THE COLLISION SINGULARITY IN A PERTURBED TWO-BODY PROBLEM

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Abstract. It is shown that, in the neighborhood of a collision singularity, the motion in a perturbed two-body problem $\ddot{\mathbf{r}} = -\mu r^{-3}\,\mathbf{r} + \mathbf{P}$, where $\mathbf{P}$ remains bounded, has the same basic properties as the motion in the neighborhood of a collision in the unperturbed two-body problem $\mathbf{P} = \mathbf{0}$.

Notation

$\mathbf{ab}$	scalar product of vectors $\mathbf{a}$ and $\mathbf{b}$; also $\mathbf{aa} = \mathbf{a}^2 = a^2$;
$\mathbf{a} \times \mathbf{b}$	vector product;
$\lvert\mathbf{a}\rvert = a$	absolute value;
$\mathbf{v} = \dot{\mathbf{r}}$	velocity.
c, $\mathbf{c}$, c_k, $\mathbf{c}_k$	are constants and b, $\mathbf{b}$, b_k, $\mathbf{b}_k$ functions, bounded for the arguments under consideration; without subscript they may change their value from one occurrence to the next, while with subscripts they have specific values.
c_k^2	denotes correspondingly a positive (or nonnegative) constant.

1. Introduction

The basic properties of the motion in the two-body and N-body problems in the neighborhood of a binary collision are well known: approaching from $t<0$ the instant of collision $t=0$, the mutual distance r and position $\mathbf{r}$ of the colliding bodies approach zero as $t^{2/3}$, while $-\dot{r}$ and $\mathbf{v}$ grow as $-t^{-1/3}$. In the N-body problem, the equation of motion (for the body colliding with that resting at the origin) can be written as

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} + \mathbf{P},$$

where $\mathbf{P}$ has, in the neighborhood of the collision point, the special form $\mathbf{P} = r\mathbf{b}$.

We shall investigate the general problem where we assume only that $\mathbf{P} = \mathbf{P}(\mathbf{r}, \mathbf{v}, t)$ is bounded (and, say, continuous or R-integrable on the considered trajectories), and we will show that in the neighborhood of a collision singularity the motion has essentially the same properties as that in the two-body problem. In certain simple cases this has been shown (explicitly or implicitly) before: for $\mathbf{P}(\mathbf{r}, t) = -\mathrm{grad}_{\mathbf{r}} U$, with $\mathbf{P}$, U, and $\partial U/\partial t$ bounded, see Sperling (1968), and for $\mathbf{P} = r\mathbf{b}$ – an identical proof holds for $\mathbf{P} = \mathbf{const} + r\mathbf{b}$ and $\mathbf{P} = \sqrt{r}\,\mathbf{b}$ – see Arenstorf (1969) and Sperling (1969). In the present general case the proof is, however, more delicate.

The method of this paper could be used to get another proof of the boundedness of the 'cluster energy h_l' in Sperling (1969).

2. Formulation of the Problem

All quantities in the following are real. Let the particle m_1 rest at the origin and m_2 move about m_1 according to Newton's gravitational law, with an additional 'perturbing' acceleration $\mathbf{P}$ relative to m_1 acting on it. The position vector $\mathbf{r}$ of m_2 with respect to m_1 satisfies the equation

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} + \mathbf{P}, \quad \mu = \gamma(m_1 + m_2); \tag{2.1}$$

consider the solution $\mathbf{r} = \mathbf{r}(t)$ of (2.1) on $t_0 \leqq t < 0$ and assume that

$$r(t) > 0 \quad \text{on} \quad [t_0, 0[$$

and

$$\liminf r = 0 \quad \text{as} \quad t \to 0.$$

Furthermore, let condition (a) or (b) or both be satisfied:

(a) $\mathbf{r}$ and $\mathbf{P}$ are bounded, i.e.: $0 < r < r^*$ on $[t_0, 0[$ and $P = |\mathbf{P}(\mathbf{r}, \mathbf{v}, t)| \leqq P^*$ for all $r \leqq r^*$, $\mathbf{v}$ arbitrary, $t \in [t_0, 0[$.

