fact 5.5. *Given an indecomposable ordinal $\rho < \omega_1$, the following are equivalent for every poset \mathcal{P} .*

- (1) \mathcal{P} is ρ -proper.
- (2) For every set X , \mathcal{P} preserves ρ -stationary subsets of ${}^\rho([X]^{\aleph_0})$; i.e., if $S \subseteq {}^\rho([X]^{\aleph_0})$ is ρ -stationary, then $\Vdash_{\mathcal{P}} S$ is ρ -stationary.

Using the above fact we can prove:

Fact 5.6. *For every countable indecomposable ordinals ρ , ρ -PR and $<\omega_1$ -PR are closed under isomorphisms, two-step iterations, lottery sums, restrictions and complete subalgebras, and contain all countably closed forcings.*

Definition 5.7. A forcing notion \mathcal{P} is ρ -semiproper iff there is a cardinal θ such that $\mathcal{P} \in H_\theta$ for which there is a club $D \subseteq [H_\theta]^{\aleph_0}$ with the property that for every continuous chain $(N_i)_{i \leq \rho}$ of countable elementary submodels of H_θ containing \mathcal{P} and every $p \in N_0 \cap \mathcal{P}$, there is an extension q of p such that q is (N_i, \mathcal{P}) -semi-generic for all $i \leq \rho$. This means now that for every $i \leq \rho$ and every \mathcal{P} -name $\dot{\alpha} \in N_i$ for an ordinal in ω_1^V , $q \Vdash_{\mathcal{P}} \dot{\alpha} \in N_i$.

Remark 5.8. \mathcal{P} is ρ -semiproper if and only if for every cardinal θ such that $\mathcal{P} \in H_\theta$ there is a club $D \subseteq [H_\theta]^{\aleph_0}$ as in the above definition.

ρ -SP denotes the class of ρ -semiproper posets. Also, we write $<\omega_1$ -SP to denote the class of those posets that are in ρ -SP for every indecomposable ordinal $\rho < \omega_1$. We say that \mathcal{P} is *semiproper* if it is 1-semiproper, and denote 1-SP also by SP.

As before we have:

Fact 5.9. ‘RO(P) is ρ -semiproper’ is both a Σ_2 property in parameters ρ and ω_1 and a Π_2 property in the same parameters. The same holds for ‘RO(P) is $<\omega_1$ -semiproper’.

Let X be a set such that $\omega_1 \subseteq X$. We say that $S \subseteq {}^\rho([X]^{\aleph_0})$ is ρ -semi-stationary if for every club $D \subseteq [X]^{\aleph_0}$ there are continuous ρ -chains $\sigma = (x_i : i \leq \rho)$ and $\sigma' = (x'_i : i \leq \rho)$ such that

- $\sigma \in S$,
- $\text{range}(\sigma') \subseteq D$, and
- for each $i \leq \rho$, $x_i \subseteq x'_i$ and $x_i \cap \omega_1 = x'_i \cap \omega_1$.

We have the following characterization of ρ -semiproperness (for any given indecomposable $\rho < \omega_1$).

Fact 5.10. Given an indecomposable ordinal $\rho < \omega_1$, the following are equivalent for every poset \mathcal{P} .

- (1) \mathcal{P} is ρ -semiproper.
- (2) \mathcal{P} preserves ρ -semi-stationary subsets of ${}^\rho([X]^{\aleph_0})$ for every set X ; i.e., if $S \subseteq {}^\rho([X]^{\aleph_0})$ is ρ -semi-stationary, then S remains ρ -semi-stationary after forcing with \mathcal{P} .

Again we get that:

Fact 5.11. For all countable indecomposable ordinals ρ , ρ -SP and $<\omega_1$ -SP are closed under isomorphisms, two-step iterations, lottery sums, restrictions and complete subalgebras, and contain all countably closed forcings.

Definition 5.12. Given a regular cardinal $\kappa \geq \omega_1$, a poset \mathcal{P} *preserves stationary subsets of κ* if every stationary subset of κ remains stationary after forcing with \mathcal{P} .

SSP denotes the class of partial orders preserving stationary subsets of ω_1 . More generally, given a cardinal λ , SSP(λ) denotes the class of partial orders preserving stationary subsets of κ for every uncountable regular cardinal $\kappa \leq \lambda$.

Recall that a Suslin tree is an ω_1 -tree T (i.e., T is a tree of height ω_1 all of whose levels are countable) without uncountable chains or antichains (a subset of T is called an *antichain* iff it consists of pairwise incomparable nodes). We will consider the above properties in conjunction with the preservation of some combination of the two following properties.

Definition 5.13. A poset \mathcal{P} *preserves Suslin trees* if $\Vdash_{\mathcal{P}} T$ is Suslin for every Suslin tree T in the ground model.

Definition 5.14. A poset \mathcal{P} is ${}^\omega\omega$ -*bounding* iff every function $f : \omega \rightarrow \omega$ added by \mathcal{P} is bounded by a function $g : \omega \rightarrow \omega$ in the ground model; i.e., iff for every \mathcal{P} -generic filter G and every $f : \omega \rightarrow \omega$, $f \in V[G]$, there is some $g : \omega \rightarrow \omega$, $g \in V$, such that $f(n) < g(n)$ for all n .

STP denotes the class of all posets preserving Suslin trees and ${}^\omega\omega$ -bounding the class of ${}^\omega\omega$ -bounding posets.

By the same arguments we gave for ρ -properness one gets:

Fact 5.15. ‘ $\text{RO}(P)$ preserves Suslin trees’ and ‘ \mathcal{P} is ${}^\omega\omega$ -bounding’ are Σ_2 properties in parameters ω_1 and ω^ω , and also Π_2 properties in the same parameters. Moreover, STP and ${}^\omega\omega$ -bounding are both closed under isomorphisms, two-step iterations, lottery sums, restrictions and complete subalgebras, and contain all countably closed forcings.

In [22, XI] Shelah isolates a property he calls S -condition, and for which he proves the following.

Lemma 5.16. *Assume \mathcal{P} is a forcing notion satisfying the S -condition. Then:*

- (1) \mathcal{P} preserves stationary subsets of ω_1 ;²⁹
- (2) if CH holds, then \mathcal{P} adds no new reals.

As shown in [22, XI–4], among the forcing notions satisfying the S -condition are Namba forcing (and natural variations thereof), all countably closed forcing notions, and the natural poset which, for a fixed stationary $S \subseteq \{\alpha < \omega_2 : \text{cf}(\alpha) = \omega\}$, adds an ω_1 -club through S with countable conditions.

Given a tree T and a node η of T , let $\text{succ}_T(\eta)$ denote the set of immediate successors of η in T . It will be convenient to define the following game $\mathcal{P}_p^{\mathcal{P}}$ (for a partial order \mathcal{P} and a \mathcal{P} -condition p).

Definition 5.17. Given a partial order \mathcal{P} such that $|\mathcal{P}| \geq \aleph_2$, $\mathcal{G}^{\mathcal{P}}$ is the following game of length ω between players I and II, with player I playing at even stages and player II playing at odd stages.

- (1) At any given stage n of the game, the corresponding player picks a pair T^n , $(p_\eta^n)_{\eta \in T^n}$, where T^n is a tree consisting of finite sequences of ordinals in $|\mathcal{P}|$ without infinite branches and where $(p_\eta^n)_{\eta \in T^n}$ is a sequence of conditions in \mathcal{P} extending p such that p_ν^n extends p_η^n in \mathcal{P} whenever ν extends η in T^n .
- (2) If $n > 0$, then
 - (a) T^n and $(p_\eta^n)_{\eta \in T^n}$ end-extend T^{n-1} and $(p_\eta^{n-1})_{\eta \in T^{n-1}}$, respectively,
 - (b) every terminal node in T^{n-1} has a proper extension in T^n , and
 - (c) every node in $T^n \setminus T^{n-1}$ extends a unique terminal node in T^{n-1} .
- (3) Player I starts by playing $T_0 = \{\emptyset\}$ and $p_\emptyset^0 \in \mathcal{P}$.
- (4) At any given even stage $n > 0$ of the game, player I picks, for every terminal node η of T^{n-1} , a finite sequence ν_η of ordinals in $|\mathcal{P}|$ such that ν_η extends η properly. He then builds T^n as

$$T^{n-1} \cup \{\nu_\eta \upharpoonright k : k \leq |\nu_\eta|, \eta \text{ a terminal node of } T^{n-1}\}.$$

Player I also has to choose of course $(p_\eta^n)_{\eta \in T^n}$ in such a way that (1) and (2) are satisfied.

- (5) At any given odd stage n of the game, player II chooses, for every terminal node η of T^{n-1} , a regular cardinal $\kappa_\eta^n \in [\aleph_2, |\mathcal{P}|]$, and builds T^n from T^{n-1} by adding to T^{n-1} a next level where, for each terminal node η of T^{n-1} , the set of immediate successors of η in T^n is $\{\eta \hat{\ } \langle \alpha \rangle : \alpha < \kappa_\eta^n\}$. Player II also has to choose of course $(p_\eta^n)_{\eta \in T^n}$ in such a way that (1) and (2) are satisfied.

After ω moves, the players have naturally built a tree $T = \bigcup_n T^n$ of height ω whose nodes are finite sequences of ordinals in $|\mathcal{P}|$, together with a sequence $(p_\eta)_{\eta \in T} = \bigcup (p_\eta^n)_{\eta \in T^n}$ of \mathcal{P} -conditions such that for all nodes η, ν in T , if ν extends η in T , then p_ν extends p_η in \mathcal{P} . Finally, player II wins the game iff for every subtree T' of T in V , if $|\text{succ}_{T'}(\eta)| = |\text{succ}_T(\eta)|$ for every $\eta \in T'$, then there is a condition in \mathcal{P} forcing that there is an ω -branch b through T' such that $p_{b \upharpoonright n} \in \dot{G}$ for all $n < \omega$.

²⁹See [22, XI–Thm. 3.6]. This theorem says that forcing notions with the S -condition do not collapse ω_1 . However, its proof actually establishes that such forcings in fact preserve stationary subsets of ω_1 .

The definition of the S -condition is the following.³⁰

Definition 5.18. A partial order \mathcal{P} satisfies the S -condition if and only if $|\mathcal{P}| \geq \aleph_2$ and player II has a winning strategy σ in the game $\mathcal{G}^{\mathcal{P}}$ such that for every partial run of the game, the output of σ at any given sequence $\eta \in {}^{<\omega}\mathcal{P}$ depends only on η , $(p_{\eta \upharpoonright k})_{k \leq |\eta|}$ and $\{k < |\eta| : |\text{succ}_T(\eta \upharpoonright k)| > 1\}$, where T denotes the tree built by the players up to that point.

S -cond is the class of complete boolean algebras \mathbf{B} satisfying the S -condition.

One has:

Fact 5.19. ‘ $\text{RO}(P)$ satisfies the S -condition’ is a Σ_2 property in parameter ω_2 and also a Π_2 property in the same parameter.

S -cond is closed under isomorphisms, two-step iterations, lottery sums, restrictions and complete subalgebras, and contains all countably closed forcings.

Proof. As in the case of all other classes dealt with in this section, if $\text{RO}(P) \in H_\theta$, then $H_\theta \models$ ‘ $\text{RO}(P)$ satisfies the S -condition’ if and only if $\text{RO}(P)$ satisfies the S -condition.

The remaining properties of S -cond other than the closure under complete subalgebras are left to the reader.

We prove now that S -cond is a class closed under complete subalgebras; it will be convenient for this to resort to the algebraic properties of complete injective homomorphisms with adjoints outlined in Theorem 6.7 of the appendix.

Assume \mathbf{B} is a complete subalgebra of some \mathbf{C} satisfying the S -condition. Let $\pi : \mathbf{C} \rightarrow \mathbf{B}$ be the adjoint map of the inclusion map of \mathbf{B} into \mathbf{C} . Let σ be the winning strategy for player II in $\mathcal{G}^{\mathbf{C}^+}$. Define σ' to be the strategy for player II in $\mathcal{G}^{\mathbf{B}^+}$ obtained by the following procedure:

- Player I and II build partial plays $\langle T_n, \{b_\eta : \eta \in T_n\} \rangle$ in $\mathcal{G}^{\mathbf{B}^+}$ and partial plays $\langle T_n, \{c_\eta : \eta \in T_n\} \rangle$ in $\mathcal{G}^{\mathbf{C}^+}$ according to these prescriptions:
 - for all $\eta \in T_n$ and all n we have that $\pi(c_\eta) = b_\eta$;
 - for all terminal nodes $\eta \in T_{2n}$ with $\eta = \langle \gamma_0, \dots, \gamma_m \rangle$, we have that

$$c_\eta = b_\eta \wedge c_{\langle \gamma_0, \dots, \gamma_{m-1} \rangle};$$

$$- \langle T_{2n+1}, \{c_\eta : \eta \in T_{2n+1}\} \rangle = \sigma(\langle T_{2n}, \{c_\eta : \eta \in T_{2n}\} \rangle);$$

- Player II defines $\sigma'(\langle T_{2n}, \{b_\eta : \eta \in T_{2n}\} \rangle) = \langle T_{2n+1}, \{b_\eta : \eta \in T_{2n+1}\} \rangle$.

