the proof of Theorem 8.7 and in desperation look at the proof of Theorem 8.11.)

8.5 Theorem

A space X is T_1 if and only if each point of X is closed.

Proof Suppose X is a T_1-space. Let $x \in X$ and $y \in X - \{x\}$. Then there is an open set U_y containing y but not x. Therefore

$$\bigcup_{y \in X - \{x\}} U_y = X - \{x\}$$

which shows that $X - \{x\}$ is a union of open sets and hence is open. Thus $\{x\}$ is closed.

Conversely if $\{x\}$ and $\{y\}$ are closed then $X - \{x\}$ and $X - \{y\}$ are open sets, one containing x but not y and the other containing y but not x; i.e. X is a T_1-space.

As a corollary we get the following result.

8.6 Corollary

In a Hausdorff space each point is a closed subset.

In fact something much more general holds.

8.7 Theorem

A compact subset A of a Hausdorff space X is closed.

Proof We may suppose that $A \neq \emptyset$ and $A \neq X$ since otherwise it is already closed and there is nothing to prove. Choose a point $x \in X - A$. For each $a \in A$ there is a pair of disjoint open sets U_a, V_a with x in U_a and a in V_a. The set $\{V_a; a \in A\}$ covers A and since A is compact there is a finite subcover, say

$$\{V_{a(1)}, V_{a(2)}, \ldots, V_{a(n)}\}$$

which covers A. The set $U = U_{a(1)} \cap U_{a(2)} \cap \ldots \cap U_{a(n)}$ is an open set containing x which is disjoint from each of the $V_{a(i)}$ and hence $U \subseteq X - A$. Thus each point $x \in X - A$ has an open set containing it which is contained in $X - A$, which means that $X - A$ is open and A is closed.

Theorem 8.7 leads to an important result.

8.8 Theorem

Suppose that $f: X \to Y$ is a continuous map from a compact space X