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On the Singularities of Generalized Solutions to n-Body-Type Problems

Vivina Barutello[1], Davide L. Ferrario[2], and Susanna Terracini[2]

[1]Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Turin, Italy and [2]Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, via Cozzi 53, 20125 Milan, Italy

Correspondence to be sent to: davide.ferrario@unimib.it

The validity of Sundman-type asymptotic estimates for collision solutions is established for a wide class of dynamical systems with singular forces, including the classical n-body problems with Newtonian, quasi-homogeneous and logarithmic potentials. The solutions are meant in the generalized sense of Morse (locally-in space and time-minimal trajectories with respect to compactly supported variations) and their uniform limits. The analysis includes the extension of the Von Zeipel’s theorem and the proof of isolatedness of collisions. Furthermore, such asymptotic analysis is applied to prove the absence of collisions for locally minimal trajectories.

1 Introduction

Many systems of interacting bodies of interest in celestial and other areas of classical mechanics have the form

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}(t, x), \qquad i = 1, \dots, n, \tag{1.1}$$

where the forces $\frac{\partial U}{\partial x_i}$ are undefined on a singular set Δ. This is, for example, the set of collisions between two or more particles in the n-body problem. Such singularities play a fundamental role in the phase portrait (see, e.g. [19]) and strongly influence the

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global orbit structure, as they can be held responsible, among others, of the presence of chaotic motions (see, e.g. [15]) and of motions becoming unbounded in a finite time [34, 53].

There are two major steps in the analysis of the impact of the singularities in the n-body problem: the first consists in performing the asymptotic analysis along a single collision (total or partial) trajectory and goes back, in the classical case, to the works by Sundman [49], Wintner [52] and, in more recent years by Sperling, Pollard, Saari, Diacu and other authors (see for instance [16, 23, 42, 43, 46, 48]). The second step consists in blowing-up the singularity by a suitable change of coordinates introduced by McGehee in [35] and replacing it by an invariant boundary—the collision manifold—where the flow can be extended in a smooth manner. It turns out that, in many interesting applications, the flow on the collision manifold has a simple structure: it is a gradient-like, Morse-Smale flow featuring a few stationary points and heteroclinic connections (see, for instance, the surveys [15, 37]). The analysis of the extended flow allows us to obtain a full picture of the behavior of solutions near the singularity, despite the fact that the flow fails to be fully regularizable (except in a few cases).

The geometric approach, via the McGehee coordinates and the collision manifold, can be successfully applied also to obtain asymptotic estimates in some cases, such as the collinear three-body problem [35], the anisotropic Kepler problem [12, 13, 21, 22], the three-body problem both in the planar isosceles case [14] and the full perturbed three body, as described in [15, 18]. Besides the quoted cases, however, one needs to establish the asymptotic estimates before blowing up the singularity, in order to prove convergence of the blow-up family. The reason is quite technical and mainly rests in the fact that a singularity of the n-body problems need not be isolated, for the possible occurrence of partial collisions in a neighborhood of the total collision. In the literature, this problem has been usually overcome by extending the flow on partial collisions via some regularization technique (such as Sundman's, in [14], or Levi-Civita's in [31]). Such a device works well only when partial collisions are binary, which are the only singularities to be globally removable. Thus, the extension of the geometrical analysis to the full n-body problems finds a strong theoretical obstruction: partial collisions must be regularizable, what is known to hold true only in few cases. Other interesting cases in which the geometric method is not effective are that of quasi-homogeneous potentials (where there is a lack of regularity for the extended flow) and that of logarithmic potentials (for the failure of the blow-up technique).

In this paper we extend the classical asymptotic estimates near collisions in three main directions.

(1) We take into account a very general notion of solution for the dynamical system (1.1), which fits particularly well to solutions found by variational techniques. Our notion of solution includes, besides all classical noncollision trajectories, all the *locally minimal solutions* (with respect to compactly supported variations) that are often termed minimal in the sense of Morse. Furthermore, we include in the set of *generalized solutions* all the limits of classical and locally minimal solutions.

(2) We extend our analysis to a wide class of potentials including not only homogeneous and quasi-homogeneous potentials, but also those with weaker singularities of logarithmic type.

(3) We allow potentials to strongly depend on time (we only require its time derivative to be controlled by the potential itself—see assumption (U1)). In this way, for instance, we can take into account models where masses vary in time.