Now assume $\langle T, \{b_\eta : \eta \in T\} \rangle$ is built according to a play of $\mathcal{G}^{\mathbf{B}^+}$ in which II follows σ' . Fix a subtree $T' \subseteq T$ as given by the winning condition for II in $\mathcal{G}^{\mathbf{B}^+}$. Given a V -generic filter H for \mathbf{B} , we must find some infinite branch η of T' in $V[H]$ such that $b_{\eta \upharpoonright n} \in H$ for all n . To find this branch let $\langle T, \{c_\eta : \eta \in T\} \rangle$ be the tree built in tandem with $\langle T, \{b_\eta : \eta \in T\} \rangle$ according to the rules we used to define σ' . Fix $G \supseteq H$ V -generic for \mathbf{C} . Since \mathbf{C} satisfies the S -condition, we can find some infinite branch η of T' in $V[G]$ such that $c_{\eta \upharpoonright n} \in G$ for all n . This gives that $b_{\eta \upharpoonright n} = \pi(c_{\eta \upharpoonright n}) \in H$ for all n . Hence, in $V[G]$ there is an infinite branch η of T' such that $b_{\eta \upharpoonright n} \in H$ for all n . Therefore the tree $T^* = \{\eta \in T' : b_\eta \in H\} \in V[H]$ is ill-founded in $V[G]$. But then the same is true in $V[H]$ by absoluteness of ill-foundedness. Finally, any infinite branch of T^* witnesses the winning condition for II using σ' relative to T, T', H .

Since this can be done for all possible choices of $T \supseteq T'$ in V with T constructed using σ' , and for all V -generic filters H for \mathbf{B} , we have that \mathbf{B} satisfies the S -condition. \square

³⁰Shelah’s definition is more general, but the present form suffices for our purposes.

5.2. Incompatible bounded category forcing axioms. In this section we isolate \aleph_1 -many classes Γ of forcing notions with $\lambda_\Gamma = \omega_1$, all of which are absolutely well-behaved (in some cases modulo the existence of unboundedly many measurable cardinals), and which are pairwise incompatible. Our main results are the following.

Theorem 5.20. (1) *Each of the following classes Γ is absolutely well-behaved and such that $\lambda_\Gamma = \omega_1$.*

- (a) PR
- (b) $\text{PR} \cap \text{STP}$
- (c) $\text{PR} \cap {}^\omega\omega$ -bounding
- (d) $\text{PR} \cap \text{STP} \cap {}^\omega\omega$ -bounding
- (e) ρ -PR for every countable indecomposable ordinal ρ such that $1 < \rho < \omega_1$.
- (f) $<\omega_1$ -PR
- (g) S -cond

(2) *Suppose there is a proper class of measurable cardinals. Then each of the following classes Γ is absolutely well-behaved and such that $\lambda_\Gamma = \omega_1$.*

- (a) ρ -SP for every countable indecomposable ordinal $\rho < \omega_1$.
- (b) $(\rho\text{-SP}) \cap \text{STP}$ for every countable indecomposable ordinal $\rho < \omega_1$.
- (c) $(\rho\text{-SP}) \cap {}^\omega\omega$ -bounding for every countable indecomposable ordinal $\rho < \omega_1$.
- (d) $(\rho\text{-SP}) \cap \text{STP} \cap {}^\omega\omega$ -bounding for every countable indecomposable ordinal $\rho < \omega_1$.
- (e) $<\omega_1$ -SP
- (f) $(<\omega_1\text{-SP}) \cap \text{STP}$
- (g) $(<\omega_1\text{-SP}) \cap {}^\omega\omega$ -bounding
- (h) $(<\omega_1\text{-SP}) \cap \text{STP} \cap {}^\omega\omega$ -bounding

Theorem 5.21. *Suppose there is a supercompact cardinal δ such that $V_\delta \prec V$. Suppose Γ and Γ' are any two different classes of forcing notions mentioned in Theorem 5.20. Then $\text{BCFA}(\Gamma)$ implies $\neg\text{BCFA}(\Gamma')$.*

Remark 5.22. For some choices of Γ and Γ' , the incompatibility of $\text{BCFA}(\Gamma)$ and $\text{BCFA}(\Gamma')$ can be proved just assuming the existence of an inaccessible cardinal δ such that $V_\delta \prec V$, or just the existence of an inaccessible δ such that $V_\delta \prec V$ together with the existence of a proper class of Woodin cardinals.

As will be clear from the proofs, Theorems 5.20 and 5.21 are just selected samples of a zoo of possibly incompatible instances of $\text{BCFA}(\Gamma)$. In particular, it should be possible to combine (some of) the classes mentioned in Theorem 5.20 with other classes of forcing notions, besides STP and ${}^\omega\omega$ -bounding, so long as these classes have a suitable iteration theory and reasonable closure properties, are both Σ_2 definable and Π_2 definable, and the resulting classes contain SSP -freezing posets.

We should point out that the following natural question—in the present context—remains open.

Question 5.23. Is there, under any reasonable large cardinal, any indecomposable $\rho < \omega_1$, $\rho > 1$, for which any of the following classes is absolutely well-behaved?

- (1) $(\rho\text{-PR}) \cap \text{STP}$
- (2) $(\rho\text{-PR}) \cap {}^\omega\omega$ -bounding
- (3) $(\rho\text{-PR}) \cap \text{STP} \cap {}^\omega\omega$ -bounding
- (4) $(<\omega_1\text{-PR}) \cap \text{STP}$
- (5) $(<\omega_1\text{-PR}) \cap {}^\omega\omega$ -bounding
- (6) $(<\omega_1\text{-PR}) \cap \text{STP} \cap {}^\omega\omega$ -bounding

The following question, of a more foundational import, addresses the possibility of there being absolutely well-behaved classes Γ such that $\lambda_\Gamma > \omega_1$.

Question 5.24. Are there, under some reasonable large cardinal assumption, any cardinal $\lambda \geq \omega_2$ and any class Γ of forcing notions such that Γ is absolutely well-behaved and such that $\lambda_\Gamma = \lambda$? Are there, again under some reasonable large cardinal assumption, any cardinal $\lambda \geq \omega_2$ and any class Γ of forcing notions with $\lambda_\Gamma = \lambda$ and such that Γ is absolutely well-behaved and such that $\text{BCFA}(\Gamma)$ is compatible with—or, even, implies— $\text{BCFA}(\Gamma')$ for any absolutely well-behaved class Γ' with $\lambda_{\Gamma'} = \omega_1$?

5.3. Proof of theorems 5.20 and 5.21.

5.3.1. *Four freezing posets.* In this section we introduce four instances of SSP-freezing posets. We feel free to confuse posets with complete boolean algebras, as the context will dictate which is the correct intended meaning of the concept. When proving SSP-freezability, we will actually be showing the following sufficient condition (for $\lambda = \omega_1$).

Lemma 5.25. *Let $\lambda \geq \omega_1$ be a cardinal, \mathbf{B} a forcing notion, and $\dot{\mathbf{C}}$ a \mathbf{B} -name for a forcing notion. Suppose that p is a set, and that if G is a \mathbf{B} -generic filter, then $\mathbf{C} = \dot{\mathbf{C}}_G$ forces that there is some $A_G \subseteq \lambda$ coding G in an absolute way mod. $\text{SSP}(\lambda)$, in the sense that there is some Σ_1 formula $\varphi(x, y, z)$ such that, if H is a $\mathbf{B} * \dot{\mathbf{C}}$ -generic filter over V such that $H \cap \mathbf{B} = G$, then*

- (1) $(H_{\lambda+}; \in, \mathbf{NS}_\lambda)^{V[H]} \models \varphi(G, A_G, p)$, and
- (2) in every outer model M of $V[H]$ such that $\mathcal{P}(\lambda)^{V[H]} \cap (\mathbf{NS}_\lambda)^M = (\mathbf{NS}_\lambda)^{V[H]}$, if

$$(H_{\lambda+}; \in, \mathbf{NS}_\lambda)^M \models \varphi(G_0, A_{G_0}, p)$$

and

$$(H_{\lambda+}; \in, \mathbf{NS}_\lambda)^M \models \varphi(G_1, A_{G_1}, p),$$

then $G_0 = G_1$.

Then the natural inclusion

$$i : \mathbf{B} \longrightarrow \mathbf{B} * \dot{\mathbf{C}}$$

SSP(λ)-freezes \mathbf{B} .

Proof. Suppose, towards a contradiction, that $b_0, b_1 \in \mathbf{B}$ are incompatible, \mathbf{D} is a complete boolean algebra, $k_0 : (\mathbf{B} * \dot{\mathbf{C}}) \upharpoonright b_0 \longrightarrow \mathbf{D}$, $k_1 : (\mathbf{B} * \dot{\mathbf{C}}) \upharpoonright b_1 \longrightarrow \mathbf{D}$ are complete homomorphisms, K is \mathbf{D} -generic and, for each $\epsilon \in \{0, 1\}$, $H_\epsilon = k_\epsilon^{-1}(K)$ and every stationary subset of λ in $V[H_\epsilon]$ remains stationary in $V[K]$. For each $\epsilon \in \{0, 1\}$, let G_ϵ be the filter on \mathbf{B} generated by $H_\epsilon \cap (\mathbf{B} \upharpoonright b_\epsilon)$, and let $A_\epsilon \subseteq \lambda$ be such that

$$(H_{\lambda+}; \in, \mathbf{NS}_\lambda)^{V[H_\epsilon]} \models \varphi(G_\epsilon, A_\epsilon, p)$$

Since

$$(H_{\lambda+}; \in, \mathbf{NS}_\kappa)^{V[K]} \models \varphi(G_0, A_0, p) \wedge \varphi(G_1, A_1, p),$$

we have that $G_0 = G_1$ by (2). But this is impossible since $b_0 \in G_0$, $b_1 \in G_1$, and since b_0 and b_1 are incompatible conditions in \mathbf{B} . \square

Our first freezing poset comes essentially from [19].

Given a set X , the Ellentuck topology on $[X]^{\aleph_0}$ is the topology on $[X]^{\aleph_0}$ generated by the sets of the form $[s, Y]$, for $Y \in [X]^{\aleph_0}$ and $s \in [Y]^{<\omega}$, where $[s, Y] = \{Z \in [Y]^{\aleph_0} : s \subseteq Z\}$.

The following lemma, except for the conclusion that \mathcal{P} preserves Suslin trees, is due to Moore [19]. The conclusion that \mathcal{P} preserves Suslin trees is due to Miyamoto [16].

Lemma 5.26. *Let X be a set, θ a cardinal such that $X \in H_\theta$, and Σ a function with domain $[H_\theta]^{\aleph_0}$ such that for every countable $M \preceq [H_\theta]^{\aleph_0}$,*

- $\Sigma(M) \subseteq [X]^{\aleph_0}$ is open in the Ellentuck topology, and
- $\Sigma(M)$ is M -stationary (meaning that for every function $F : [X]^{<\omega} \longrightarrow X$, if $F \in M$, then there is some $Z \in \Sigma(M) \cap M$ such that $F''[Z]^{<\omega} \subseteq Z$).

Let $\mathcal{P} = \mathcal{P}_{X,\theta,\Sigma}$ be the set, ordered by reverse inclusion, of all countable \subseteq -continuous \in -chains $p = (M_i^p)_{i \leq \nu}$ of countable elementary substructures of H_θ such that for every limit ordinal $i \leq \nu$ there is some $i_0 < i$ with the property that $M_k^p \cap X \in \Sigma(M_i^p)$ for all k such that $i_0 < k < i$.

Then

- (1) \mathcal{P} is proper, preserves Suslin trees, and does not add new reals.
- (2) Whenever G is \mathcal{P} -generic over V and $M_i^G = M_i^p$ for $p \in G$ and $i \in \text{dom}(p)$, $(M_i^G)_{i < \omega_1}$ is in $V[G]$ the \subseteq -increasing enumeration of a club of $[H_\theta^V]^{\aleph_0}$ and is such that:

For every limit ordinal $i < \omega_1$ there is some $i_0 < i$ with the property that $M_k^G \cap X \in \Sigma(M_i^G)$ for all k such that $i_0 < k < i$.

Remark 5.27. In most interesting cases, the forcing $\mathcal{P}_{X,\theta,\Sigma}$ in the above lemma is not ω -proper.

In [19], Moore defines the Mapping Reflection Principle (MRP) as the following statement: Given X , θ , and Σ as in the hypothesis of Lemma 5.26, there is a \subseteq -continuous \in -chain $(M_i)_{i < \omega_1}$ of countable elementary substructures of H_θ such that for every limit ordinal $i < \omega_1$ there is some $i_0 < i$ with the property that $M_k \cap X \in \Sigma(M_i)$ for every k such that $i_0 < k < i$.