(b) $\mathbf{P}$ is bounded, i.e.: $P = |\mathbf{P}(\mathbf{r}, \mathbf{v}, t)| \leqq P^*$ for all $\mathbf{r}$ arbitrary, $\mathbf{v}$ arbitrary, $t \in [t_0, 0[$.

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LEMMA: $\liminf r = 0$ implies $\lim r = 0$ as $t \to 0$.

PROOF: Assume contrariwise that $\liminf r = 0$ and $\limsup r > Q > 0$; then there exist an arbitrarily small $|t_1|$ and t_2, t_3 with $t_1 < t_2 < t_3 < 0$ such that

$$\begin{aligned} r(t_1) &= r_1 = \tfrac{1}{10} Q \\ r(t_2) &= r_2 = Q \qquad r(t) \geqq \tfrac{1}{10} Q \quad \text{on} \quad [t_1, t_3] \\ r(t_3) &= r_3 = \tfrac{1}{10} Q, \end{aligned} \tag{3.1}$$

and Equation (2.1) implies that

$$|\ddot{\mathbf{r}}| < \frac{100\mu}{Q^2} + 2c_1^2 = 2c_2 > 0. \tag{3.2}$$

Choose t_1 so small that

$$t_1^2 < \frac{9Q}{1000c_2}; \tag{3.3}$$

since on $[t_1, t_3]$

$$\mathbf{r}(t) = \mathbf{r}_1 + \mathbf{v}_1(t - t_1) + \int_{t_1}^{t} (t - \tau)\, \ddot{\mathbf{r}}(\tau)\, d\tau, \tag{3.4}$$

we find for $t=t_2$

$$Q \leqq \tfrac{1}{10} Q + v_1 |t_2 - t_1| + c_2 |t_2 - t_1|^2,$$

hence

$$v_1 \geqq \tfrac{9}{10} Q |t_2 - t_1|^{-1} - c_2 |t_2 - t_1| > \frac{8Q}{10|t_2 - t_1|}. \tag{3.5}$$

At $t=t_3$, Equation (3.4) yields

$$\begin{aligned} r_3 &\geqq v_1 |t_3 - t_1| - \tfrac{1}{10} Q - c_2 |t_3 - t_1|^2 \\ &> \tfrac{8}{10} Q \frac{|t_3 - t_1|}{|t_2 - t_1|} - \tfrac{1}{10} Q - \tfrac{9}{1000} Q > \tfrac{6}{10} Q, \end{aligned}$$

contradicting the assumption $r_3 = \frac{1}{10} Q$ (cf. (3.1)).

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Define the 'angular momentum' $\mathbf{K}$ by

$$\mathbf{K} = \mathbf{r} \times \mathbf{v}. \tag{4.1}$$

THEOREM: $\lim \mathbf{K} = \mathbf{0}$ as $t \to 0$.

PROOF: From Equation (2.1) we find that

$$\dot{\mathbf{K}} = \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{P},$$

implying that $\mathbf{K}$ is bounded on $[t_0, 0[$ and $\lim \mathbf{K} = \mathbf{K}(0)$ as $t \to 0$ exists;

$$\mathbf{K} = \mathbf{K}(0) + \int_0^t \mathbf{r} \times \mathbf{P}\, \mathrm{d}t = \mathbf{K}(0) + t\mathbf{b} \quad \text{on} \quad [t_0, 0[. \tag{4.2}$$

