PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

LECTURE 14

28/6/2022

① Establish a natural correspondance ①
between $C(St(B), 2^\omega) = \{ f : St(B) \to 2^\omega : f \text{ continuous} \}$

$\backsimeq$

$\{ \tau \in V^B : \tau \text{ is a } B\text{-name for an element of } 2^\omega \}$

$[\![\tau \in \check{\omega} \to \check{2} \text{ is a function}]\!]_B = 1_B$

in this correspondance universal Baireness detects subset R of $(2^\omega)^k$ which allows to show that $(2^\omega, R) \underset{\text{Boolean}}{\overset{?}{\sqsubseteq}} \left(C(St(B), 2^\omega)/G, R^*/G \right)$

$$R^* \leftarrow \mathcal{L}(\beta_1 \dots \beta_k) = \{G : R(\beta_1(G) \dots \beta_k(G))\}$$

(2)

$$\cancel{\mathcal{B}}(B) \cong \mathrm{CLOP}(St(B))$$

$$\shortparallel$$

$$\mathrm{RO}(St(B))$$

$$\beta_1 \times \dots \times \beta_k : St(B) \to (2^\omega)^k$$

$$\underbrace{\cancel{(\beta_1 \dots G_k)}}_{G} \mapsto (\beta_1(G) \dots \beta_k(G))$$

Def: Given $\tau \in V^B$ s.t. $[\![\tau : \check{\omega} \to \check{2} \text{ is a function}]\!]_B = 1_B$ (2)

we let $f_\tau : St(B) \to 2^\omega$

$G \mapsto z$ with $z(n) = i$ iff $[\![\tau(\check{n}) = \check{i}]\!] \in G$

$\{z \in 2^\omega : z(n) = i\}$

$f_\tau[N_{n,i}] = \{G : [\![\tau(\check{n}) = \check{i}]\!] \in G\} = N_{[\![\tau(\check{n}) = \check{i}]\!]}$

$[\![\tau(\check{n}) = \check{0}]\!] \vee [\![\tau(\check{n}) = \check{1}]\!] = 1_B$

$[\![\tau(\check{n}) = \check{0}]\!] \wedge [\![\tau(\check{n}) = \check{1}]\!] = 0_B$

Given $f : St(B) \to 2^\omega$ (3)

$\sigma_f \in V^B \quad [\![\sigma_f : \check{\omega} \to \check{2}]\!]_B = 1_B$

$[\![\sigma_f(\check{n}) = \check{i}]\!] = f^{-1}[N_{n,i}]$

$CLOP(St(B)) \cong B$

$1_B = [\![\sigma_f : \check{\omega} \to \check{2} \text{ is a function}]\!]$

$\Uparrow$

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$\sigma_f = \{ \langle \langle \check{n}, \check{i} \rangle, f^{-1}[N_{n,i}] \rangle : n \in \omega, i \in 2 \}$

1_B

$\shortparallel$

$[\![\sigma_f(\check{n}) = \check{0}]\!] \vee [\![\sigma_f(\check{n}) = \check{1}]\!] = f^{-1}[N_{n,0}] \cup f^{-1}[N_{n,1}] = St(B)$

$[\![\sigma_f(\check{n}) = \check{0}]\!] \wedge [\![\sigma_f(\check{n}) = \check{1}]\!] = \qquad \cap \qquad = \emptyset = 0_B$

④

<u>Fact</u> Given τ s.t. $[\![\tau : \check{\omega} \to 2]\!] = 1_B$

$$\cancel{\tau_{b_{G}}}\ [\![G_{b_\tau} = \tau]\!]_B = 1_B$$

Given $g : St(B) \to 2^\omega$ continuous

$$b_{G} = g$$

$$g(G)(n) = c \iff [\![G(\check{n}) = \check{c}]\!] \in G \iff G \in b_{G}^{-1}[N_{n,c}]$$

$$\Updownarrow$$

$$b_{G}(G)(n) = c$$

$$C(St(B), 2^\omega) \approx \{\tau \in V^B : [\![\tau : \check{\omega} \to \check{2}]\!] = 1_B\}$$