It follows from Lemma 5.26 that MRP is a consequence of PFA, and of the forcing axiom for the class of forcing notions in $\text{PR} \cap \text{STP}$ not adding new reals.

We will call a partial order \mathcal{R} an *MRP-poset* if there are X , θ and Σ as in the hypothesis of Lemma 5.26 such that $\mathcal{R} = \mathcal{P}_{X,\theta,\Sigma}$.

Proposition 5.28. *Given a forcing notion \mathcal{P} , there is \mathcal{P} -name $\dot{\mathcal{Q}}$ for a forcing notion such that*

- (1) $\dot{\mathcal{Q}}$ is forced to be of the form $\text{Coll}(\omega_1, \mathcal{P}) * \dot{\mathcal{R}}$, where $\dot{\mathcal{R}}$ is a $\text{Coll}(\omega_1, \mathcal{P})$ -name for an MRP-poset, and
- (2) $\mathcal{P} * \dot{\mathcal{Q}}$ SSP-freezes \mathcal{P} , as witnessed by the inclusion map.

Proof. By Lemma 5.25, it suffices to prove that \mathcal{P} forces that in $V^{\text{Coll}(\omega_1, \mathcal{P})}$ there is an MRP-poset $\dot{\mathcal{R}}$ such that $\text{Coll}(\omega_1, \mathcal{P}) * \dot{\mathcal{R}}$ codes the generic filter for \mathcal{P} in an absolute way mod. SSP in the sense of that lemma. For this, let us work in $V^{\mathcal{P} * \text{Coll}(\omega_1, \mathcal{P})}$. Let \dot{B}_G be a subset of ω_1 coding the generic filter for \mathcal{P} in some canonical way, let $\vec{C} = (C_\delta : \delta \in \text{Lim}(\omega_1)) \in V$ be a ladder system on ω_1 (i.e., every C_δ is a cofinal subset of δ of order type ω), and let $(S_\alpha)_{\alpha < \omega_1} \in V$ be a partition of ω_1 into stationary sets. Given $X \subseteq Y$, countable sets of ordinals, such that $Y \cap \omega_1$ and $\text{ot}(Y)$ are both limit ordinals and such that X is bounded in $\text{sup}(Y)$, let $c(X, Y)$ mean

$$|C_{\text{ot}(Y)} \cap \text{sup}(\pi_Y \text{``} X)| < |C_{Y \cap \omega_1} \cap X \cap \omega_1|,$$

where π_Y is the collapsing function of Y .

Let θ be a large enough cardinal and let Σ be the function sending a countable $N \preceq H_\theta$ to the set of $Z \in [\omega_2 \cap N]^{\aleph_0}$ such that $c(X, \omega_2 \cap N)$ iff the unique $\alpha < \omega_1$ such that $N \cap \omega_1 \in S_\alpha$ is in $B_{\dot{G}}$. Now, $X = \omega_2$, θ and Σ satisfy the hypothesis of Lemma 5.26 (s. [19]).³¹ Let $\mathcal{R} = \mathcal{P}_{\omega_2, \theta, \Sigma}$. By Lemma 5.26, \mathcal{R} adds $(Z_i^{\dot{G}})_{i < \omega_1}$, a strictly \subseteq -increasing enumeration of a club of $[\omega_2^V]^{\aleph_0}$, such that for every limit ordinal $i < \omega_1$, if $Z_i^{\dot{G}} \cap \omega_1 \in S_\alpha$, then there is a tail of $k < i$ such that $c(Z_k^{\dot{G}}, Z_i^{\dot{G}})$ if and only if $\alpha \in B_{\dot{G}}$.

Let $\kappa = \omega_2$, let H be $(\mathcal{P} * \text{Coll}(\omega_1, \mathcal{P})) * \dot{\mathcal{R}}$ -generic, let $G = H \cap \mathcal{P}$, and let A_G be a subset of ω_1 which canonically codes B_G and $(Z_i^{\dot{G}})_{i < \omega_1}$. If M is any outer model such that every stationary subset of ω_1 in $V[H]$ remains stationary in M , then B_G is the unique subset

³¹ Σ is in essence the mapping used by Moore in [19] to prove that BPFA implies $2^{\aleph_1} = \aleph_2$.

B of ω_1 for which there is, in M , a set $A \subseteq \omega_1$ coding B together with an \subseteq -increasing enumeration $(Z_i)_{i < \omega_1}$ of a club of $[\kappa]^{\aleph_0}$ with the property that for every limit ordinal $i < \omega_1$, if $Z_i \cap \omega_1 \in S_\alpha$, then there is a tail of $k < i$ such that $c(Z_k, Z_i)$ if and only if $\alpha \in B$. Indeed, if $B' \in M$ were another such set, as witnessed by $A' \subseteq M$, $\alpha \in B \Delta B'$, and $(Z'_i)_{i < \omega_1} \in M$ were an \subseteq -increasing enumeration of a club of $[\kappa]^{\aleph_0}$ with the property that for every limit ordinal $i < \omega_1$, if $Z'_i \cap \omega_1 \in S_\alpha$, then there is a tail of $k < i$ such that $c(Z_k, Z_i)$ if and only if $\alpha \in B'$, then we would be able to find some i such that $Z_i = Z'_i$, $Z_i \cap \omega_1 \in S_\alpha$, and such that $Z_k = Z'_k$ for all k in some cofinal subset J of i . But then we would have that $c(Z_k, Z_i)$, for all k in some final segment of J , both holds and fails.

Finally, it is immediate to see that there is a Σ_1 formula $\varphi(x, y, z)$ such that $\varphi(G, A_G, p)$ expresses the above property of G and A_G over $(H_{\omega_2}; \in)^M$ for any M as above, for $p = (\kappa, \vec{C}, (S_\alpha)_{\alpha < \omega_1})$. \square

Using coding techniques from [8], one can prove the following stronger version of Lemma 5.28. However, we do not have any use for this stronger form, so we will not give the proof here.

Lemma 5.29. *Given a partial order \mathcal{P} there is \mathcal{P} -name \dot{Q} for a partial order with the following properties.*

- (1) \dot{Q} is forced to be of the form $\text{Coll}(\omega_1, \mathcal{P}) * \dot{\mathcal{R}}$ where $\dot{\mathcal{R}}$ is a $\text{Coll}(\omega_1, \mathcal{P})$ -name for a forcing of the form $\dot{\mathcal{R}}_0 * \dot{\mathcal{R}}_1$, where $\dot{\mathcal{R}}_0$ has the countable chain condition and $\dot{\mathcal{R}}_1$ is forced to be an MRP-poset.
- (2) Suppose $b_0, b_1 \in \text{RO}(\mathcal{P})$ are incompatible, \mathbf{B} is a complete boolean algebra, and $k_\epsilon : \text{RO}(\mathcal{P} * \dot{Q}) \upharpoonright b_\epsilon \rightarrow \mathbf{B}$ are complete homomorphisms for $\epsilon \in \{0, 1\}$. Then \mathbf{B} collapses ω_1 .

Our second freezing poset comes from [24, Section 1], where the following is proved, using a result of Todorćević from [23].

Lemma 5.30. *There is a sequence $((K_0^\xi, K_1^\xi) : \xi < \omega_1)$ of colourings of $[\kappa]^2$, for $\kappa = \text{cf}(2^{\aleph_0})$, with the property that in any ω_1 -preserving outer model in which $|\kappa| = \aleph_1$, if $B \subseteq \omega_1$, then there is a c.c.c. partial order \mathcal{R} forcing the existence of \aleph_1 -many decompositions $\kappa = \bigcup_{n < \omega} X_n^\xi$, for $\xi < \omega_1$, such that for all $\xi < \omega_1$:*

- for some fixed $i_\xi = 0, 1$, X_n^ξ is $K_{i_\xi}^\xi$ -homogeneous for all $n < \omega$;
- $\xi \in B$ if and only if $i_\xi = 0$.

Proposition 5.31. *Given a forcing notion \mathcal{P} , there is \mathcal{P} -name \dot{Q} for a forcing notion with the following properties.*

- (1) Letting $\mu = |\mathcal{P}| + \text{cf}(2^{\aleph_0})$, \dot{Q} is forced to be a forcing of the form $\text{Coll}(\omega_1, \mu) * \dot{\mathcal{R}}$, where $\dot{\mathcal{R}}$ is a $\text{Coll}(\omega_1, \mu)$ -name for a c.c.c. forcing.
- (2) $\mathcal{P} * \dot{Q}$ SSP-freezes \mathcal{P} , as witnessed by the inclusion map. In fact, if $b_0, b_1 \in \text{RO}(\mathcal{P})$ are incompatible, \mathbf{B} is a complete boolean algebra, and $k_\epsilon : \text{RO}(\mathcal{P} * \dot{Q}) \upharpoonright b_\epsilon \rightarrow \mathbf{B}$ is a complete homomorphism for $\epsilon \in \{0, 1\}$, then \mathbf{B} collapses ω_1 .

Proof. Let us work in $V^{\mathcal{P} * \text{Coll}(\omega_1, \mu)}$. Let \dot{B}_G be a subset of ω_1 coding the generic filter for \mathcal{P} in some canonical way, let $\vec{K} = ((K_0^\xi, K_1^\xi) : \xi < \omega_1)$ be a sequence of colourings of $[\kappa]^2$, for $\kappa = \text{cf}^V(2^{\aleph_0})$, as given by Lemma 5.30, and let \mathcal{R} be a c.c.c. partial order forcing the existence of \aleph_1 -many decompositions $\kappa = \bigcup_{n < \omega} X_n^\xi$ such that for all $\xi < \omega_1$,

- there is i_ξ such that $[X_n^\xi]^2 \subseteq K_{i_\xi}^\xi$ for all $n < \omega$ and
- $\xi \in B$ if and only if for all $n < \omega$, $[X_n^\xi]^2 \subseteq K_0^\xi$.

Let A_G be a subset of ω_1 which canonically codes B_G and $\{(X_n^\xi)_{n < \omega} : \xi < \omega_1\}$. If M is any outer model in which ω_1^V has not been collapsed, then B_G is the unique $B \subseteq \omega_1$

for which there is, in M , a set $A \subseteq \omega_1$ coding B together with \aleph_1 -many decompositions $\{(X_n^\xi)_{n < \omega} : \xi < \omega_1\}$ of κ such that for all $n < \omega$ and $\xi < \omega_1$, $\xi \in B$ if and only if $[X_n^\xi]^2 \subseteq K_0^\xi$. Indeed, if $B' \in M$ were another such set, as witnessed by $A' \subseteq M$, $\xi \in B \Delta B'$, and \aleph_1 -many decompositions $\{(Y_n^\xi)_{n < \omega} : \xi < \omega_1\} \in M$ of κ , then there would be some n and m such that $X_n^\xi \cap Y_m^\xi$ has more than one element, and is in fact uncountable. But then, for every $s \in [X_n^\xi \cap Y_m^\xi]^2$, we would have that s is both in K_0^ξ and K_1^ξ , which is impossible.

Finally, it is immediate to see that there is a Σ_1 formula $\varphi(x, y, z)$ such that $\varphi(G, A_G, p)$ expresses the above property of G and A_G over $(H_{\omega_2}; \in)^M$ for any M as above, for $p = \vec{K}$. \square

The following principle, as well as Lemma 5.33, are due to Woodin ([30]).

Definition 5.32. ψ_{AC} is the following statement: Suppose S and T are stationary and co-stationary subsets of ω_1 . Then there are $\alpha < \omega_2$ and a club C of $[\alpha]^{\aleph_0}$ such that for every $X \in C$, $X \cap \omega_1 \in S$ if and only if $\text{ot}(X) \in T$.

The AC-subscript in the above definition hints at the fact that ψ_{AC} implies $L(\mathcal{P}(\omega_1)) \models AC$ (which comes from an argument similar to the one in the proof of Lemma 5.34).

Our third freezing poset is essentially the following forcing for adding a suitable instance of ψ_{AC} by initial segments, using a measurable cardinal κ (i.e., turning κ into an ordinal α as required by the conclusion of ψ_{AC}).

Lemma 5.33. *Let κ be a measurable cardinal and let S and T be stationary and co-stationary subsets of ω_1 . Let $\mathcal{Q} = \mathcal{Q}_{\kappa, S, T}$ be the set, ordered by reverse inclusion, of all countable \subseteq -continuous \in -chains $p = (M_i^p)_{i < \nu}$ of countable elementary substructures of H_κ such that for every $i \leq \nu$, $M_i^p \cap \omega_1 \in S$ if and only if $\text{ot}(M_i^p \cap \kappa) \in T$.*

- (1) \mathcal{Q} is $< \omega_1$ -semiproper, preserves Suslin trees, and does not add new reals.
- (2) if G is \mathcal{Q} -generic over V and $M_i^G = M_i^p$ whenever $p \in G$ and $i \in \text{dom}(p)$, then $(M_i^G)_{i < \omega_1}$ is the \subseteq -increasing enumeration of a club of $[H_\kappa^V]^{\aleph_0}$ such that for every limit ordinal $i < \omega_1$, $M_i^G \cap \omega_1 \in S$ if and only if $\text{ot}(M_i^G \cap \kappa) \in T$.

