PHD COURSE
ON
MODEL COMPANIONSHIP RESULTS
FOR
SET THEORY

LECTURE 15 - 29/6/2022

$Th(H_{\omega_1}, \in_{\Delta_0})$ in models of $MK + LC$ ①

<u>Recall</u> MK is ~~GB~~ with Comprehension ~~changed in~~ for all $\in$-formulae ~~nsu~~ in signature $\{\in, Set\}$

$$\forall X\, (Set(X) \Leftrightarrow \exists Y\, X \in Y)$$

$$\forall \cancel{X} \left[(F \text{ is a function} \wedge Set(dom(F))) \to Set(ran(F)) \right]$$

$$\forall X\, \exists F\, \big(dom(F) \in Ord \vee dom(F) = Ord \wedge F \text{ is a function} \wedge ran(F) = X \wedge F \text{ is a bijection} \big)$$

in MK Ord is ~~an~~ regular

__Fact__ in MK $(V, \in \cap V^2)$ ~~$\in$~~ is a ~~n element~~ definable part ~~of~~ of

$\quad\quad = \{x : Set(x)\}$ in its models

__Notation__ A model of MK is

$(V, V, \in)$

sets and proper classes

sets

in such a model $V, \in \cap V^2$ are elements of V

it is not hard to check then that $\{\alpha \in Ord ; (V_\alpha, \in) \prec (V, \in)\} \in V$

From now on we work in

MK + Ord is Mahlo i.e

$\mathrm{Reg} = \{\kappa \in \mathrm{Ord} : \kappa$ is a regular cardinal$\}$ is stat.

$\uparrow$

$$\forall C \, (C \subseteq \mathrm{Ord} \wedge C \text{ club} \rightarrow \mathrm{Reg} \cap C \neq \emptyset)$$

$T =$ MK + Ord is Mahlo + $\forall X\,(\mathrm{Set}(X) \leftrightarrow X$ is constructible$)$

is a consistent theory.

For example if $(V, \in) \models \delta$ is Mahlo + ZFC

$(V_{\delta+1} \cap L, L_\delta, \in) \models T$

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Key intuition: Standard model of MK is

$(V_{\delta+1}, V_\delta, \in)$ with δ strongly inacc.

and in our context $(V_{\delta+1}, V_\delta, \in)$ with δ Mahlo.

if we force in V with some $P \subseteq V_\delta$

~~then we can~~ ~~and~~ and G is V-generic for P

iff G is $V_{\delta+1}$-generic for P (P a cba $P \subseteq V_\delta$)

and it is naturally to define $V[G]$ as

$(V_{\delta+1}[G], V_\delta[G], \in)$

$V_{\delta+1}[G] = \{\tau_G : \tau \in V^B_{\delta+1}\}$, $V_\delta[G] = \{\tau_G : \tau \in V_\delta\}$

$\{\tau_G : \tau \subseteq V_\delta \quad \tau \in V^B\}$

$\{\tau_G : \tau \subseteq V^P \cap V_{\delta+1}\}$

$\{F : V^P \to P : \mathrm{dom}(F) \subseteq V_\delta\}$

if we do class forcing over a model of MK Cohen's forcing theorem holds i.e.

$p \Vdash \varphi(\tau_1 \ldots \tau_n)$ iff for all G V-generic for P

$$(\mathcal{V}[G], V[G], \in) \models \varphi((\tau_1)_G, \ldots (\tau_n)_G)$$

but $(\mathcal{V}[G], V[G], \in)$ need not model MK

for example if $P = \mathrm{Coll}(\omega, \mathrm{Ord})$

$$(\mathcal{V}[G], V[G], \in) \not\models \text{Replacement}.$$

We are interested in forcing with $\mathrm{Coll}(\omega, <\mathrm{Ord})$ in models of MK.

if G is V-generic for $\mathrm{Coll}(\omega, <\mathrm{Ord})$, then

$\{\tau_G : \tau \in V, \tau \text{ is a } B\text{-name}\}$

$\shortparallel$

$(V[G], \in) \models ZFC^- +$ All sets are countable

i.e. it is a model of the theory of

$H(\aleph_1)$

Previously we saw that if $\mathfrak{M} \sqsupseteq H_{\omega_2}^V$ and $\mathfrak{M} \equiv_1 H_{\omega_2}$ with respect to $\in_{\Delta_0}$ then $\mathfrak{M} \sqsubseteq \left(H_{\omega_2}^{V^B}/G, \in\right)$

if we work in $\epsilon_{\Delta_0} \cup \{R_\varphi : \varphi \text{ a projective}\} = \epsilon_{proj}$

the same holds true i.e. if (7)

$\mathcal{M} \sqsupseteq (H_{\omega_1}, \epsilon_{proj})$ and $\mathcal{M} \equiv_1 H_{\omega_1}$ w.r.t. ϵ_{proj}

then there is $\mathbb{B}$ s.t. $\mathcal{M} \sqsubseteq \left(H_{\omega_1}^{\mathbb{B}}/G, \epsilon_{proj}/G\right)$

Consider $\exists x\,(x \in P(\omega) \setminus L)$ in ϵ_{Δ_0} this is Σ_2

but in ϵ_{proj} it is Σ_1 and clearly if one makes it true by forcing it remains true afterwards.