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$$SS$$

$$\{\sigma_{f_\tau} : [\![\tau : \check{\omega} \to \check{2}]\!] = 1_B\}$$

if R is a boolean relation $\mathcal{R}$ on

$$\cancel{M}_\Sigma\ C(St(B), 2^\omega) \qquad [\![f = g]\!]_B = \mathrm{Reg}^{C(St(B), 2^\omega)}(\{G : f(G) = g(G)\})$$

$$\cap$$

$$(C(St(B), 2^\omega), [\![\cdot = \cdot]\!]) \qquad RO(St(B)) \cong B$$

is a B-model for $\{=\}$ $\quad$ B is a cba $\to$ $\|$

$$CLOP(St(B)) \cong B$$

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$$[\![f = g]\!] \wedge [\![g = h]\!]$$

$$\mathrm{Reg}(\{q : f(q) = g(q)\}) \cap \mathrm{Reg}(\{q : h(q) = g(q)\})$$

$$\cap|$$

$$\mathrm{Reg}(\{q : f(q) = h(q)\})$$

$$\|$$

$$[\![f = h]\!]$$

⑦

Now if $R \subseteq (2^\omega)^k$ is UB

we get thm

$$\left(C(St(\mathbb{B}), 2^\omega) \cancel{/\equiv} [- = -] , R^B \right)$$

$$R^B(f_2 \dots f_k) = \operatorname{Reg}\big(\underbrace{\{G : R(f_2(G) \dots f_k(G))\}}_{A}\big) \quad \operatorname{Int}(\operatorname{Cl}(A)) =$$

$$[\![f_1 = g]\!] \wedge [\![R(f_1, f_2 \dots f_k)]\!] \leq [\![R(g, f_2, \dots f_k)]\!]$$

$$\neg [\![R(f_2 \dots f_k)]\!] = \cancel{St(B) \setminus} \operatorname{Reg}\big(\{G : \neg R(f_2(G) \dots f_k(G))\}\big)$$

$$= St(B) \setminus \operatorname{Reg}(\{G : \overset{N}{R}(f_2(G) \dots f_k(G)\})$$

$$[\![b = g]\!] = \mathrm{Reg}\left(\{ G : b(G) = g(G) \}\right)$$ ⑧

$$\|$$

$$\cancel{[\![\sigma_b = \sigma_g]\!] = \bigvee \{ G}$$

$$[\![\sigma_b = \sigma_g]\!]_B^{V^B}$$

$$(2^\omega)^B$$

$$\|$$

$$\left(C(St(B), 2^\omega), [\![\cdot = \cdot]\!] \right) \cong_B \left(\{ \tau \in V^B : [\![\tau : \check{\omega} \to \check{2}]\!] = 1_B \}, [\![\cdot = \cdot]\!]_B^{V^B} \right)$$

$$b \mapsto \sigma_b$$

$$\left(C(St(B), 2^\omega), [\![\cdot = \cdot]\!], R^B \right) \cong \left((2^\omega)^B, [\![\cdot = \cdot]\!]_B^{V^B}, \dot{R}^B \right)$$

$$\left[\!\left[\dot{R}^{B}(\sigma_1 \ldots \sigma_k) \right]\!\right]_B^{\cancel{(\omega)}^B} = \mathrm{Reg}\left(\left\{ G : R\left(f_{\sigma_1}(G), \ldots, f_{\sigma_k}(G)\right)\right\}\right)$$

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$$\shortparallel$$

$$\left[\!\left[R^{B}(f_1 \ldots f_k) \right]\!\right]$$

$$\nparallel$$

$$\dot{R}^{B} = \left\{ \cancel{\langle \sigma_1 \ldots \sigma_k \rangle} \langle \mathrm{otuple}(\sigma_1 \ldots \sigma_k), \left[\!\left[R^{B}(f_{\sigma_1} \ldots f_{\sigma_k}) \right]\!\right] \rangle : \sigma_1 \ldots \sigma_k \in (2^{\omega})^{B} \right\}$$

$$(2^{\omega}, R_1 \ldots R_k) \models \Xi \left(V^{B}, [\cdot = \cdot]^{V^{B}}, \dot{R}_1^{B} \ldots \dot{R}_k^{B} \right)$$

$$\tau \longmapsto \check{\tau}$$

$$\cancel{R_k(\tau_1 \ldots \tau_n)} \prec_1 \left((2^{\omega})^{B}, [\cdot = \cdot]^{V^{B}}, \dot{R}_1^{B} \ldots \dot{R}_k^{B} \right)$$