Substituting

$$\mathbf{r}\ddot{\mathbf{r}} = -\mu/r + \mathbf{rP}, \tag{4.3}$$

which results from Equation (2.1), into the identity

$$r\ddot{r} = \mathbf{r}\ddot{\mathbf{r}} + v^2 - \dot{r}^2, \tag{4.4}$$

after having used the identity

$$K^2 = (\mathbf{r} \times \mathbf{v})^2 = r^2 v^2 - r^2 \dot{r}^2, \tag{4.5}$$

we get

$$\ddot{r} = K^2/r^3 - \mu/r^2 + b, \quad b = \mathbf{rP}/r. \tag{4.6}$$

Assume now that contrary to the assertion $\mathbf{K}(0) \neq \mathbf{0}$, i.e., $K^2 > \frac{1}{2} K(0)^2 > 0$ for all sufficiently small t; we conclude that, as $t \to 0$,

$\ddot{r}$ is ultimately positive, hence

$\dot{r}$ ultimately increases and does not change sign.

Now let t^* be so small that $\dot{r}$ does not change sign on $[t^*, 0[$; multiply Equation (4.6) by $\dot{r}\not\equiv 0$, integrate from t^* to t, and evaluate the first and third term on the right by the mean value theorem; we find

$$\tfrac{1}{2}\dot{r}^2 = -b^2\frac{1}{r^2} + \frac{\mu}{r} + br + c, \quad 0 < K(0)^2 < 4b^2;$$

for $t\to 0$, i.e., $r\to 0$, the right side becomes negative, which is absurd. Thus

$$\mathbf{K} = t\mathbf{b}. \tag{4.7}$$

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LEMMA: $r\dot{r} = \mathbf{rv}\to 0$ as $t\to 0$.

PROOF, first case: Assume that ultimately $r\to 0$ monotonically.

Then ultimately $K = rb$, since

$$K = \left|\int_0^t \mathbf{r}\times\mathbf{P}\,\mathrm{d}t\right| \leqq \int rP\,\mathrm{d}t \leqslant r\int P\,\mathrm{d}t,$$

observing that ultimately $\max r(\tau) = r(t)$ for $\tau\in[t, 0[$. Substituting $K = rb$ into (4.6), we find

$$\ddot{r} = -\frac{\mu}{r^2} + \frac{b}{r} + c \tag{5.1}$$

and conclude that

$\ddot{r}$ is ultimately negative, hence

$\dot{r}$ ultimately decreases and does not change sign.

Again, let t^* be so small that $\dot{r}$ does not change sign on $[t^*, 0[$, multiply Equation (5.1) by $\dot{r}\not\equiv 0$, integrate and evaluate; then

$$\tfrac{1}{2}\dot{r}^2 = \frac{\mu}{r} + b\log r + br + c,$$

implying that

$$r\dot{r}^2 \to 2\mu \quad \text{as} \quad t\to 0, \tag{5.2}$$

i.e., $r\dot{r}\to 0$.

Second case: Assume now that ultimately $r\to 0$ non-monotonically; then there exists a sequence $\{t_\nu\}\to 0$ such that $\dot{r}_\nu = \dot{r}(t_\nu) = 0$ and $\dot{r}$ does not change sign on $[t_\nu, t_{\nu+1}]$. Consider such an interval $[t_k, t_l]$; r is monotonic on it, and without restriction we can assume that the maximum of r occurs at t_k (otherwise we integrate from t_l to t): $r_k = r(t_k)\geqq r(t)$, $t\in[t_k, t_l]$. Now multiply Equation (4.6) by $\dot{r}$, integrate from t_k to $t\in[t_k, t_l]$, and evaluate the first and third term by the mean value theorem; then, since $\dot{r}(t_k) = 0$,

$$\tfrac{1}{2}\dot{r}^2 = -\tfrac{1}{2}K(\tau)^2\left(\frac{1}{r^2} - \frac{1}{r_k^2}\right) + \mu\left(\frac{1}{r} - \frac{1}{r_k}\right) + b(r - r_k),\ \tau\in[t_k, t],$$

and

$$(r\dot{r})^2 = -K(\tau)^2\left(1-\left(\frac{r}{r_k}\right)^2\right) + 2\mu r\left(1-\frac{r}{r_k}\right) + br^2(r-r_k) \quad \text{on} \quad [t_k, t_l],$$

implying the assertion $r\dot{r} \to 0$ because of $r \leqq r_k$ and $K^2(\tau) \to 0$, $r \to 0$, as $t \to 0$.