Our main results on the asymptotics near total collisions (at the origin) are Theorems 4.18 and 4.20 (for quasi-homogeneous potentials) and Theorems 4.27 and 4.28 (when the potential is of logarithmic type) which extend the classical Sundman–Sperling asymptotic estimates [48, 49] in the directions above (see also [18, 20]).

As a consequence of the asymptotic estimates, the presence of a total collision prevents the occurrence of partial ones for neighboring times.

This observation plays a central role when extending the asymptotic estimates to the full n-body problem, since it allows us to reduce from partial (even simultaneous) collisions to total ones by decomposing the system in colliding clusters. Our results also lead to the extension of the concept of singularity for the dynamical system (1.1) to the class of generalized solutions. We shall prove an extension of the Von Zeipel's theorem: when the moment of inertia is bounded then every singularity of a generalized solution admits a limiting configuration, hence all singularities are collisions. The results on total collisions are then fully extended to partial ones in Theorem 5.2.

A further motivation for the study of generalized solutions comes from the variational approach to the study of selected trajectories to the n-body problem. Indeed, the exclusion of collisions is a major problem in the application of variational techniques as it results in the recent literature, where many different arguments have been introduced to prove that the trajectories found in such a way are collisionless (see [1, 2, 4, 6–8, 10, 11, 25, 26, 38–40, 47, 50, 51]). As a first application, we shall be able to extend some of these techniques in order to prove that action-minimizing trajectories are free of collisions for a wider class of interaction potentials. For example, in the case of quasi-homogeneous potentials, once collisions are isolated, the blow-up technique can be successfully applied to prove that locally minimal solutions are, in many circumstances, *free of collisions*. In order to do that, we can use the method of *averaged variations* introduced by Marchal and developed in [9, 26, 33]. It has to be noticed that, when dealing with logarithmic-type potentials, the blow-up technique is not available, since converging blow-up sequences do not exist; we can anyway prove that the average over all possible variations is negative by taking advantage of the harmonicity of the function $\log|x|$ in $\mathbb{R}^2$. With this result we can then extend to quasi-homogeneous and logarithmic potentials all the analysis of the (equivariant) minimal trajectories carried in [26].

Besides the direct method, other variational techniques—Morse and minimax theory—have been applied to the search of periodic solutions in singular problems [1, 3, 32, 45]. In the quoted papers, however, only the case of *strong force* interaction (see [27]) has been treated. Let us consider a sequence of solutions to penalized problems where an infinitesimal sequence of strong force terms is added to the potential: then its limit enjoys the same conservation laws as the generalized solutions. Hence, our main results apply also to this class of trajectories. We believe that our study can be usefully applied to develop a Morse theory that takes into account the topological contribution of collisions. Partial results in this direction are given in [5, 44], where the contribution of collisions to the Morse index is computed.

2 Singularities of Locally Minimal Solutions

2.1 Locally minimal solutions

We fix a scalar product on the *configuration space* $\mathbb{R}^k$, with associated norm, and we denote by $I(x) = |x|^2$ the *moment of inertia* associated to the configuration $x \in \mathbb{R}^k$ and

$$\mathcal{E} := \{x \in \mathbb{R}^k : |x|^2 = 1\},$$

the inertia ellipsoid. We define the radial and "angular" variables associated to $x \in \mathbb{R}^k$ as

$$r := |x| = I^{\frac{1}{2}}(x) \in [0, +\infty), \quad s := \frac{x}{|x|} \in \mathcal{E}. \tag{2.1}$$

We consider the dynamical system

$$\ddot{x} = \nabla U(t, x) \tag{2.2}$$

on the time interval $(a,b) \subset \mathbb{R}$, $-\infty \le a < b \le +\infty$. Here U is a positive time-dependent potential function $U\colon (a,b) \times (\mathbb{R}^k \setminus \Delta) \to \mathbb{R}^+$, and it is supposed to be of class $\mathcal{C}^1$ on its domain; by ∇U we denote its gradient with respect to the given metric.