Previously we have seen (8)

$(H_{\omega_1}, \in_{\Delta_0})$ is T-ec where T is

universal $\in_{\Delta_0}$- ~~univers~~ theory of V.

in particular if V is L ~~we~~ get

and G is L-generic for $Coll(\omega, \omega)$

$(H^{L}_{\omega_1}, \in_{\Delta_0}) \prec_1 (H^{L[G]}_{\omega_1}, \in_{\Delta_0})$ but not fully elementary since

$(H^{L[G]}_{\omega_1}, \in_{\Delta_0}) \models \exists r \left(r \subseteq \omega \wedge r \notin L \right)$ (a Σ_1-sentence for $\in_{\Delta_0}$)

On the other hand if projective sets are UB and LC holds one can show

$$\left(H_{\omega_2}, \in_{proj}\right) \prec \left(H_{\omega_2}^{V[G]}, \in_{proj}\right)$$

$$\Downarrow$$

$$\left(H_{\omega_1}, \in\right) \prec \left(H_{\omega_1}^{V[G]}, \in\right)$$

<u>Def</u>: (Projective Resurrection) (PR) holds ~~if~~

(in $(V, V, \in_{\Delta_0})$) if

$V[G] = \{\tau_G : \tau$ is a set and τ is a B-name$)$

$$\left(H^V_{\omega_1}, \in_{\Delta_0}\right) \prec \left(\cancel{V[G]}, \in_{\Delta_0}\right)$$

whenever G is ~~Coll(ω~~ V-generic for $Coll(\omega, <Ord)$.

i.e. for any $a_1 .. a_n \in H_{\omega_1}$ and $\phi(x_1 .. x_n)$ $\in$-formula $(H_{\omega_1}, \in_{\Delta_0}) \models \phi(a_1 .. a_n)$ iff $\underline{[\![\phi(\check{a}_1, .. \check{a}_n)]\!]_1 = 1}$

$[\![f(\check{a}_1 \ldots \check{a}_n)]\!]^{V}_{\mathrm{Coll}(\omega, <0_{\mathrm{Pd}})}$

$\cancel{V^B}$ $\cancel{V^B}$

$\cancel{V}$

$V^B = \{ \tau \in V : \tau$ is a B-name and τ is a set $\}$

$\cap$

$V \ldots$

$$\mathrm{Coll}(\omega, <\mathrm{Ord}) = \{ p : \omega \times \mathrm{Ord} \to \mathrm{Ord} : |p| < \aleph_0 \text{ and } \forall \langle n, \alpha \rangle \in \mathrm{dom}(p)\ p(n, \alpha) \in \alpha \}$$

~~If~~ if G is V-generic for $\mathrm{Coll}(\omega, <\mathrm{Ord})$,

in $V[G] \models \bigcup G = g$ is s.t.

$g_\alpha(n) = g(n, \alpha)$ defines a surj. of ω onto α.

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$RO(Coll(\omega, <\cancel{\kappa}))$ = $\{X :$ $\downarrow X = X$ and X is reg ~~xm~~ in order top. of $Coll(\omega, <\kappa)\}$

Key point: $Coll(\omega, <Ord)$ is $<Ord$-CC

in $RO(P)$ a ~~dense set can be descrb~~ ~~to~~

an X s.t. $\downarrow X = X$ can be described by some A max ant. of $\downarrow X$

because in $RO(P)$ $\bigvee X = \bigvee A$

$RO(Coll(\omega, <Ord)) = \{ A : A \subseteq Coll(\omega, <Ord)$ is a max art. $\}$ (15)

and $A \approx B$ iff $\downarrow A \cap \downarrow B$ is dense in $\downarrow A$ and in $\downarrow B$

by picking in each equivalence class a max. art. we get that $RO(Coll(\omega, <Ord))$ is described by a class B_{Ord} s.t.

$(V, V, \in) \models (B_{Ord}, \wedge_{B_{Ord}}, \vee_{B_{Ord}}, 0_{B_{Ord}}, 1_{B_{Ord}}, \neg_{B_{Ord}})$ is a ba

<u>Remark</u> PR is a weak form of Woodin's generic absoluteness results.