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Define the 'energy' h by

$$h = \tfrac{1}{2}v^2 - \mu/r\,; \tag{6.1}$$

using the Equation of motion (2.1), we derive

$$\dot{h} = \mathbf{vP}\,. \tag{6.2}$$

THEOREM: h remains bounded as $t \to 0$.

PROOF: Integrate Equation (4.3) from t_1 to t_2, $t_0 < t_1 < t_2 < 0$, and evaluate the left side by partial integration; then

$$\int_{t_1}^{t_2} \mathbf{r}\ddot{\mathbf{r}}\,\mathrm{d}t = \mathbf{r}_2\mathbf{v}_2 - \mathbf{r}_1\mathbf{v}_1 - \int v^2\,\mathrm{d}t$$

$$= -\mu\int\frac{\mathrm{d}t}{r} + \int \mathbf{rP}\,\mathrm{d}t\,.$$

Substitute for μ/r from Equation (6.1) and observe that by Section 5 Lemma $\mathbf{rv}$ is bounded; we get

$$\int_{t_1}^{t_2} v^2\,\mathrm{d}t = -2\int_{t_1}^{t_2} h\,\mathrm{d}t + 2b_1\,,$$

$$b_1 = \mathbf{r}_2\mathbf{v}_2 - \mathbf{r}_1\mathbf{v}_1 - \int_{t_1}^{t_2} \mathbf{rP}\,\mathrm{d}t\,. \tag{6.3}$$

Assume now that contrary to our assertion $\limsup|h| = \infty$ as $t \to 0$. We can choose t_1 and t_2 so close to $t = 0$ that the following conditions are satisfied:

$$2\,|t_1|\int_{t_1}^{0} P^2\,\mathrm{d}t < \tfrac{1}{4} \tag{6.4a}$$

$$|h_2| - |h_1| > 1\,, \tag{6.4b}$$

$$\left(|h_1| + 2\,|b_1|\int_{t_1}^{0} P^2\,\mathrm{d}t\right)|h_2|^{-1} < \tfrac{1}{4} \tag{6.4c}$$

$$|h_2| = |h(t_2)| = \max|h(t)|\,, \quad t \in [t_1, t_2]\,. \tag{6.4d}$$

The following sequence of inequalities will lead to a contradiction: Using Equation (6.2),

$$1 < |h_2| - |h_1| \leqq |h_2 - h_1| = \left|\int_{t_1}^{t_2} \mathbf{vP}\, \mathrm{d}t\right|$$

$$\leqq \int vP\, \mathrm{d}t < \left(\int vP\, \mathrm{d}t\right)^2$$

$$\leqq \left(\int P^2\, \mathrm{d}t\right)\left(\int v^2\, \mathrm{d}t\right)$$

the last step by Schwarz's inequality; substitute for the last integral from Equation (6.3) and evaluate:

$$|h_2| - |h_1| < 2\left(\int P^2\, \mathrm{d}t\right)\left(\int_{t_1}^{t_2} |h|\, \mathrm{d}t + |b_1|\right)$$

$$< 2\left(\int P^2\, \mathrm{d}t\right) |h_2|\, |t_2 - t_1| + |b_1|$$

or finally

$$|h_2| < \left[2\,|t_2 - t_1| \int_{t_1}^{t_2} P^2\, \mathrm{d}t + \left(|h_1| + 2\,|b_1| \int_{t_1}^{t_2} P^2\, \mathrm{d}t\right) |h_2|^{-1}\right] |h_2| < \tfrac{1}{2}\,|h_2|\,,$$

which is absurd.