Remark 2.1. In the case of n-body-type systems as described in equation (1.1), given $m_1, \dots, m_n$, $n \ge 2$ positive real numbers, we define the scalar product induced by the *mass metric* on the *configuration space* $\mathbb{R}^{nd}$ between $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, as

$$x \cdot y = \sum_{i=1}^{n} m_i \langle x_i, y_i \rangle, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. We denote by $|\cdot|$ the norm induced by the mass scalar product (2.3). Then $\nabla U(t,x)$ denotes the gradient of the potential, in the mass metric, with respect to the spatial variable x, that is,

$$\nabla U(t,x) = M^{-1} \frac{\partial U}{\partial x}(t,x),$$

where $(\frac{\partial U}{\partial x})_i = \frac{\partial U}{\partial x_i}$, $i = 1, \dots, n$, and $M = [M_{ij}]$, $M_{ij} = m_i \delta_{ij} \mathbf{1}_d$ ($\mathbf{1}_d$ is the d-dimensional identity matrix) for every $i, j = 1, \dots, n$. $\square$

Furthermore, we suppose that Δ is the closed singular set for U of an attractive type, in the sense that

(U0) $\displaystyle \lim_{x \to \Delta} U(t,x) = +\infty$, uniformly in t;
$U + |\nabla U(t,x)| \in L^\infty((a,b) \times K)$, for every compact $K \subset \mathbb{R}^k \setminus \Delta$.

Borrowing the terminology from the study of the singularities of the n-body problem, the set Δ will be often referred as *collision set* and it is required to be a *cone*, that is,

$$x \in \Delta \quad \Rightarrow \quad \lambda x \in \Delta, \quad \forall \lambda \in \mathbb{R}.$$

We observe that being a cone implies that $0 \in \Delta$. When $x(t^*) = x^* \in \Delta$ for some $t^* \in (a,b)$, we will say that x has an *interior collision* at $t = t^*$ and that t^* is a *collision instant* for x. When $\lim_{t \to t^*} x(t) = x^* \in \Delta$ and $t^* = a$ or $t^* = b$ (when finite), we will talk about a *boundary collision*. In particular, if $x^* = 0 \in \Delta$, we will say that x has a *total collision at the origin* at $t = t^*$. A collision instant t^* is termed *isolated* if there exists $\delta > 0$ such that, for every $t \in (t^* - \delta, t^* + \delta) \cap (a,b)$, $x(t) \notin \Delta$.

We consider the following assumptions on the potential U.

(U1) There exists a constant $C_1 \geq 0$ such that, for every $(t,x) \in (a,b) \times (\mathbb{R}^k \setminus \Delta)$,

$$\left|\frac{\partial U}{\partial t}(t,x)\right| \leq C_1(U(t,x)+1).$$

(U2) There exist constants $\tilde{\alpha} \in (0,2)$ and $C_2 \geq 0$ such that

$$\nabla U(t,x) \cdot x + \tilde{\alpha} U(t,x) \geq -C_2.$$

We then define the *Lagrangian action functional* on the interval (a,b) as

$$\mathcal{A}(x,[a,b]) := \int_a^b K(\dot{x}) + U(t,x)\,dt, \tag{2.4}$$

where

$$K(\dot{x}) := \tfrac{1}{2}|\dot{x}|^2 \tag{2.5}$$

is the *kinetic energy*. We observe that $\mathcal{A}(\cdot,[a,b])$ is bounded and $\mathcal{C}^1$ on the Hilbert space $H^1((a,b),\mathbb{R}^k \setminus \Delta)$. In terms of the variables r and s introduced in equation (2.1), the action functional reads as

$$\mathcal{A}(rs,[a,b]) := \int_a^b \frac{1}{2}(\dot{r}^2 + r^2|\dot{s}|^2) + U(t,rs)\,dt$$

and the corresponding Euler–Lagrange equations, whenever $x \in H^1((a,b),\mathbb{R}^k \setminus \Delta)$, are

$$\begin{aligned} &-\ddot{r} + r|\dot{s}|^2 + \nabla U(t,rs)\cdot s = 0,\\ &-2r\dot{r}\dot{s} - r^2\ddot{s} + r\nabla_T U(t,rs) = \mu s, \end{aligned} \tag{2.6}$$

where $\mu = r^2|\dot{s}|^2$ is the Lagrange multiplier due to the presence of the constraint $|s|^2 = 1$ and the vector $\nabla_T U(t,rs)$ is the tangent component to the ellipsoid $\mathcal{E}$ of the gradient $\nabla U(t,rs)$, that is, $\nabla_T U(t,rs) = \nabla U(t,rs) - \nabla U(t,rs)\cdot s$.