Proof. The proofs of all assertions, except the fact that \mathcal{Q} preserves Suslin trees, are standard. For the reader's convenience, we sketch the proof that \mathcal{Q} is $< \omega_1$ -semiproper, though. We also prove that \mathcal{Q} preserves Suslin trees.

We get the $< \omega_1$ -semiproperness of \mathcal{Q} as follows: the main point is that if \mathcal{U} is a normal measure on κ , N is an elementary submodel of some H_θ such that $\mathcal{U} \in N$ and $|N| < \kappa$, and $\eta \in \bigcap(\mathcal{U} \cap N)$, then

$$N[\eta] := \{f(\eta) : f \in N, f \text{ a function with domain } \kappa\}$$

is an elementary submodel of H_θ such that $N \cap \kappa$ is a proper initial segment of $N[\eta] \cap \kappa$ ($\eta \in N[\eta]$ is above every ordinal in $N \cap \kappa$, and any $\gamma \in N[\eta] \cap \eta$ is of the form $f(\eta)$, for some regressive function $f : \kappa \rightarrow \kappa$ in N which, by normality of \mathcal{U} , is constant on some set in $\mathcal{U} \cap N$). If N is countable, then by iterated applications of this construction, taking unions at nonzero limit stages, one obtains a \subseteq -continuous and \subseteq -increasing sequence $(N_\nu)_{\nu < \omega_1}$ of elementary submodels of H_θ such that $N_0 = N$ and such that $N_{\nu'} \cap \kappa$ is a proper end-extension of $N_\nu \cap \kappa$ for all $\nu < \nu' < \omega_1$. Since $(\text{ot}(N_\nu \cap \kappa) : \nu < \omega_1)$ is then a strictly increasing and continuous sequence of countable ordinals, we may find, by stationarity of S and $\omega_1 \setminus S$, some $\nu < \omega_1$ such that $N \cap \omega_1 \in S$ if and only if $\text{ot}(N_\nu) \in T$.

This observation yields the $< \omega_1$ -semiproperness of \mathcal{Q} since, given $\alpha < \omega_1$ and an \in -chain $(N^\xi)_{\xi < \alpha}$ of countable elementary submodels of some H_θ such that $\mathcal{U}, S, T \in N^0$, one can run the above construction for each N^ξ by working inside $N^{\xi+1}$.

The preservation of Suslin trees can be proved by the following version of the argument in [16] for showing that MRP-forcings preserve them. Suppose U is a Suslin tree, \dot{A} is

a \mathcal{Q} -name for a maximal antichain of U , and N is a countable elementary submodel of some large enough H_θ containing U , \dot{A} , and all other relevant objects. By moving to an ω_1 -end-extension of N if necessary as in the proof of $<\omega_1$ -semiproperness, we may assume that $N \cap \omega_1 \in S$ if and only if $\text{ot}(N \cap \kappa) \in T$. Let $(u_n)_{n < \omega}$ enumerate all nodes in U of height $N \cap \omega_1$. Given a condition $p \in \mathcal{Q}$ in N , we may build an (N, \mathcal{Q}) -generic sequence $(p_n)_{n < \omega}$ of conditions in N extending p and such that for every n there is some $v \in U$ below u_n such that p_{n+1} forces $v \in \dot{A}$. By the choice of N , we have in the end that $p^* = \bigcup_n p_n \cup \{(N \cap \omega_1, N \cap \kappa)\}$ is a condition in \mathcal{Q} extending p . But, by construction of $(p_n)_{n < \omega}$, p^* forces \dot{A} to be contained in the countable set $U \cap N$: If $u \in \dot{A} \setminus N$, and u_n is the unique node of height $N \cap \omega_1$ such that $u_n \leq_U u$, then u_n is forced to extend some node in \dot{A} of height less than $N \cap \omega_1$, which is a contradiction since \dot{A} was supposed to be a name for an antichain.

It remains to show how to find p_{n+1} given p_n . Working in N , we first extend p_n to some p'_n in some suitable dense subset $D \in N$ of \mathcal{Q} . Since U is a Suslin tree, we have that u_n is totally (U, N) -generic, in the sense that for every antichain B of U in N , u_n extends a unique node in B . Also, the set $E \in N$ of $u \in U$ for which there is some $v \in U$ below u and some $q \in \mathcal{Q}$ extending p'_n and forcing that $v \in \dot{A}$ is dense in U . It follows that we may find some $u \in E \cap N$ below u_n , as witnessed by some $q \in \mathcal{Q} \cap N$ and some $v \in U \cap N$. But then we may let $p_{n+1} = q$. \square

Given a measurable cardinal κ , we will call a partial order \mathcal{R} a ψ_{AC}^κ -poset if \mathcal{R} is of the form $\mathcal{Q}_{\kappa, S, T}$ for stationary and co-stationary subsets S, T of ω_1 .

Proposition 5.34. *Given a forcing notion \mathcal{P} and a measurable cardinal κ , there is \mathcal{P} -name $\dot{\mathcal{Q}}$ for a forcing notion such that*

- (1) $\dot{\mathcal{Q}}$ is forced to be of the form $\text{Coll}(\omega_1, \mathcal{P}) * \dot{\mathcal{R}}$, where $\dot{\mathcal{R}}$ is a $\text{Coll}(\omega_1, \mathcal{P})$ -name for a ψ_{AC}^κ -poset, and
- (2) $\mathcal{P} * \dot{\mathcal{Q}}$ SSP-freezes \mathcal{P} , as witnessed by the inclusion map.

Proof. Let $(S_\alpha : \alpha < \omega_1)$ be a partition of ω_1 into stationary sets and let T be a stationary and co-stationary subset of ω_1 . Working in $V^{\mathcal{P} * \text{Coll}(\omega_1, \mathcal{P})}$, let $B_{\dot{G}} \neq \emptyset$ be a subset of ω_1 coding \dot{G} in a canonical way. We may assume that $B_{\dot{G}} \neq \omega_1$. Let $\dot{\mathcal{R}}$ be $\mathcal{Q}_{\kappa, S, T}$ for $S = \bigcup_{\alpha \in B_{\dot{G}}} S_\alpha$. By Lemma 5.33, $\dot{\mathcal{R}}$ adds a club $C_{\dot{G}}$ of $[\kappa]^{\aleph_0}$ with the property that for each $X \in C_{\dot{G}}$, $X \cap \omega_1 \in \bigcup_{\alpha \in B_{\dot{G}}} S_\alpha$ if and only if $\text{ot}(X) \in T$. Now it is easy to see that $\mathcal{P} * \dot{\mathcal{R}}$ codes \dot{G} in an absolute way in the sense of Lemma 5.25. The main point is that if H is a $(\mathcal{P} * \text{Coll}(\omega_1, \mathcal{P})) * \dot{\mathcal{R}}$ -generic filter, $G = H \cap \mathcal{P}$, and M is an outer model such that every stationary subset of ω_1 in $V[H]$ remains stationary in M , then in M there is no $B' \subseteq \omega_1$ such that $B' \neq B_G$ and such that there is a club C of $[\kappa]^{\aleph_0}$ with the property that for all $X \in C$, $X \cap \omega_1 \in \bigcup_{\alpha \in B'} S_\alpha$ if and only if $\text{ot}(X) \in T$. Otherwise, if $\alpha \in B' \Delta B_G$, then there would be some $X \in C \cap C_G$ such that $X \cap \omega_1 \in S_\alpha$. But then we would have that $\text{ot}(X)$ is both in T and in $\omega_1 \setminus T$. \square

Let us move on now to our fourth freezing poset.

Given cardinals $\mu < \lambda$ with μ regular, let

$$S_\mu^\lambda = \{\xi < \lambda : \text{cf}(\xi) = \mu\}$$

Let $\vec{S} = (S_\alpha)_{\alpha < \omega_1}$ be a sequence of pairwise disjoint stationary subsets of $S_\omega^{\omega_2}$, let $U \subseteq S_\omega^{\omega_3}$ be such that both U and $S_\omega^{\omega_3} \setminus U$ are stationary, and let $B \subseteq \omega_1$. Then $\mathcal{S}_{\vec{S}, U, B}$ is the partial order, ordered by end-extension, consisting of all strictly \subseteq -increasing and \subseteq -continuous sequences $(Z_\nu)_{\nu \leq \nu_0}$, for some $\nu_0 < \omega_1$, such that for all $\nu \leq \nu_0$ and all $\alpha < \omega_1$,

- $Z_\nu \in [\omega_3]^{\aleph_0}$, and
- if $\text{sup}(Z_\nu \cap \omega_2) \in S_\alpha$, then $\text{sup}(Z_\nu) \in U$ if and only if $\alpha \in B$.

The proof of the following lemma appears in [10] essentially.³²

Lemma 5.35. *Let $\vec{S} = (S_\alpha)_{\alpha < \omega_1}$ be a sequence of pairwise disjoint stationary subsets of $S_\omega^{\omega_2}$, let $U \subseteq S_\omega^{\omega_3}$ be such that both U and $S_\omega^{\omega_3} \setminus U$ are stationary, and let $B \subseteq \omega_1$. Then $\dot{S}_{\vec{S}, U, B}$ preserves stationary subsets of ω_1 , as well as the stationarity of all S_α , and forces the existence of strictly \subseteq -increasing and \subseteq -continuous enumeration $(Z_\nu : \nu < \omega_1)$ of a club of $[\omega_3^V]^{\aleph_0}$ such that for all $\nu, \alpha < \omega_1$, if $\text{sup}(Z_\nu \cap \omega_2^V) \in S_\alpha$, then $\text{sup}(Z_\nu) \in U$ if and only if $\alpha \in B$.*

Proposition 5.36. *For every partial order \mathcal{P} and for all cardinals $\kappa^1 > \kappa^0 \geq \delta \geq |\mathcal{P}|$, if $\Vdash_{\mathcal{P} * \text{Coll}(\omega_1, \delta)} \kappa^0 = \omega_2$, $\Vdash_{\mathcal{P} * \text{Coll}(\omega_1, \delta)} \kappa^1 = \omega_3$, $(S_\alpha)_{\alpha < \omega_1} \in V$ is a sequence of pairwise disjoint stationary subsets of $S_\omega^{\kappa^0}$, and $U \subseteq S_\omega^{\kappa^1}$ is a stationary set in V such that $S_\omega^{\kappa^1} \setminus U$ is also stationary in V , then there is a $\mathcal{P} * \text{Coll}(\omega_1, \delta)$ -name \dot{B} for a subset of ω_1 such that*

- (1) \mathcal{P} forces $\text{Coll}(\omega_1, \delta) * \dot{S}_{\vec{S}, U, \dot{B}}$ to have the S -condition, and such that
- (2) $\mathcal{P} * (\text{Coll}(\omega_1, \delta) * \dot{S}_{\vec{S}, U, \dot{B}})$ SSP-freezes \mathcal{P} , as witnessed by the inclusion map

$$i : \mathcal{P} \longrightarrow \mathcal{P} * (\text{Coll}(\omega_1, \delta) * \dot{S}_{\vec{S}, U, \dot{B}})$$

Proof. Working in $V^{\mathcal{P} * \text{Coll}(\omega_1, \delta)}$, let $(S_\alpha)_{\alpha < \omega_1} \in V$ and $U \in V$ be as stated, and let $B_{\dot{G}}$ be a subset of ω_1 coding the generic filter G for \mathcal{P} in a canonical way. Let $\dot{\mathcal{R}}$ be a \mathcal{P} -name for $\text{Coll}(\omega_1, \delta) * \dot{S}_{\vec{S}, U, B_{\dot{G}}}$.

Claim 5. \mathcal{P} forces that $\dot{\mathcal{R}}$ has the S -condition.

Proof. Since $\text{Coll}(\omega_1, \delta)$ has the S -condition, it suffices to prove that $\mathcal{P} * \text{Coll}(\omega_1, \delta)$ forces $\dot{S}_{\vec{S}, U, \dot{B}}$ to have the S -condition. Let us work in $V^{\mathcal{P} * \text{Coll}(\omega_1, \delta)}$. Let σ be the following strategy for player II in $\mathcal{G}^{\dot{\mathcal{R}}}$: Whenever it is her turn to play, player II will alternate between the following courses of action (a), (b) (i.e., she will opt for (a) or (b) depending on the parity of the finite set $\{k < |\eta| : |\text{succ}_T(\eta \upharpoonright k)| > 1\}$, with the notation used in Definition 5.18).

