**12.2 Lemma**

Let W, X be topological spaces and suppose that $W = A \cup B$ with A, B both closed subsets of W. If $f: A \to X$ and $g: B \to X$ are continuous functions such that $f(w) = g(w)$ for all $w \in A \cap B$ then $h: W \to X$ defined by

$$h(w) = \begin{cases} f(w) & \text{if } w \in A, \\ g(w) & \text{if } w \in B \end{cases}$$

is a continuous function.

*Proof* Note that h is well defined. Suppose that C is a closed subset of X, then

$$\begin{aligned} h^{-1}(C) &= h^{-1}(C) \cap (A \cup B) \\ &= (h^{-1}(C) \cap A) \cup (h^{-1}(C) \cap B) \\ &= f^{-1}(C) \cup g^{-1}(C). \end{aligned}$$

Since f is continuous, $f^{-1}(C)$ is closed in A and hence in W since A is closed in W. Similarly $g^{-1}(C)$ is closed in W. Hence $h^{-1}(C)$ is closed in W and h is continuous.

**12.3 Definition**

A space X is said to be *path connected* if given any two points $x_0, x_1$ in X there is a path in X from $x_0$ to $x_1$.

Note that by Lemma 12.1 it is sufficient to fix $x_0 \in X$ and then require that for all $x \in X$ there is a path in X from $x_0$ to x. (Some books use the term *arcwise connected* instead of path connected.)

For example $\mathbb{R}^n$ with the usual topology is path connected. The reason is that given any pair of points $a, b \in \mathbb{R}^n$ the mapping $f: [0,1] \to \mathbb{R}^n$ defined by $f(t) = tb + (1-t)a$ is a path from a to b. More generally any convex subset of $\mathbb{R}^n$ is path connected. A subset E of $\mathbb{R}^n$ is *convex* if whenever a, $b \in E$ then the set $\{ tb + (1-t)a; 0 \leq t \leq 1 \}$ is contained in E, i.e. E is convex if the straight-line segment joining any pair of points in E is in E itself. See Figure 12.1 for an example of a convex and of a non-convex subset of $\mathbb{R}^2$.

Figure 12.1

A convex subset. A non-convex subset.