7. Properties of the Collision Singularity

Substituting from Equation (4.3) into Equation (4.4) and replacing v^2 from Equation (6.1), we find the Lagrange-Jacobi equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} r^2 = \frac{2\mu}{r} + 4h + 2\mathbf{rP}\,, \tag{7.1}$$

which we write with

$$R = r^2 \tag{7.2}$$

as

$$\ddot{R} = 2\mu/\sqrt{R} + b\,. \tag{7.3}$$

Since $R \to 0$ as $t \to 0$, $\ddot{R}$ is ultimately positive, hence $\dot{R}$ ultimately increases and does not change sign; assume now that this holds on the considered interval. Multiplying Equation (7.3) by $\dot{R}$ and integrating from t to 0, we find, since $\dot{R}(0) = 0$ by Section 5

Lemma and observing that $\dot{R}$ does not change sign on the integration interval,

$$\dot{R}^2 = 8\mu\sqrt{R} + bR\,, \tag{7.4}$$

$$\int_0^R R^{-1/4}(1 + bR^{1/2})\,\mathrm{d}R = (8\mu)^{1/2}\int_0^t \mathrm{d}t\,,$$

$$\tfrac{4}{3}R^{3/4} + bR^{5/4} = (8\mu)^{1/2}\,t\,, \quad R = (\tfrac{9}{2}\mu)^{2/3}\,t^{4/3}(1 + bR^{1/2})$$

and finally

$$r = (\tfrac{9}{2}\mu)^{1/3}\,t^{2/3}\;1 + bt^{2/3}) \tag{7.5}$$

and, by Equation (7.4),

$$\dot{r} = (\tfrac{4}{3}\mu)^{1/3}\,t^{-1/3}(1 + bt^{2/3})\,. \tag{7.6}$$

Substituting from (7.5) into the Equation of motion (2.1) results in

$$\ddot{\mathbf{r}} = -\tfrac{2}{9}t^{-2}(1 + bt^{2/3})\,\mathbf{r} + \mathbf{P}\,. \tag{7.7}$$

Introduce new independent and dependent variables ϑ and u by

$$\vartheta = t^{1/3}\,, \quad \mathbf{r} = \vartheta\mathbf{u}\,; \tag{7.8}$$

then Equation (7.7) transforms into

$$\mathbf{u}'' = b\mathbf{u} + 9\vartheta^3\mathbf{P}\,, \quad ' = \mathrm{d}/\mathrm{d}\vartheta\,. \tag{7.9}$$

Equation (7.5) implies that $t^{-2/3}\mathbf{r}$ remains bounded, hence so does $\vartheta^{-1}\mathbf{u}$ as t, $\vartheta\to 0$; Equation (7.9) can now be written as

$$\mathbf{u}'' = \vartheta\mathbf{b}\,. \tag{7.10}$$

$\mathbf{u}''\to\mathbf{0}$ implies that $\lim\mathbf{u}' = \mathbf{u}'(0)$ as t, $\vartheta\to 0$ exists;
integrating Equation (7.10) and transforming back to t and $\mathbf{r}$, we find

$$\mathbf{r} = \mathbf{u}'(0)\,t^{2/3} + t^{4/3}\mathbf{b} \tag{7.11}$$

and, substituting this into Equation (7.7) and integrating,

$$\dot{\mathbf{r}} = \mathbf{v} = \tfrac{2}{3}\mathbf{u}'(0)\,t^{-1/3} + t^{1/3}\mathbf{b}\,, \tag{7.12}$$

and comparison of Equation (7.11) with Equation (7.5) yields

$$|\mathbf{u}'(0)| = (\tfrac{9}{2}\mu)^{1/3}\,.$$

Equations (7.5), (7.11), (7.6), and (7.12) describe the behavior of r, $\mathbf{r}$, $\dot{r}$, and $\mathbf{v}$ as $t<0$ approaches $t=0$.