- (a) Player II chooses $\kappa_\eta = \kappa^0$, $\text{succ}_T(\eta) = \{\eta \frown \langle \alpha \rangle : \alpha < \kappa^0\}$, and $(p_{\eta \frown \langle \alpha \rangle})_{\alpha < \kappa^0}$ where, for each $\alpha < \kappa^0$, $p_{\eta \frown \langle \alpha \rangle}$ is a condition extending p_η and such that $\alpha \in \bigcup \text{range}(p_{\eta \frown \langle \alpha \rangle})$.
- (b) Player II chooses $\kappa_\eta = \kappa^1$, $\text{succ}_T(\eta) = \{\eta \frown \langle \alpha \rangle : \alpha < \kappa^1\}$, and $(p_{\eta \frown \langle \alpha \rangle})_{\alpha < \kappa^1}$ where, for each $\alpha < \kappa^1$, $p_{\eta \frown \langle \alpha \rangle}$ is a condition extending p_η and such that $\alpha \in \bigcup \text{range}(p_{\eta \frown \langle \alpha \rangle})$.

Let now T be the tree built along a run of $\mathcal{G}^{\dot{\mathcal{R}}}$ in which player II has played according to σ , let T' be a subtree of T such that $|\text{succ}_{T'}(\eta)| = |\text{succ}_T(\eta)|$ for every $\eta \in T'$, and let N be a countable elementary substructure of some large enough H_θ containing all relevant objects (which includes our run of $\mathcal{G}^{\dot{\mathcal{R}}}$ and T'), such that $\text{sup}(N \cap \kappa^0) \in S_0$, and such that $\text{sup}(N \cap \kappa^1) \in U$ if $0 \in B_{\dot{G}}$ and $\text{sup}(N \cap \kappa^1) \notin U$ if $0 \notin B_{\dot{G}}$.

Such an N can be easily found (s. [10]): Indeed, suppose, for concreteness, that $0 \in B_{\dot{G}}$. Then, letting $F : [H_\theta]^{<\omega} \longrightarrow H_\theta$ be a function generating the club of countable elementary submodels of H_θ containing all relevant objects, we may find, using the stationarity of U , an ordinal $\alpha \in U$ such that the closure X_0 of $[\alpha]^{<\omega}$ under F is such that $X_0 \cap \kappa^1 = \alpha$. We may of course assume that $\alpha > \kappa^0$. Since $\text{cf}(\alpha) = \omega$, we may pick a countable cofinal subset Y of α . Using now the stationarity of S_0 , we may find $\beta \in S_0$ with the property that the closure X_1 of $[\beta \cup Y]^{<\omega}$ under F is such that $X_1 \cap \kappa^0 = \beta$. Since $\text{cf}(\beta) = \omega$, we may now pick a countable subset Z of β . But then, letting N be the closure of $Y \cup Z$ under F , we have that $\text{sup}(N \cap \kappa^0) = \beta$ and $\text{sup}(N \cap \kappa^1) = \alpha$, and so N is as desired.

³²See also the argument in the proof of Proposition 5.36 that \dot{Q} satisfies the S -condition.

Letting $(p_\eta)_{\eta \in T'}$ be the tree of $\dot{\mathcal{R}}$ -conditions corresponding to T' , it is now easy to find a cofinal branch b through T' such that for all $n < \omega$, $b \upharpoonright n \in N$, and such that

$$\sup\left(\bigcup_{n < \omega} (\cup \text{range}(p_{b \upharpoonright n}) \cap \kappa^0)\right) = \sup(N \cap \kappa^0)$$

and

$$\sup\left(\bigcup_{n < \omega} (\cup \text{range}(p_{b \upharpoonright n}))\right) = \sup(N \cap \kappa^1).$$

Let

$$\nu = \sup\{\text{dom}(p_{b \upharpoonright n}) : n < \omega\}^{33}$$

and

$$X = \bigcup_{n < \omega} (\cup \text{range}(p_{b \upharpoonright n}) \cap \kappa^1).$$

It follows now that, letting $p_b = \bigcup_n p_{b \upharpoonright n} \cup \{\langle \nu, X \rangle\}$, p_b is a condition in $\dot{\mathcal{R}}$ forcing that $p_{b \upharpoonright n}$ is in the generic filter for all n . □

Going back to V , the proof that $\mathcal{P} * (\text{Coll}(\omega_1, \delta) * \dot{\mathcal{S}}_{\vec{S}, U, \dot{B}})$ SSP-freezes \mathcal{P} (as witnessed by the inclusion map) is very much like the proofs of Propositions 5.28 and 5.34. Suppose H is a generic filter for $\mathcal{P} * (\text{Coll}(\omega_1, \delta) * \dot{\mathcal{S}}_{\vec{S}, U, \dot{B}})$, $G = H \cap \mathcal{P}$, and M is any outer model such that every stationary subset of ω_1 in $V[H]$ remains stationary in M . Suppose, towards a contradiction, that in M there is some subset $B' \neq B_G$ of ω_1 for which there is an \subseteq -increasing and \subseteq -continuous enumeration $(Z'_\nu : \nu < \omega_1)$ of a club of $[\kappa^1]^{\aleph_0}$ such that for all ν , $\alpha < \omega_1$, if $\sup(Z'_\nu \cap \kappa^0) \in S_\alpha$, then $\sup(Z'_\nu) \in U$ if and only if $\alpha \in B'$. If $\alpha \in B' \Delta B_G$, there is some ν such that $Z_\nu = Z'_\nu$ and $\sup(Z_\nu \cap \kappa^0) \in S_\alpha$. But then we have both $\sup(Z_\nu) \in U$ and $\sup(Z_\nu) \notin U$.

Finally, the existence of a Σ_1 definition—with $p = (\lambda, \vec{S}, U)$ as parameter—as required by Lemma 5.25 is easy. □

5.3.2. Iteration Lemmas. We need the following preservation lemmas, due to Shelah ([22, III, resp. VI], see also [11]).

Lemma 5.37. *Suppose $\rho < \omega_1$ is an indecomposable ordinal and*

$$(\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma)$$

is a countable support iteration such that for all $\beta < \gamma$,

$$\Vdash_{\mathcal{P}_\beta} \dot{\mathcal{Q}}_\beta \in \rho\text{-PR}$$

Then $\mathcal{P}_\gamma \in \rho\text{-PR}$.

Lemma 5.38. *Suppose $\rho < \omega_1$ is an indecomposable ordinal and*

$$(\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma)$$

is a countable support iteration such that for all $\beta < \gamma$,

$$\Vdash_{\mathcal{P}_\beta} \dot{\mathcal{Q}}_\beta \in (\rho\text{-PR}) \cap {}^\omega\omega\text{-bounding}$$

Then $\mathcal{P}_\gamma \in (\rho\text{-PR}) \cap {}^\omega\omega\text{-bounding}$.

We will also use the following preservation result due to Miyamoto.

³³Incidentally, note that we cannot guarantee that $\nu = N \cap \omega_1$.

Lemma 5.39. *Suppose $\rho < \omega_1$ is an indecomposable ordinal and*

$$(\mathcal{P}_\alpha, \dot{Q}_\beta : \alpha \leq \gamma, \beta < \gamma)$$

is a countable support iteration such that for all $\beta < \gamma$,

$$\Vdash_{\mathcal{P}_\beta} \dot{Q}_\beta \in (\rho\text{-PR}) \cap \text{STP}$$

Then $\mathcal{P}_\lambda \in (\rho\text{-PR}) \cap \text{STP}$.

Shelah defines a certain variant of the notion of countable support iteration, which he calls *revised countable support (RCS) iteration*. Variants of the notion of RCS iteration have been proposed by Miyamoto and others (for example a detailed account of RCS-iterations in line with Donder and Fuchs' approach is given in [5]).³⁴ In the following, any mention of revised countable support iteration will refer to either Shelah's or Miyamoto's version.

The first preservation result involving RCS iterations we will need is the following lemma, proved in [22, XI].

Lemma 5.40. *Suppose $\langle \mathcal{P}_\alpha, \dot{Q}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ is an RCS iteration such that the following holds for all $\beta < \gamma$.*

(1) *If β is even, $\Vdash_{\mathcal{P}_\beta} \dot{Q}_\beta = \text{Coll}(2^{|\mathcal{P}_\beta|}, \omega_1)$.*

(2) *If β is odd, $\Vdash_{\mathcal{P}_\beta} \dot{Q}_\beta$ has the S -condition.*

Then \mathcal{P}_γ has the S -condition.

The following is a well-known result of Shelah.

Lemma 5.41. *Suppose $\rho < \omega_1$ is an indecomposable ordinal and*

$$(\mathcal{P}_\alpha, \dot{Q}_\beta : \alpha \leq \gamma, \beta < \gamma)$$

is a revised countable support iteration such that for all $\beta < \gamma$,

$$\Vdash_{\mathcal{P}_\beta} \dot{Q}_\beta \in \rho\text{-SP}$$

Then $\mathcal{P}_\gamma \in \rho\text{-SP}$.

We will also need the following lemmas due to Miyamoto [17, 18].

Lemma 5.42. *Suppose CH holds, $\rho < \omega_1$ is an indecomposable ordinal, and*

$$(\mathcal{P}_\alpha, \dot{Q}_\beta : \alpha \leq \gamma, \beta < \gamma)$$

is a revised countable support iteration such that for all $\beta < \gamma$,

$$\Vdash_{\mathcal{P}_\beta} \dot{Q}_\beta \in (\rho\text{-SP}) \cap {}^\omega\omega\text{-bounding}$$

Then $\mathcal{P}_\gamma \in (\rho\text{-SP}) \cap {}^\omega\omega\text{-bounding}$.

Lemma 5.43. *Suppose $\rho < \omega_1$ is an indecomposable ordinal and*

$$(\mathcal{P}_\alpha, \dot{Q}_\beta : \alpha \leq \gamma, \beta < \gamma)$$

is a revised countable support iteration such that for all $\beta < \gamma$,

$$\Vdash_{\mathcal{P}_\beta} \dot{Q}_\beta \in (\rho\text{-SP}) \cap \text{STP}$$

Then $\mathcal{P}_\gamma \in (\rho\text{-SP}) \cap \text{STP}$.

³⁴It is not clear whether these notions are equivalent in any reasonable sense.

5.3.3. *Absolutely well-behaved classes.*

Lemma 5.44. *The following classes Γ are absolutely well-behaved and such that $\lambda_\Gamma = \omega_1$.*

- (1) ρ -PR for every countable indecomposable ordinal ρ .
- (2) $<\omega_1$ -PR
- (3) $\text{PR} \cap \text{STP}$
- (4) $\text{PR} \cap {}^\omega\omega$ -bounding
- (5) $\text{PR} \cap \text{STP} \cap {}^\omega\omega$ -bounding

Proof. Given any of these classes Γ , all conditions in the definition of absolutely well-behaved class—except for the fact that Γ has the Γ -freezability property—are clearly satisfied for Γ . In particular, Γ is defined by both a Σ_2 property and a Π_2 property by Fact 5.4,³⁵ it is closed under isomorphisms, two-step iterations, lottery sums, restrictions and complete subalgebras, and it contains all countably closed forcings by Fact 5.6. The iterability property follows from Lemmas 5.37, 5.38 and 5.39: the winning strategy for player II is to play the countable support limit at all limit stages (notice that at stages of cofinality ω_1 this limit is the direct limit). As to the freezability property, it turns out that Γ has in fact the SSP-freezability property. This follows immediately from Proposition 5.28 together with Lemma 5.26 for PR, as well as for the classes in (3), (4) and (5), and from Proposition 5.31 for ρ -PR, for any given indecomposable ordinal $\rho < \omega_1$ such that $\rho > 1$, and for $<\omega_1$ -PR. Finally, it is clear that $\lambda_\Gamma = \omega_1$ holds for each of these classes Γ . \square

We move on now to our first class not contained in PR.

Lemma 5.45. *S -cond is an absolutely well-behaved class Γ with $\lambda_\Gamma = \omega_1$.*

Proof. Let $\Gamma = S$ -cond. Except for the freezability condition, all conditions in the definition of absolutely well-behaved class are clearly satisfied by Γ : Γ is defined by both a Σ_2 property and a Π_2 property, it is closed under isomorphisms, two-step iterations, lottery sums, restrictions and complete subalgebras, and it contains all countably closed forcings by Fact 5.19. The iterability condition follows immediately from Lemma 5.40: the winning strategy for player II is to play the revised countable support limit at all limit stages (notice that at stages of cofinality ω_1 this limit is the direct limit), and to play at all non-limit stages $\alpha + 2n$ the algebra $\mathbb{B}_{\alpha+2n} = \mathbb{B}_{\alpha+2n-1} * \dot{C}$, where \dot{C} is a $\mathbb{B}_{\alpha+2n-1}$ -name for the boolean completion of $\text{Coll}(\omega_1, 2^{|\mathbb{B}_{\alpha+2n-1}|})$. As to the freezability condition, we have that S -cond has in fact, by Proposition 5.36, the SSP-freezability condition—which implies the Γ -freezability condition by Lemma 5.16 (1). Finally, it is clear that $\lambda_\Gamma = \omega_1$. \square

Lemma 5.46. *Suppose there is a proper class of measurable cardinals. Given any indecomposable ordinal $\rho < \omega_1$, each of the following classes Γ is absolutely well-behaved and such that $\lambda_\Gamma = \omega_1$.*

- (1) ρ -SP
- (2) $(\rho\text{-SP}) \cap \text{STP}$
- (3) $(\rho\text{-SP}) \cap {}^\omega\omega$ -bounding
- (4) $(\rho\text{-SP}) \cap \text{STP} \cap {}^\omega\omega$ -bounding

Also, each of the following classes is ω_1 -suitable with respect to the same theory.