<u>Recall</u>:

<u>Thm</u> (Woodin) Assume G is V-generic for some $P \in V$ and there are class many Woodin in V. Then

$$\left(H_{\omega_1}^{V}, \in_{\Delta_0}\right) \prec \left(H_{\omega_1}^{V[G]}, \in_{\Delta_0}\right)$$

$$\not\cong \qquad \sqcup\!\!\!\!\!\,\,\,/\!/$$

$$H_{\omega_1}^{V[H]}$$

Fact ~~PFA~~ $\boxed{\text{Proj. Resurrection}}$ + Ord is Mahlo entails that $\forall B \in V$ $\exists C \leq B$ i.e. B is complete subalgebra of C (16)

s.t. ~~H_{ω_1}~~ if G is V-generic for C and $H \in V[G]$ is V-generic for B then

$$(H_{\omega_1}^V, \in_{\Delta_0}) \prec (H_{\omega_1}^{V[G]}, \in_{\Delta_0})$$

$$(H_{\omega_1}^V, \in_{\Delta_0}) \prec_1 (H_{\omega_1}^{V[H]}, \in_{\Delta_0})$$

$$(H_{\omega_1}^{V[H]}, \in_{\Delta_0}) \sqsubseteq (H_{\omega_1}^{V[G]}, \in_{\Delta_0})$$

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Thm If $(V, V, \in) \models MK + \mathrm{Ord}$ is Mahlo

δ is inaccessible and $(V_\delta, \in) \prec (V, \in)$, and

if G is V-generic for $\mathrm{Coll}(\omega, <\delta)$ then

$V[G] \models PR$ i.e. if H is

$V[G]$-generic for $\mathrm{Coll}(\omega, <\mathrm{Ord})$ then

$$\left(H_{\omega_1}^{V[G]}, \in\right) \prec \left(\cancel{H_{\omega_1}^{V[G][H]}}\ V[G][H], \in\right)$$

Cor Assume $(\mathcal{V}, V, \in) \models MK + \mathrm{Ord}$ is Mahlo $+$ PK

Let G_0 be V-generic for $P \in V$

and $\delta > |P|$ be inacc. s.t. $(V_\delta, \in) \prec (V, \in)$

then let G_1 be $V[G_0]$-generic for $\mathrm{Coll}(\omega, <\delta)$

and G_2 be $V[G_0][G_1]$-generic for $\mathrm{Coll}(\omega, <\mathrm{Ord})$

$$(H_{\omega_1}^{V}, \in_{\Delta_0}) \prec (V[G_2], \in_{\Delta_0}) = (\underbrace{V[G_0][G_1]}_{\prec}[G_2], \in_{\Delta_0})$$

$$\prec$$

$$(H_{\omega_1}^{V[G_0]}, \in_{\Delta_0}) \sqsubseteq (H_{\omega_1}^{V[G_0][G_1]}, \in_{\Delta_0}) = (H_{\omega_1}^{V[G_1]}, \in_{\Delta_0})$$

if $(V_\delta, \in) \prec (V, \in)$

$(V_\delta, \in) \models p \Vdash \varphi(\tau_1 \dots \tau_n)$ iff $(V, \in) \models p \Vdash \varphi(\tau_1 \dots \tau_n)$

if $\tau_1 \dots \tau_n \in V_\delta \cap V^{\mathrm{Coll}(\omega, <\delta)}$

Now $\mathrm{Id} : \mathrm{Coll}(\omega, <\delta) \hookrightarrow \mathrm{Coll}(\omega, <\mathrm{Ord})$

is a complete embedding if δ is a regular cardinal

~~Also if G is V-generic for $Coll(\omega, <\delta)$~~ 20

~~and H is~~

~~Also~~ If H is ~~$Coll(\omega, <Ord)$~~

V-generic for $Coll(\omega, <Ord)$ then

$G = H \cap Coll(\omega, <\delta)$ is V-generic for $Coll(\omega, <\delta)$

and we get that $\forall V_{\delta+1}$

$$V_\delta[G] \models \varphi((\tau_1)_G \ldots (\tau_n)_G) \iff \exists p \in G \; \boxed{V_\delta \models p \Vdash_{Coll(\omega, <\delta)} \varphi(\tau_1 \ldots \tau_n)}$$

$$\iff \exists p \in H \; \boxed{V \models p \Vdash_{Coll(\omega, <Ord)} \varphi(\tau_1 \ldots \tau_n)} \iff V[H] \models \varphi(\tau_1 \ldots \tau_n)$$

$V_\delta[G] = H_{\omega_2}^{V[G]} \prec (N[H], \in_{\Delta_0})$ (21)

~~X~~ **Thm.** if $T \supseteq ZFC_\tau$ and

$\mathcal{M}$ is T-ec

and ψ is ~~q~~ Σ_1 s.t.

$T \models \forall x \forall \vec{z} \exists y \, \psi(x, y, \vec{z})$

then $\mathcal{M} \models Rep(\psi)$