Definition 2.2. A path $x \in H^1_{\mathrm{loc}}((a,b),\mathbb{R}^k)$ is a *locally minimal solution* for the dynamical system (2.2) if, for every $t_0 \in (a,b)$, there exists $\delta_0 > 0$ such that the restriction of x to the interval $I_0 = [t_0 - \delta_0, t_0 + \delta_0]$, is a local minimizer for $\mathcal{A}(\cdot, I_0)$ with respect to compactly supported variations (fixed ends). □

Remark 2.3. Since locally minimal solutions are in the local Sobolev space, they are continuous (actually locally Hölder) in the interval (a, b). We observe that a priori a locally minimal solution x can have a large collision set, $x^{-1}(\Delta)$; this set, though closed and of Lebesgue measure zero, can very well admit many accumulation points. For this reason, the Euler–Lagrange equations (2.6) and the dynamical system (2.2) do not hold for a locally minimal trajectory, not even in a distributional sense. On the other hand, one easily proves that a locally minimal solution satisfies the differential system (2.2) in the complement of the collision set $x^{-1}(\Delta)$. □

Remark 2.4. When the potential is of class $\mathcal{C}^2$ outside the collision set Δ, then every classical noncollision solution in the interval (a, b) is a locally minimal solution. □

Definition 2.5. A path x is a *generalized solution* for the dynamical system (2.2) if there exists a sequence x_n of locally minimal solutions such that

(1) $x_n \to x$ uniformly on compact subsets of (a, b);
(2) for almost all $t \in (a, b)$ the associated total energy $h_n(t) := K(\dot{x}_n(t)) - U(t, x_n(t))$ converges.

To avoid trivialities, we shall assume that the collision set $x^{-1}(\Delta)$ is not the full interval (a, b). □

Remark 2.6. Though generalized solutions could be quite weird objects, still they inherit some of the properties of locally minimal solutions. At first, we remark that it is a continuous path and it still satisfies the differential system (2.2) outside the (closed) collision set $x^{-1}(\Delta)$ (cf. Remark 2.3). This can be easily seen by selecting a noncollision instant $\bar{t} \notin x^{-1}(\Delta)$ such that the energy $h_n(\bar{t}) : K(\dot{x}_n(\bar{t})) - U(t, x_n(\bar{t}))$ converges, and passing to the limit in the differential system (2.2). □

In the framework of classical solutions to n-body systems, a solution x on the interval (t_1, t^*), has a *singularity* at t^* (finite) if it is not possible to extend it as a classical solution to a larger interval (t_1, t_3) with $t_3 > t^*$. The Painlevé's theorem [17, 41] asserts that the occurrence of a singularity at a finite time t^* is equivalent to the fact that the minimal of the mutual distances becomes infinitesimal as $t \to t^*$. This fact reads as follows.

Painlevé's theorem 1. Let $\bar{x}$ be a classical solution for the n-body dynamical system on the interval (t_1, t^*). If $\bar{x}$ has a singularity at $t^* < +\infty$, then the potential associated to the problem diverges to $+\infty$ as t approaches t^*. □

Painlevé's theorem does not necessarily imply that a collision (that is a singularity such that the whole configuration admits a definite limit) occurs when there is a singularity at a finite time; indeed, these two facts are equivalent only if each particle approaches a definite configuration (on this subject we refer to [42, 43, 46]). This result has been stated by Von Zeipel in 1908 (see [54] and also [36]) and definitely proved by Sperling in 1970 (see [48]): in the n-body problem, the occurrence of singularities (in finite time) that are not collisions is then equivalent to the existence of an unbounded motion.

Von Zeipel's theorem 1. If $\bar{x}$ is a classical solution for the n-body dynamical system on the interval (t_1, t^*) with a singularity at $t^* < +\infty$ and $\limsup_{t\to t^*} I(\bar{x}(t)) < +\infty$, then $\bar{x}(t)$ has a definite limit configuration x^* as t tends to t^*. □

We will come back later on the proof of this result (in Corollary 3.3 and in Section 5). One natural way of extending the notion of singularity to generalized solutions could be to say that a (classical, locally minimal, generalized) solution x on the interval (t_1, t^*), has a *singularity* at t^* (finite) if it is not possible to extend $\bar{x}$ as a (respectively, classical, locally minimal, generalized) solution to a larger interval (t_1, t_3) with $t_3 > t^*$. On the other hand, as done in Lemma 5.1, one can very easily extend Painlevé's theorem to the wider class of locally minimal, or even generalized solutions. In other words, for generalized solutions,

$$\limsup_{t\to t^*} U(t, \bar{x}(t)) = +\infty \;\Rightarrow\; \liminf_{t\to t^*} U(t, \bar{x}(t)) = +\infty.$$

Notice that the above limit makes sense, because $\bar{x}$ is a continuous function and therefore so is $U(t, \bar{x}(t))$ as an extended valued function. This observation motivates the next weaker definition.