- (1) $<\omega_1$ -SP
- (2) $(<\omega_1\text{-SP}) \cap \text{STP}$
- (3) $(<\omega_1\text{-SP}) \cap {}^\omega\omega$ -bounding
- (4) $(<\omega_1\text{-SP}) \cap \text{STP} \cap {}^\omega\omega$ -bounding

³⁵By Remark 3.12 this is enough to guarantee clause (5) in the definition of well-behaved class. And of course the same applies in the proofs of Lemmas 5.45 and 5.46.

Proof. Each of these Γ is defined both by a Σ_2 property and by a Π_2 property by Fact 5.9, and is closed under preimages by complete injective homomorphisms, two-step iterations and products, and contains all countably closed forcings by Fact 5.11. The iterability condition for each of these classes follows from (some combination of) Lemmas 5.41, 5.42, and 5.43: the winning strategy for player II is to play the revised countable support limit at all limit stages (notice that at stages of cofinality ω_1 this limit is the direct limit). The freezability condition follows from Lemma 5.33, together with Proposition 5.34 (for the case $\rho = 1$, one could as well invoke Lemma 5.26 together with Proposition 5.28 instead). Finally, the fact that $\lambda_\Gamma = \omega_1$ is again immediate. \square

The standard proof, due to Shelah ([21]), that SPFA implies $\text{SSP} = \text{SP}$ actually shows the following.

Proposition 5.47. *FA($(\langle \omega_1 \text{-SP} \rangle \cap \text{STP} \cap \omega\omega\text{-bounding})$) implies $\text{SSP} = \text{SP}$.*

Proof. This follows from the fact that the natural semiproper forcing $\mathcal{Q}_{\mathcal{P}}$ (s. [21]) such that an application of $\text{FA}_{\aleph_1}(\{\mathcal{Q}_{\mathcal{P}}\})$ yields the semiproperness of a given SSP \mathcal{P} is in fact $\langle \omega_1 \text{-semiproper}$, does not add new reals, and preserves Suslin trees. The proof of the first two assertions is straightforward, and the preservation of Suslin trees follows by an argument as in the final part of the proof of Lemma 5.33. \square

Corollary 5.48. *Suppose FA($(\langle \omega_1 \text{-SP} \rangle \cap \text{STP} \cap \omega\omega\text{-bounding})$) holds. Then:*

- (1) *BCFA(SSP) holds iff BCFA(SP) does.*
- (2) *BCFA(SSP \cap STP) holds iff BCFA(SP \cap STP) does.*
- (3) *BCFA(SSP \cap $\omega\omega\text{-bounding}$) holds iff BCFA(SP \cap $\omega\omega\text{-bounding}$) does.*
- (4) *BCFA(SSP \cap STP \cap $\omega\omega\text{-bounding}$) holds iff BCFA(SP \cap STP \cap $\omega\omega\text{-bounding}$) does.*

5.3.4. *Pairwise incompatibility of BCFA(Γ) for absolutely well-behaved Γ .* Each one of the incompatibilities contained in Theorem 5.21 follows from two or more of the lemmas in this subsection put together.

Given an ordinal $\alpha < \omega_2$, a function $g : \omega_1 \rightarrow \omega_1$ is a *canonical function for α* if there is a surjection $\pi : \omega_1 \rightarrow \alpha$ and a club $C \subseteq \omega_1$ such that for all $\nu \in C$, $g(\nu) = \text{ot}(\pi \restriction \nu)$. Let *Club Bounding* denote the following statement: For every function $f : \omega_1 \rightarrow \omega_1$ there is some $\alpha < \omega_2$ such that $\{\nu < \omega_1 : f(\nu) < g(\nu)\}$ contains a club whenever g is a canonical function for α .

Lemma 5.49. *Suppose δ is an inaccessible cardinal such that $V_\delta \prec V$. Let Γ be any absolutely well-behaved class, defined with a parameter in V_δ , and such that $\lambda_\Gamma = \omega_1$. Suppose $\Gamma \subseteq \text{PR}$ holds after forcing with $\text{Add}(\omega_1, 1)$. If BCFA(Γ) holds, then Club Bounding fails.*

Proof. To see that Club Bounding fails in V , we first force with $\text{Add}(\omega_1, 1)$. Let G be the corresponding generic filter. We then force with $\Gamma_\delta^{V[G]}$. By Theorem 3.14, it holds in $V[G]$ that $\Gamma_\delta^{V[G]}$ forces BCFA(Γ). Also, it is immediate to check that G adds a function $f : \omega_1 \rightarrow \omega_1$ such that

$$\{X \in [\alpha]^{\aleph_0} : X \cap \omega_1 \in \omega_1, \text{ot}(X) < f(X \cap \omega_1)\}$$

is a stationary subset of $[\alpha]^{\aleph_0}$ for every ordinal α . Since every proper forcing will preserve the stationarity of these sets and since $\Gamma_\delta^{V[G]}$ is proper in $V[G]$ by our assumption, it follows that Club Bounding fails in $(V[G])^{\Gamma_\delta}$. Since the failure of Club Bounding is expressible over H_{ω_2} , and since we have that $H_{\omega_2}^{V^{\Gamma_\delta}} = H_{\omega_2}^{V^{\text{Add}(\omega_1, 1) * \Gamma_\delta}} \prec H_{\omega_2}^{V^\Gamma}$ and $H_{\omega_2}^V \prec H_{\omega_2}^{V^\Gamma}$, it follows that Club Bounding fails in V . \square

Lemma 5.50. *BFA $_{\aleph_1}((\langle \omega_1 \text{-PR} \rangle \cap \omega\omega\text{-bounding})$) implies that there are no Suslin trees.*

Proof. Suppose, towards a contradiction, that T is a Suslin tree and the forcing axiom $\text{BFA}_{\aleph_1}((\langle \omega_1\text{-PR} \rangle \cap {}^\omega\omega\text{-bounding})$ holds. Without loss of generality we may assume that T is a normal Suslin tree. We have that T is a c.c.c. forcing which is ${}^\omega\omega$ -bounding as in fact it does not add new reals. But forcing with T adds an ω_1 -branch through T . Hence, by $\text{BFA}_{\aleph_1}(T)$, T has an ω_1 -branch and so it is not Suslin, which is a contradiction. \square

Recall that \mathfrak{d} is the minimal cardinality of a family $\mathcal{F} \subseteq {}^\omega\omega$ with the property that for every $f : \omega \rightarrow \omega$ there is some $g \in \mathcal{F}$ such that $g(n) < f(n)$ for a tail of $n < \omega$.

Lemma 5.51. *Suppose δ is an inaccessible cardinal such that $V_\delta \prec V$. Let Γ be any absolutely well-behaved class, defined with a parameter in V_δ , and such that $\lambda_\Gamma = \omega_1$. Suppose $\Gamma \subseteq {}^\omega\omega$ -bounding. If $\text{BCFA}(\Gamma)$ holds, then $\mathfrak{d} = \omega_1$.*

Proof. We know that Γ_δ forces $\text{BCFA}(\Gamma)$, and collapses $(2^{\aleph_0})^V$ to \aleph_1 . Since $\Gamma \subseteq {}^\omega\omega$ -bounding, we thus have that $V^{\Gamma_\delta} \models \mathfrak{d} = \omega_1$. But ‘ $\mathfrak{d} = \omega_1$ ’ is expressible over H_{ω_2} , and therefore $V \models \mathfrak{d} = \omega_1$ by the same absoluteness argument as in the proof of Lemma 5.49. \square

Lemma 5.52. $\text{BFA}_{\aleph_1}((\langle \omega_1\text{-PR} \rangle \cap \text{STP})$ implies $\mathfrak{d} > \omega_1$.

Proof. This is immediate since Cohen forcing, being countable, preserves Suslin trees. \square

Lemma 5.53. *Suppose $\text{BFA}_{\aleph_1}((\langle \omega_1\text{-SP} \rangle \cap \text{STP} \cap {}^\omega\omega\text{-bounding})$ holds and there is a measurable cardinal. Then ψ_{AC} holds, and therefore Club Bounding holds as well.*

Proof. $\text{BFA}_{\aleph_1}((\langle \omega_1\text{-SP} \rangle \cap \text{STP} \cap {}^\omega\omega\text{-bounding})$ implies ψ_{AC} by Lemma 5.33. To see that ψ_{AC} implies Club Bounding, see for example Fact 3.1 in [4]. \square

The following lemma is an immediate consequence of Lemma 5.53.

Lemma 5.54. *Suppose δ is an inaccessible cardinal such that $V_\delta \prec V$. Suppose there is a proper class of measurable cardinals. Let Γ be an absolutely well-behaved class with $\lambda_\Gamma = \omega_1$ defined from a parameter in V_δ and such that $(\langle \omega_1\text{-SP} \rangle \cap \text{STP} \cap {}^\omega\omega\text{-bounding}) \subseteq \Gamma$ holds in any generic extension by a member of Γ . If $\text{BCFA}(\Gamma)$ holds, then ψ_{AC} , and therefore also Club Bounding, hold as well.*

Proof. Γ_δ forces $\text{BFA}_{\aleph_1}(\Gamma)$ and therefore also $\text{BFA}_{\aleph_1}((\langle \omega_1\text{-SP} \rangle \cap \text{STP} \cap {}^\omega\omega\text{-bounding})$. Hence, $H_{\omega_2}^{V^{\Gamma_\delta}} \models \psi_{AC}$ by Lemma 5.53. But then $V \models \psi_{AC}$ by the usual absoluteness argument. \square

Remark 5.55. The conclusion, in Lemma 5.54, that Club Bounding holds is equiconsistent with the existence of an inaccessible limit of measurable cardinals [9]. In fact, if there is no inner model with an inaccessible limit of measurable cardinals, then Club Bounding fails.

Lemma 5.56 follows trivially from the fact that $\text{BFA}_{\aleph_1}(\langle \omega_1\text{-PR} \rangle)$ implies MA_{ω_1} .

Lemma 5.56. $\text{BFA}_{\aleph_1}(\langle \omega_1\text{-PR} \rangle)$ implies $2^{\aleph_0} = 2^{\aleph_1}$.

A partial order \mathbb{P} is said to have the σ -bounded chain condition if $\mathbb{P} = \bigcup_{n < \omega} \mathbb{P}_n$ and for each n there is some $k_n < \omega$ such that for every $X \in [\mathbb{P}_n]^{k_n}$ there are distinct $p, p' \in \mathbb{P}_n$ which are compatible in \mathbb{P} . Also, a partial order \mathbb{P} is *Knaster* if every uncountable subset of \mathbb{P} contains an uncountable subset consisting of pairwise compatible conditions in \mathbb{P} .

It is easy to see, and a well-known fact, that random forcing preserves Suslin trees. This follows from the fact that random forcing has the σ -bounded chain condition, that every forcing with the σ -bounded chain condition is Knaster, and that every Knaster forcing preserves Suslin trees.

Lemma 5.57 follows from the above, together with the fact that random forcing is ${}^\omega\omega$ -bounding and adds a new real.

Lemma 5.57. $\text{BFA}_{\aleph_1}((\langle \omega_1\text{-PR} \rangle \cap \text{STP} \cap {}^\omega\omega\text{-bounding})$ implies $\neg \text{CH}$.

Lemma 5.58. *Suppose δ is an inaccessible cardinal such that $V_\delta \prec V$. Let Γ be any absolutely well-behaved class, defined with a parameter in V_δ , and such that $\lambda_\Gamma = \omega_1$. Suppose $\Gamma \subseteq S\text{-cond}$ holds after forcing with $\text{Add}(\omega_1, 1)$. If $\text{BCFA}(\Gamma)$ holds, then so does CH .*

Proof. Let G be $\text{Add}(\omega_1, 1)$ -generic, and note that $V[G]$ satisfies CH . Then force $\text{BCFA}(\Gamma)$ over $V[G]$ via Γ_δ . Let V_1 be this extension. Since, in $V[G]$, Γ_δ has the S -condition, forcing with $\Gamma_\delta^{V[G]}$ over $V[G]$ did not add new reals thanks to Lemma 5.16 (2). In particular, $V_1 \models \text{CH}$. But then CH holds in V by the usual absoluteness argument. \square

It turns out that $\text{BCFA}(\Gamma)$, where Γ is any absolutely well-behaved class such that $\lambda_\Gamma = \omega_1$ and $\Gamma \subseteq S\text{-cond}$ holds after adding a Cohen subset of ω_1 actually implies \diamond . The proof is essentially the same as above, using the following recent result due to Magidor, together with the fact that if $V \subseteq V_1 \subseteq W$ are models with the same ω_1 and $\vec{X} \in V$ is a \diamond -sequence in W , then \vec{X} is also a \diamond -sequence in V_1 .