8. Regularization of the Equations of Motion

Equation (7.5) implies the convergence of Sundman's integral

$$\int_{t^*}^{t} \frac{\mathrm{d}t}{r} \quad \text{as} \quad t^*\to 0\,.$$

i.e., the existence (and finiteness) of Sundman's variable

$$s = \int_0^t \frac{\mathrm{d}t}{r}. \tag{8.1}$$

Following Sundman, we introduce s instead of t as the new independent variable and, denoting differentiation with respect to s by a prime $'$, transform Equations (2.1) and (7.1) into

$$\mathbf{r}'' = \frac{1}{r}(r'\mathbf{r}' - \mu\mathbf{r}) + r^2\mathbf{P} \tag{8.2}$$

and

$$r'' = \mu + 2rh + r\mathbf{rP}\,; \tag{8.3}$$

setting

$$\mathbf{w} = \frac{1}{r}(r'\mathbf{r}' - \mu\mathbf{r}) = \frac{1}{r}(r^2\dot{r}\mathbf{v} - \mu\mathbf{r}), \tag{8.4}$$

we find by means of Equations (7.5), (7.6), (7.11), and (7.12) that

$$\lim \mathbf{w} \quad \text{exists as} \quad t, s \to 0, \tag{8.5}$$

and differentiating and using Equation (8.3),

$$\mathbf{w}' = rr'\mathbf{P} + \frac{\mathbf{r}'}{r}(r'' - \mu)$$

and finally

$$\mathbf{w}' = rr'\mathbf{P} + (2h + rP)\,\mathbf{r}' \tag{8.6}$$

follow.

By Equations (7.5) and (7.12) $\lim \mathbf{r}' = \lim r\mathbf{v} = \mathbf{0}$ exists as $t, s \to 0$. The system of 'regularized' equations now reads

$$\begin{aligned} \mathbf{r}'' &= \mathbf{w} + \mathbf{r}^2\mathbf{P} \\ \mathbf{w}' &= (\mathbf{rr}')\,\mathbf{P} + (2h + \mathbf{rP})\,\mathbf{r}' \\ h' &= \mathbf{r}'\mathbf{P} \end{aligned} \tag{8.7a}$$

$$\begin{aligned} r'' &= \mu + 2rh + r\mathbf{rP} \\ t' &= r\,. \end{aligned} \tag{8.7b}$$

We have used quotation marks above since $\mathbf{P}$ has been defined only for $t<0$; therefore our results hold only up to the instant of collision $t=0$. If we impose on $\mathbf{P}(\mathbf{r}, \mathbf{v}, t)$ further suitable conditions, e.g., boundedness and continuity for $|t| < T\,(>0)$, we see that the motion can be continued to $t>0$; if we assume that $\mathbf{P}(\mathbf{r}, \mathbf{v}, t)$ is holomorphic for $|\mathbf{r}| < \mathscr{R}$, all $\mathbf{v}$, $|t| < T$, then we conclude in the usual manner that $\mathbf{r}(s)$, $\mathbf{w}(s)$, $h(s)$, $r(s)$, and $t(s)$ are holomorphic at $s=0$, and particularly that the motion possesses a real analytic continuation into $t>0$, provided that $\mathbf{P}$ is real for real arguments.

Other systems of regularized equations can be derived by transforming also the dependent variables, following Levi-Civita, Stiefel and Kustaanheimo, or others; we shall not go into details here.

References

Arenstorf, R. F.: 1969, 'Regularization Theory for the Elliptic Restricted Three-Body Problem' (to be published in *J. Differential Equations*).

Sperling, H. J.: 1968, 'The Binary Collision in the *N*-Body Problem', *Icarus* **9**, 305–314.

Sperling, H. J.: 1969, 'On the Real Singularities of the *N*-Body Problem' (to be published in *J. Reine Angew. Math*).