Definition 2.7. We say that the (generalized) solution $\bar{x}$ for the dynamical system (2.2) has a *singularity* at $t = t^*$ if

$$\lim_{t\to t^*} U(t, \bar{x}(t)) = +\infty.$$

□

Definition 2.8. The singularity t^* is said to be a *collision* for the generalized solution $\bar{x}$ if it admits a limit configuration as t tends to t^*. □

2.2 Approximation of locally minimal solutions

Let $\bar{x}$ be a locally minimal solution on the interval (a, b) and let $I_0 \subset (a, b)$ be an interval such that $\bar{x}$ is a (local) minimizer for $\mathcal{A}(\cdot, I_0)$ with respect to compactly supported variations. Generally, local minimizers need not to be isolated; we illustrate below a penalization argument to select a particular solution from the possibly large set of local minimizers. To begin with, we define the auxiliary functional on the space $H^1(I_0, \mathbb{R}^k)$,

$$\bar{\mathcal{A}}(x, I_0) := \int_{I_0} K(\dot{x}) + U(t, x) + \frac{|x - \bar{x}|^2}{2}\, dt. \tag{2.7}$$

When the interval I_0 is sufficiently small, $\bar{x}$ is actually the global minimizer for the penalized functional $\bar{\mathcal{A}}(\cdot, I_0)$ defined in equation (2.7). Of course, we may assume that

$$\bar{\mathcal{A}}(\bar{x}, I_0) = \mathcal{A}(\bar{x}, I_0) < +\infty, \tag{2.8}$$

which is equivalent to require that $\bar{\mathcal{A}}(\cdot, I_0)$ takes a finite value at least at one point.

Proposition 2.9. Let $\bar{x}$ be a locally minimal solution on the interval (a, b), let $\delta_0 > 0$ and $t_0 \in (a, b)$ be such that $\bar{x}$ is a local minimizer for $\mathcal{A}(\cdot, I_0)$, where $I_0 = [t_0 - \delta_0, t_0 + \delta_0] \subset (a, b)$. Then there exists $\bar{\delta} = \bar{\delta}(\bar{x}) > 0$ such that whenever $\delta_0 \leq \bar{\delta}$, $\bar{x}$ is the unique global minimizer for $\bar{\mathcal{A}}(\cdot, I_0)$ with the fixed boundary conditions $\bar{x}_{|\partial I_0}$. □

Proof. For every $x \in H^1_{\mathrm{loc}}(I_0, \mathbb{R}^k)$, the inequality $\mathcal{A}(x, I_0) \leq \bar{\mathcal{A}}(x, I_0)$ holds true, and it is an equality only if $x = \bar{x}$. Since $\bar{x}$ is a local minimizer for $\mathcal{A}(x, I_0)$, one easily infers, by a simple convexity argument and the semicontinuity of the action functional, the existence of $\varepsilon > 0$ such that

$$\|x - \bar{x}\|_\infty < \varepsilon \ \text{ and } \ x_{|\partial I_0} = \bar{x}_{|\partial I_0} \quad \Rightarrow \quad \mathcal{A}(\bar{x}, I_0) \leq \mathcal{A}(x, I_0).$$

We conclude that, for every $x \in H^1_{\mathrm{loc}}(I_0, \mathbb{R}^k)$, such that $0 < \|x - \bar{x}\|_\infty < \varepsilon$, the following chain of inequalities holds:

$$\bar{\mathcal{A}}(\bar{x}, I_0) = \mathcal{A}(\bar{x}, I_0) \leq \mathcal{A}(x, I_0) < \bar{\mathcal{A}}(x, I_0);$$

hence $\bar{x}$ is a strict local minimizer for $\bar{\mathcal{A}}(\cdot, I_0)$, independently on δ_0.