Theorem 5.59. (Magidor) *Suppose \diamond holds. Then there is a \diamond -sequence that remains a \diamond -sequence after any forcing with the S -condition.*

The following well-known fact can be proved by an argument as in the final part of the proof of Lemma 5.33.

Fact 5.60. *If \mathcal{P} is countably closed, then \mathcal{P} preserves Suslin trees.*

The proof of the following lemma is like the proofs of Lemmas 5.49, 5.51, and 5.58, using the well-known fact that $\text{Add}(\omega_1, 1)$ adds a Suslin tree T .

Lemma 5.61. *Suppose δ is an inaccessible cardinal such that $V_\delta \prec V$. Let Γ be any absolutely well-behaved class, defined with a parameter in V_δ , and such that $\lambda_\Gamma = \omega_1$. Suppose $\Gamma \subseteq \text{STP}$ holds after forcing with $\text{Add}(\omega_1, 1)$. If $\text{BCFA}(\Gamma)$ holds, then there is a Suslin tree.*

It will be convenient to consider the following families of Club-Guessing principles on ω_1 (s. [2]).

Definition 5.62. Let $\tau < \omega_1$ be a nonzero ordinal.

- (1) τ -TWCG denotes the following statement: There is a sequence

$$\vec{C} = (C_\delta : \delta = \omega^\tau \cdot \eta \text{ for some nonzero } \eta < \omega_1)$$

such that $|\{C_\delta \cap \gamma : \gamma < \omega_1\}| \leq \aleph_0$ for every $\delta \in \text{dom}(\vec{C})$, and such that for every club $C \subseteq \omega_1$ there is some $\delta \in \text{dom}(\vec{C})$ with $\text{ot}(C_\delta \cap C) = \omega^\tau$.

- (2) τ -TCG denotes the following statement: There is a sequence

$$\vec{C} = (C_\delta : \delta = \omega^\tau \cdot \eta \text{ for some nonzero } \eta < \omega_1)$$

such that $|\{C_\delta \cap \gamma : \gamma < \omega_1\}| \leq \aleph_0$ for every $\delta \in \text{dom}(\vec{C})$, and such that for every club $C \subseteq \omega_1$ there is some $\delta \in \text{dom}(\vec{C})$ with $C_\delta \subseteq C$.

In the above definition, TWCG and TCG stand for *thin weak club-guessing* and *thin club-guessing*, respectively.

Lemma 5.63. *Let $\tau < \omega_1$ be a nonzero ordinal. Then*

$$\text{BFA}_{\aleph_1}((\omega^\tau\text{-PR}) \cap \text{STP} \cap {}^\omega\omega\text{-bounding})$$

implies the failure of τ' -TWCG for every τ' such that $\tau < \tau' < \omega_1$.

Proof. Let us consider the following natural forcing $\mathcal{P}_{\vec{C}}$ for killing an instance

$$\vec{C} = (C_\delta : \delta = \omega^\tau \cdot \eta \text{ for some nonzero } \eta < \omega_1)$$

of τ' -TWCG: $\mathcal{P}_{\vec{C}}$ is the set, ordered by reverse end-extension, of countable closed subset c of ω_1 such that $\text{ot}(C_\delta \cap c) < \omega^{\tau'}$ for every $\delta \in \text{dom}(\vec{C})$. It is proved in [2] that $\mathcal{P}_{\vec{C}}$ is ω^τ -proper, does not add new reals, and adds a club $C \subseteq \omega_1$ such that $\text{ot}(C_\delta \cap C) < \omega^{\tau'}$ for every $\delta \in \text{dom}(\vec{C})$. Hence, it only remains to prove that $\mathcal{P}_{\vec{C}}$ preserves Suslin trees. This can be shown by an argument similar to the main argument in the proof in [16] that MRP-posets preserve Suslin tree. We present the argument here for the reader's convenience.

Suppose U is a Suslin tree, \dot{A} is a \mathcal{Q} -name for a maximal antichain of U , and N is a countable elementary submodel of some large enough H_θ containing U , \dot{A} , and all other relevant objects. Let $\delta = N \cap \omega_1$. As in the last part of the proof of Lemma 5.33, let $(u_n)_{n < \omega}$ enumerate all nodes in U of height δ . Given a condition $c \in \mathcal{P}_{\vec{C}} \cap N$, we aim to build an $(N, \mathcal{P}_{\vec{C}})$ -generic sequence $(c_n)_{n < \omega}$ of conditions in N extending c such that for every n there is some $v \in U$ below u_n such that c_{n+1} forces $v \in \dot{A}$. We will make sure that $C_\delta \cap \bigcup_{n < \omega} c_n \subseteq c$, which will guarantee that $c^* = \bigcup_{n < \omega} c_n \cup \{\delta\} \in \mathcal{P}_{\vec{C}}$. But this will be enough, as then c^* will be an extension of c in $\mathcal{P}_{\vec{C}}$ forcing $\dot{A} \subseteq U \cap N$.

It thus remains to show how to find c_{n+1} given c_n . Working in N , we may first fix some countable $M \preceq H_\chi$ (for some large enough χ) containing c_n and all other relevant objects (including some relevant dense set $D \subseteq \mathcal{P}_{\vec{C}}$ that we need to meet), and such that $[\eta, \delta_M] \cap C_\delta = \emptyset$, where $\delta_M = M \cap \omega_1$. In order to find M , we first consider a strictly \subseteq -increasing and continuous sequence $(M_\nu)_{\nu < \omega_1} \in N$ of elementary submodels containing all relevant objects. Since $(M_\nu \cap \omega_1)_{\nu < \delta}$ is a club of δ of order type δ and $\text{ot}(C_\delta) = \omega^{\tau'} < \delta$, we can then find some $\nu < \delta$ such that $M = M_\nu$ is as desired. Now, working in M , we may, first, extend c_n to a condition c'_n such that $\max(c'_n) > \eta$ and $[\max(c_n), \eta] \cap c'_n = \emptyset$, and then extend c'_n to a condition c''_n in D . Let now \bar{u} be the unique node in U below u_n of height δ_M . Since U is a Suslin tree, we have that u_n is totally (U, M) -generic. Also, the set $E \in N$ of $u \in U$ for which there is some $v \in U$ below u and some $\bar{c} \in \mathcal{P}_{\vec{C}}$ extending c''_n and forcing that $v \in \dot{A}$ is dense in U . It follows that we may find some $u \in E \cap M$ below u_n , as witnessed by some $\bar{c} \in \mathcal{P}_{\vec{C}} \cap N$ and some $v \in U \cap M$. But then we may let $c_{n+1} = \bar{c}$. \square

Lemma 5.64. *Suppose δ is an inaccessible cardinal such that $V_\delta \prec V$. Let Γ be any absolutely well-behaved class, defined with a parameter in V_δ , and such that $\lambda_\Gamma = \omega_1$.*

Let $\tau < \omega_1$ be a nonzero ordinal, and suppose $\Gamma \subseteq \omega^\tau$ -SP holds in any generic extension by countably closed forcing. If BCFA(Γ) holds, then so does τ -TCG.

Proof. By a result in [31], there is a countably closed forcing notion adding a τ -TCG-sequence. The rest of the argument is as in the proof of Lemma 5.49 (and subsequent lemmas), using the preservation of τ -TCG-sequences by any ω^τ -semiproper forcing, which is a completely standard fact. \square

5.4. Bounded category forcing axioms and stronger large cardinal assumptions.

As we know, the theory of H_{ω_2} given by the bounded category forcing axioms we have explored in this section is invariant, in the presence of reasonable large cardinals, relative to extensions via members in the corresponding class forcing CFA(Γ). In this final subsection we show that the combinatorial theory of H_{ω_2} given by these axioms is nevertheless sensitive to additional background large cardinal assumptions.³⁶

Recall that δ_2^1 is the supremum of the the set of lengths of Δ_2^1 -definable pre-well-orderings on \mathbb{R} .

³⁶See also Remark 5.55.

Proposition 5.65. *Suppose δ is an inaccessible cardinal such that $V_\delta \prec V$. Suppose there is a proper class of Woodin cardinals. Let Γ be any absolutely well-behaved class, defined with a parameter in V_δ , such that $\lambda_\Gamma = \omega_1$ and $\Gamma \subseteq \text{PR}$. Suppose $\text{BCFA}(\Gamma)$ holds. Then $\delta_2^1 < \omega_2$.*

Proof. By a result of Neeman and Zapletal [20], the existence of a proper class of Woodin cardinals yields that if \mathcal{P} is a proper poset and G is \mathcal{P} -generic over V , then the identity on $L(\mathbb{R})^V$ is an elementary embedding between $L(\mathbb{R})^V$ and $L(\mathbb{R})^{V[G]}$. Since Γ_δ is proper, it follows that $V^{\Gamma_\delta} \models \delta_2^1 < \omega_2$. Since ‘ $\delta_2^1 < \omega_2$ ’ is expressible over H_{ω_2} , we then have that $V \models \delta_2^1 < \omega_2$ by the usual absoluteness argument. \square

Recall that the Strong Reflection Principle (SRP) is the following assertion: For every set X such that $\omega_1 \subseteq X$ and every $S \subseteq [X]^{\aleph_0}$ there is a strong reflecting sequence $(x_i)_{i < \omega_1}$ for S , i.e., $x_i \in [X]^{\aleph_0}$, $(x_i)_{i < \omega_1}$ is strictly \subseteq -increasing and \subseteq -continuous, and for all i , $x_i \notin S$ if and only if there is no $y \in S$ such that $x_i \subseteq y$ and $y \cap \omega_1 = x_i \cap \omega_1$.

Lemma 5.66. *$\text{FA}_{\aleph_1}(\text{SP} \cap \text{STP} \cap \omega\omega\text{-bounding})$ implies $\delta_2^1 = \omega_2$.*

Proof. We have that $\text{FA}_{\aleph_1}(\text{SP} \cap \text{STP} \cap \omega\omega\text{-bounding})$ implies SRP since, given $S \subseteq [X]^{\aleph_0}$, the standard forcing for adding a strong reflecting sequence for S is semiproper, does not add reals, and preserves Suslin trees, where the last fact follows from an argument as in the final part of the proof of Lemma 5.33. Also, SRP implies $\neg \square_\kappa$, for every cardinal $\kappa \geq \omega_1$, and hence implies that the universe is closed under sharps. Since it also implies the saturation of NS_{ω_1} , by a classical result of Woodin ([30]) it implies $\delta_2^1 = \omega_2$. \square

Corollary 5.67. *Suppose δ is a supercompact cardinal such that $V_\delta \prec V$. Let Γ be any absolutely well-behaved class, defined with a parameter in V_δ , such that $\lambda_\Gamma = \omega_1$ and such that $\text{SP} \cap \text{STP} \cap \omega\omega\text{-bounding} \subseteq \Gamma$ forces in every generic extension via any member from Γ . Suppose $\text{BCFA}(\Gamma)$ holds. Then $\delta_2^1 = \omega_2$.*

Proof. We know that Γ_δ forces $\text{FA}_{\aleph_1}(\Gamma)$ and therefore, by our assumption, it also forces $\text{FA}_{\aleph_1}(\text{SP} \cap \text{STP} \cap \omega\omega\text{-bounding})$. But then $V \models \delta_2^1 = \omega_2$ by Lemma 5.66 together with the usual absoluteness argument. \square

Question 5.68. Does $\text{FA}_{\aleph_1}(\omega\text{-SP})$ imply $\delta_2^1 = \omega_2$?

6. APPENDIX

We collect here a few results (without proofs) translating the approach to forcing and iterations via posets to that done via complete boolean algebras; for details see the forthcoming [5] or [29] (the latter is available on ArXiv).

Given a boolean algebra \mathbf{B} and a prefilter G on \mathbf{B} (i.e., a family such that $\bigwedge F > 0_{\mathbf{B}}$ for all finite $F \subseteq G$), we denote by \mathbf{B}/G the quotient algebra obtained using the ideal $J = \{b : b \wedge c = 0_{\mathbf{B}} \text{ for all } c \in G\}$. \mathbf{B}^+ denotes the positive elements of \mathbf{B} and $\dot{G}_{\mathbf{B}} = \{\check{b}, b : b \in \mathbf{B}\}$ is the canonical \mathbf{B} -name for the V -generic filter.