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$$\left(2^{\omega}, R_1 \dots R_k, =\right) \overset{?}{\lesssim} \left(C\left(St(B), 2^{\omega}\right), [- = -]_B^{V^B}, R_1^B \dots R_k^B\right)^2$$

Let $R \subseteq (2^\omega)^k$

$$(2^\omega, R, =) \prec \left({(2^\omega)^X}/{\mathcal{G}}, =, {R^X}/{\mathcal{G}} \right)$$

Given X index set

$P(X)$-model given by $\{f : f : X \to 2^\omega\}$

$(2^\omega)^X$

$$[\![f = g]\!]^{(2^\omega)^X}_{P(X)} = \{ x \in X : f(x) = g(x) \}$$

$$[\![R^X(f_1 \ldots f_n)]\!]^{(2^\omega)^X}_{P(X)} = \{ x \in X \; R(f_1(x) \ldots f_n(x)) \}$$

$$X \hookrightarrow St(P(X))$$

$$x \mapsto G_x = \{Y \subseteq X : x \in Y\}$$

X is a dense open subset of $St(P(X))$

$St(P(X))$ is the Stone-Čech compactification of $(X, P(X))$ i.e. the unique compact Hausdorff space $Z \supseteq X$ as a dense subspace s.t. if $f : X \to Y$ is cont. with Y compact Hausdorff then $\exists!\ \beta(f) : Z \to Y$ s.t. $f = \beta(f)|X$.

$$X \hookrightarrow St(P(X))$$

(Y, σ) compact Hausdorff

$$f : X \to Y$$

$$\downarrow$$

$$\beta(f) : St(P(X)) \to Y$$

$$G \longmapsto \text{unique point in } \bigcap\{\overline{U} : U \in \sigma ; f^{-1}[U] \in G\}$$

in case Y is 2^ω

$$(2^\omega)^X \approx C(St(P(X)) ; 2^\omega)$$

and if $R \subseteq (2^\omega)^k$ is any relation

$\cancel{\beta(b)^{-1}[R}$ $\quad R^X(b_1 \cdots b_k) = \{x : R(b_1(x) \cdots b_k(x))\}$

$\subseteq X$

$b_i : St(P(X)) \to 2^\omega$

$b_i \upharpoonright X$

$\big(C(St(P(X))), [\![-=-]\!], R^X\big)$ is a $B(X)$-valued model

$\sqcup|$ $\qquad c_1$

$\vee$ $\qquad \uparrow$

$(2^\omega, =, R)$ $\qquad \iota$

Now if B is any CBA and (15)

$R \subseteq (2^\omega)^k$ is an B-Baire relation

we can embed

$$(2^\omega, =, R) \sqsubseteq (C(St(B), 2^\omega), [\![\cdot = \cdot]\!], R^B)$$

$\prec$?

$\prec_2$

holds w.l.c. if R is Borel

$$[\![R^B(f_1 \dots f_k)]\!] = Reg(\{G : R(f_1(G) \dots f_k(G))\})$$

$(2^\omega, =, R) \prec_2$

(16)

given $R \subseteq (2^\omega)^k$ if R is Borel

$$(2^\omega, =, R) \sqsubseteq \prec_2 \left(C(St(B), [\![- = -]\!]), R^B\right)$$

if R is B-Baire i.e.

$$\prec \left(C(St(P(X))), [\![- = -]\!], R^X\right)$$

$f_1 \ldots f_k$

$f : St(B) \to (2^\omega)^k$

$$\left((2^\omega)^X, [\![=]\!], R^X\right)$$

is continuous

$\{G : R(f_1(G) \ldots f_k(G))\}$

has the Baire property in $St(B)$

$\prec$ fully elementary

if R is UB

and LC axioms are in place.

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$(2^\omega, R : R \text{ is } \cancel{\text{Borel}}$ lightface Borel$) \sqsubseteq (H\omega_1, \in_{\Delta_0})$

conversely $(H\omega_1, \in_{\Delta_0})$ exits inside

$(2^\omega, R : R \text{ is lightface Borel})$ by looking

at WFE