In order to complete the proof, we show that $\bar{\mathcal{A}}(\bar{x}, I_0) < \bar{\mathcal{A}}(x, I_0)$ also for those functions $x \in H^1_{\mathrm{loc}}(I_0, \mathbb{R}^k)$ such that $\|x - \bar{x}\|_\infty \geq \varepsilon$, provided δ_0 is sufficiently small. Indeed, since the Sobolev space $H^1_{\mathrm{loc}}(I_0, \mathbb{R}^k)$ is embedded in the space of absolutely continuous functions, we can compute, by Hölder inequality,

$$\begin{aligned} |(x - \bar{x})(t)| &\leq \int_{I_0} |\dot{x}(s)| ds + \int_{I_0} |\dot{\bar{x}}(s)| ds \\ &\leq \sqrt{2\delta_0} \left(\sqrt{\int_{I_0} |\dot{x}(s)|^2 ds} + \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 ds} \right). \end{aligned} \tag{2.9}$$

By taking the supremum at both sides of equation (2.9), it follows that

$$\frac{\|x - \bar{x}\|_\infty}{\sqrt{2\delta_0}} - \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 \, ds} \leq \sqrt{\int_{I_0} |\dot{x}(s)|^2 \, ds},$$

and therefore, for every $x \in H^1(I_0, \mathbb{R}^k)$,

$$\begin{aligned} \bar{\mathcal{A}}(x, I_0) \geq \frac{1}{2} \int_{I_0} |\dot{x}(s)|^2 \, ds &\geq \frac{1}{2} \left(\frac{\|x - \bar{x}\|_\infty}{\sqrt{2\delta_0}} - \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 \, ds} \right)^2 \\ &\geq \frac{1}{2} \left(\frac{\varepsilon}{\sqrt{2\delta_0}} - \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 \, ds} \right)^2. \end{aligned} \tag{2.10}$$

Hence, by choosing δ_0 such that $2\delta_0 < \varepsilon^2 (\sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 ds} + \sqrt{\bar{\mathcal{A}}(\bar{x}, I_0)})^{-2}$, it follows that

$$\bar{\mathcal{A}}(x, I_0) \geq \frac{1}{2} \left(\frac{\varepsilon}{\sqrt{2\delta_0}} - \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 ds} \right)^2 > \bar{\mathcal{A}}(\bar{x}, I_0)$$

also for those paths $x \in H^1_{\mathrm{loc}}(I_0, \mathbb{R}^k)$ such that $\|x - \bar{x}\|_\infty \geq \varepsilon$. This concludes the proof. ■

We now wish to approximate the singular potential U with a family of smooth potentials $U_\varepsilon \colon (a, b) \times \mathbb{R}^k \to \mathbb{R}^+$, depending on a parameter $\varepsilon > 0$. To this aim, consider the function

$$\eta(s) = \begin{cases} s & \text{if } s \in [0, 1] \\ \frac{-s^2 + 6s - 1}{4} & \text{if } s \in [1, 3] \\ 2 & \text{if } s \geq 3; \end{cases}$$

notice that $\eta \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$ and, for every $s \in [0, +\infty)$,

$$\dot{\eta}(s)\, s \leq \eta(s) \qquad \text{and} \qquad \dot{\eta}(s) \leq 1.$$

Now let us define, for $\varepsilon > 0$,

$$\eta_\varepsilon(s) := \frac{1}{\varepsilon}\eta(\varepsilon s);$$

then the following inequalities hold for every $s \in [0, +\infty)$:

$$\dot{\eta}_\varepsilon(s)\, s \leq \eta_\varepsilon(s), \qquad \text{and} \qquad \dot{\eta}_\varepsilon(s) \leq 1. \tag{2.11}$$

By means of the family η_ε, we can regularize the potential U in the following way:

$$U_\varepsilon(t, x) = \begin{cases} \eta_\varepsilon(U(t, x)), & \text{if } x \in \mathbb{R}^k \setminus \Delta, \\ 2/\varepsilon, & \text{if } x \in \Delta. \end{cases} \tag{2.12}$$