Theorem 6.1. *Assume (P, \leq) is a partial order. Let $\text{RO}(P)$ denote the family of regular open sets for the topology τ_P whose open sets are the downward closed subsets of P . The function $i_p : P \rightarrow \text{RO}(P)$ given by $i_p : p \mapsto \text{Reg}(\downarrow p)$ (where $\downarrow p = \{q \in P : q \leq p\}$ and $\text{Reg}(A)$ is the interior of the closure of A for the topology generated by the sets $\downarrow p$) is an order and incompatibility preserving map of (P, \leq) into $\text{RO}(P)^+$ with image dense in $\text{RO}(P)^+$, and is such that $p \Vdash_P \phi(\tau_1, \dots, \tau_n)$ if and only if $i_P(p) \leq \llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\text{RO}(P)}$.*

6.1. Two-step iterations. Given \dot{C} , a \mathbf{B} -name for a forcing notion, by $\mathbf{B} * \dot{C}$ we intend the two-step iteration as defined in [14, Section VIII.5] (or, equivalently—when \dot{C} is a \mathbf{B} -name for a cba—according to [13, Lemma 16.3], which is actually more in line with the

approach pursued in [5] or [29]). To simplify notation we also feel free to confuse (in some occasions) a partial order with its boolean completion.

Theorem 6.2. *If $i : \mathbb{B} \rightarrow \mathbb{C}$ is an injective complete homomorphism of complete boolean algebras, then*

$$\mathbb{B} * (\mathbb{C}/_{i[\dot{G}_{\mathbb{B}}]}) \cong \mathbb{C}.$$

Conversely, if $\dot{Q} \in V^{\mathbb{B}}$ is a \mathbb{B} -name for a complete boolean algebra and G is V -generic for \mathbb{B} , then

$$(\mathbb{B} * \dot{Q})/_{i_{\mathbb{B} * \dot{c}}[G]} \cong \dot{Q}_G.$$

Proposition 6.3. *Let \dot{C}_0, \dot{C}_1 be \mathbb{B} -names for complete boolean algebras, and let \dot{k} be a \mathbb{B} -name for a complete homomorphism from \dot{C}_0 to \dot{C}_1 . Then there is a complete homomorphism $i : \mathbb{B} * \dot{C}_0 \rightarrow \mathbb{B} * \dot{C}_1$ such that*

$$\llbracket \dot{k} = i/\dot{G}_{\mathbb{B}} \rrbracket = 1_{\mathbb{B}}.$$

Moreover if \dot{k} is a \mathbb{B} name for an injective homomorphism, i is injective.

In the following diagrams we assume G is V -generic for \mathbb{B} .

$$\begin{array}{ccc} \begin{array}{c} (\dot{C}_0)_G \\ \downarrow (k)_G \\ (\dot{C}_1)_G \end{array} & V : \begin{array}{ccc} \mathbb{B} & \xrightarrow{i_0} & \mathbb{B} * \dot{C}_0 \\ & \searrow i_1 & \downarrow i \\ & & \mathbb{B} * \dot{C}_1 \end{array} & V[G] : \begin{array}{ccc} (\mathbb{B} * \dot{C}_0)/_{i_0[G_{\mathbb{B}}]} & \xrightarrow{\cong} & (\dot{C}_0)_G \\ \downarrow i/G_{\mathbb{B}} & & (k)_G \downarrow \\ (\mathbb{B} * \dot{C}_1)/_{i_1[G_{\mathbb{B}}]} & \xrightarrow{\cong} & (\dot{C}_1)_G \end{array} \end{array}$$

Proposition 6.4. *Let G be V -generic for \mathbb{B} , I be its dual ideal, and $i_j : \mathbb{B} \rightarrow \mathbb{C}_j$ be complete homomorphisms for $j = 0, 1$.*

Assume $\mathbb{C}_0/_{i_0[G]}$ and $\mathbb{C}_1/_{i_1[G]}$ are isomorphic complete boolean algebras in $V[G]$.

$$\begin{array}{ccc} V : & & V[G] : \\ \begin{array}{ccc} \mathbb{B} & \xrightarrow{i_0} & \mathbb{C}_0 \\ & \searrow i_1 & \downarrow \\ & & \mathbb{C}_1 \end{array} & & \begin{array}{ccc} \mathbb{C}_0/_{i_0[G]} & & \\ \downarrow k \cong & & \\ \mathbb{C}_1/_{i_1[G]} & & \end{array} \end{array}$$

Then $\mathbb{C}_0 \upharpoonright i_0(b)$ and $\mathbb{C}_1 \upharpoonright i_1(b)$ are isomorphic in V for some $b \in G$ and $k \cong l/G$.

$$\begin{array}{ccc} V[G] : & & V : \\ \begin{array}{ccc} \mathbb{C}_0/_{i_0[G]} = \mathbb{C}_0 \upharpoonright i_0(b)/_{i_0[G]} \\ \downarrow k \cong & & \downarrow l/G \cong \\ \mathbb{C}_1/_{i_1[G]} = \mathbb{C}_1 \upharpoonright i_1(b)/_{i_1[G]} & & \end{array} & & \begin{array}{ccc} \mathbb{B} & \xrightarrow{i_0} & \mathbb{C}_0 \xrightarrow{\text{rest}} & \mathbb{C}_0 \upharpoonright i_0(b) \\ & \searrow i_1 & \downarrow l \cong & \downarrow \\ & & \mathbb{C}_1 \xrightarrow{\text{rest}} & \mathbb{C}_1 \upharpoonright i_1(b) \end{array} \end{array}$$

Lemma 6.5. *Assume Γ is a definable class of forcings. Let $\mathbb{B}, \mathbb{C}_0, \mathbb{C}_1$ be complete boolean algebras, and let G be any V -generic filter for \mathbb{B} . Let i_0, i_1, j be Γ -correct complete homomorphisms in V forming a commutative diagram of injective complete homomorphisms of complete boolean algebras as in the following picture:*

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{i_0} & \mathbb{C}_0 \\ & \searrow i_1 & \downarrow j \\ & & \mathbb{C}_1 \end{array}$$

Then in $V[G]$, $j/G : C_0/G \rightarrow C_1/G$ is still a $\Gamma^{V[G]}$ -correct complete homomorphism.

Proposition 6.6. *Assume Γ is a definable class of forcings. Let G be V -generic for some complete boolean algebra B . Assume $k : B \rightarrow R$ is a Γ -correct homomorphism in V and $h : R/k[G] \rightarrow Q$ is a Γ -correct homomorphism in $V[G]$.*

$$V : \quad B \xrightarrow{k} R \qquad V[G] : \quad \begin{array}{c} R/k[G] \\ \downarrow h \\ Q \end{array}$$

Then there are in V :

- $C \in \Gamma$,
- a Γ -correct homomorphism $l : B \rightarrow C$,
- a Γ -correct homomorphism $\bar{h} : R \rightarrow C$,

such that:

- Q is isomorphic to $C/l[G]$ in $V[G]$,
- $\bar{h}/G \cong h$ (modulo the isomorphism of Q with $C/l[G]$) holds in $V[G]$,
- $\bar{h} \circ k = l$ holds in V ,
- $0_C \notin l[G]$.

$$V : \quad \begin{array}{ccc} B & \xrightarrow{k} & R \\ & \searrow l & \downarrow \bar{h} \\ & & C \end{array} \qquad V[G] : \quad \begin{array}{ccc} 2 \cong B/G & \xrightarrow[k/G \cong \text{Id}]{} & R/k[G] \\ & \searrow l/G \cong \text{Id} & \downarrow \bar{h}/G \cong h \\ & & C/l[G] \cong Q \end{array}$$

6.2. Limit length iterations.

Theorem 6.7. *Assume $i : B \rightarrow C$ is a complete injective homomorphism of complete boolean algebra. Then the following holds:*

- (1) i has an adjoint $\pi_i : C \rightarrow B$ defined by $\pi_i(c) = \bigwedge \{b \in B : i(b) \geq c\}$
- (2) For any $b \in B$ and $c, d \in C$, we have that:
 - (a) π_i is order preserving;
 - (b) $(\pi_i \circ i)(b) = b$, hence π_i is surjective;
 - (c) $(i \circ \pi_i)(c) \geq c$; in particular, π_i maps C^+ to B^+ ;
 - (d) π_i preserves joins, i.e., $\pi_i(\bigvee X) = \bigvee \pi_i[X]$ for all $X \subseteq C$ for which the supremum $\bigvee X$ exists in C ;
 - (e) $i(b) = \bigvee \{e : \pi_i(e) \leq b\}$;
 - (f) $\pi_i(c \wedge i(b)) = \pi_i(c) \wedge b = \bigvee \{\pi_i(e) : e \leq c, \pi_i(e) \leq b\}$;
 - (g) π_i preserves neither meets nor complements whenever i is not surjective, but $\pi_i(d \wedge c) \leq \pi_i(d) \wedge \pi_i(c)$ and $\pi_i(\neg c) \geq \neg \pi_i(c)$.

Definition 6.8. $\mathcal{F} = \{i_{\alpha\beta} : B_\alpha \rightarrow B_\beta \mid \alpha \leq \beta < \lambda\}$ is a complete iteration system of complete boolean algebras iff for all $\alpha \leq \beta \leq \gamma < \lambda$:

- (1) B_α is a complete boolean algebra and $i_{\alpha\alpha}$ is the identity on it;
- (2) $i_{\alpha\beta}$ is a complete injective homomorphism and $\pi_{\alpha\beta} : B_\beta \rightarrow B_\alpha$, given by $c \mapsto \bigwedge \{b \in B : i_{\alpha\beta}(b) \geq c\}$, is its associated adjoint;
- (3) $i_{\beta\gamma} \circ i_{\alpha\beta} = i_{\alpha\gamma}$.

Let \mathcal{F} be a complete iteration system of length λ . Then:

- The *inverse limit* of the iteration is

$$\varprojlim(\mathcal{F}) = \left\{ f \in \prod_{\alpha < \lambda} \mathbf{B}_\alpha : \forall \alpha \forall \beta > \alpha (\pi_{\alpha\beta} \circ f)(\beta) = f(\alpha) \right\}$$

and its elements are called *threads*.

- The *direct limit* is

$$\varinjlim(\mathcal{F}) = \{ f \in \varprojlim(\mathcal{F}) : \exists \alpha \forall \beta > \alpha f(\beta) = i_{\alpha\beta}(f(\alpha)) \}$$

and its elements are called *constant threads*. The support of a constant thread, $\text{supp}(f)$, is the least α such that $(i_{\alpha\beta} \circ f)(\alpha) = f(\beta)$ for all $\beta \geq \alpha$.

- The *revised countable support limit* is³⁷

$$\lim_{\text{rcs}}(\mathcal{F}) = \{ f \in \varprojlim(\mathcal{F}) : f \in \varinjlim(\mathcal{F}) \vee \exists \alpha f(\alpha) \Vdash_{\mathbf{B}_\alpha} \text{cof}(\check{\lambda}) = \check{\omega} \}.$$

Theorem 6.9. Assume $\{P_\alpha, \dot{Q}_\beta : \alpha \leq \lambda, \beta < \lambda\}$ is an iteration of posets. Let $i_{\alpha\beta} : \text{RO}(P_\alpha) \rightarrow \text{RO}(P_\beta)$ be the complete homomorphism induced by the natural inclusion of P_α into P_β . Then:

- $\mathcal{F} = \{i_{\alpha,\beta} : \alpha \leq \beta < \lambda\}$ is an iteration system of complete injective homomorphisms of complete boolean algebras.
- If $\lambda = \omega$, $\varprojlim(\mathcal{F})$ is isomorphic to the boolean completion of the full limit of $\{P_\alpha, \dot{Q}_\beta : \alpha \leq \lambda, \beta < \lambda\}$.
- For any regular λ , $\varinjlim(\mathcal{F})$ is isomorphic to the boolean completion of the direct limit of $\{P_\alpha, \dot{Q}_\beta : \alpha \leq \lambda, \beta < \lambda\}$.

Theorem 6.10 (Baumgartner). Let λ be a regular cardinal and $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system such that \mathbf{B}_α is $<\lambda$ -cc for all α and $S = \{\alpha : \mathbf{B}_\alpha \cong \text{RO}(\varinjlim(\mathcal{F} \upharpoonright_\alpha))\}$ is stationary. Then $\varinjlim(\mathcal{F})$ is $<\lambda$ -cc.

These results suffice to prove all the needed equivalences (used in this paper) between results on forcing and iterations proved in the language of partial orders and their corresponding formulation in terms of complete boolean algebras.

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³⁷This definition of revised countable support limit is due to Donder and Fuchs; it is not clear whether it is in any sense equivalent to Miyamoto or Shelah's definition of rcs-limit. It is nonetheless effective to prove the main results about semiproper iterations: specifically, the forthcoming [5] or [29] (which is available on ArXiv) give complete proofs of the preservation of semiproperness through RCS-iterations. However, in this paper we also want to preserve properties stronger than semiproperness. The proofs of such preservation results are not given in [5] or in [29]; hence we refer the reader to Miyamoto's or Shelah's iteration theorems for the preservation through RCS-limits (according to their definitions) of these stronger properties.

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