It is worthwhile noticing that each $U_\varepsilon(t, x)$ coincides with $U(t, x)$ whenever $U(t, x) \leq 1/\varepsilon$; in fact

$$\eta_\varepsilon(s) = \frac{1}{\varepsilon}\eta(\varepsilon s) = s$$

whenever $\varepsilon s \in [0, 1]$, that is, $s \in [0, 1/\varepsilon]$. Next, we consider the associated family of boundary value problems on the interval $I_0 \subset (a, b)$,

$$\begin{cases} \ddot{x} = \nabla U_\varepsilon(t, x) + (x - \bar{x}), \\ x|_{\partial I_0} = \bar{x}|_{\partial I_0} \end{cases} \tag{2.13}$$

where, as usual, $\nabla U_\varepsilon(t, x)$ is the gradient, in the mass metric, with respect to the spatial variable x. Solutions of equation (2.13) are critical points of the action functional

$$\bar{\mathcal{A}}_\varepsilon(x, I_0) := \int_{I_0} K(\dot{x}) + U_\varepsilon(t, x) + \frac{|x - \bar{x}|^2}{2}\, dt. \tag{2.14}$$

We observe that $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ is bounded and $\mathcal{C}^2$ is on $H^1_{\mathrm{loc}}(I_0, \mathbb{R}^k)$, since U_ε is smooth on the whole $\mathbb{R}^k$. We also remark that the infimum of $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ is achieved, for $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ is a positive and coercive functional on $H^1_{\mathrm{loc}}(I_0, \mathbb{R}^k)$.

In the next proposition, we prove that a locally minimal solution has the fundamental property of being the limit of a sequence of global minimizers for the (smooth) approximating functionals $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$, provided the interval $I_0 \subset (a, b)$ is chosen so small that the restriction of the minimal solution to I_0 is the unique global minimizer for $\bar{\mathcal{A}}(\cdot, I_0)$. This is a crucial result; indeed, as already observed in Remark 2.3, although the Euler-Lagrange equations and the differential system do not hold for a locally minimal solution, we will anyway be able to take advantage of the the differential equations associated with the approximating global minimizers (for the regularized problems). In this way, we will be in a position to prove the basic qualitative properties of locally minimal (and generalized) solutions, namely the conservation laws and the monotonicity formula, which will be widely exploited in the rest of the paper.

Proposition 2.10. Let $\bar{x}$ and I_0 be given by Proposition 2.9. Let $\varepsilon > 0$ and x_ε be a global minimizer for $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$, with boundary conditions $x_{\varepsilon|\partial I_0} = \bar{x}_{|\partial I_0}$. Then, up to subsequences, as $\varepsilon \to 0$,

- (i) $U_\varepsilon(t, x_\varepsilon) \to U(t, \bar{x})$ almost everywhere and in L^1;
- (ii) $x_\varepsilon \to \bar{x}$ uniformly;
- (iii) $\dot{x}_\varepsilon \to \dot{\bar{x}}$ in L^2;
- (iv) $\dot{x}_\varepsilon \to \dot{\bar{x}}$ almost everywhere;
- (v) $\frac{\partial U_\varepsilon}{\partial t}(t, x_\varepsilon) \to \frac{\partial U}{\partial t}(t, \bar{x})$ almost everywhere and in L^1.

$\square$

Proof. As we have already observed, for every $\varepsilon > 0$, the potential U_ε coincides with U on the sublevel $\{(t, x) : U(t, x) \leq 1/\varepsilon\}$ and, by its definition, for every $(t, x) \in I_0 \times (\mathbb{R}^k \setminus \Delta)$ there holds

$$U_\varepsilon(t, x) \leq U(t, x).$$

Therefore

$$\bar{\mathcal{A}}_\varepsilon(x, I_0) \leq \bar{\mathcal{A}}(x, I_0)$$

for every $x \in H^1_{\mathrm{loc}}(I_0, \mathbb{R}^k)$. It follows from equation (2.8) that

$$\bar{\mathcal{A}}_\varepsilon(x_\varepsilon, I_0) = \inf_{x \in H^1_{\mathrm{loc}}} \bar{\mathcal{A}}_\varepsilon(x, I_0) \leq \bar{\mathcal{A}}(\bar{x}, I_0) < +\infty, \tag{2.15}$$

which implies the boundedness of the family $\{\int_{I_0} |\dot{x}_\varepsilon|^2 + |x_\varepsilon - \bar{x}|^2\}_\varepsilon$. Hence, we deduce the existence of a sequence $(x_{\varepsilon_n})_{\varepsilon_n} \subset (x_\varepsilon)_\varepsilon$ such that $(\dot{x}_{\varepsilon_n})_{\varepsilon_n}$ converges weakly in L^2 and uniformly to some limit $\tilde{x}$. In addition, we observe that

$$\lim_{\varepsilon_n \to 0} U_{\varepsilon_n}(t, x_{\varepsilon_n}(t)) = U(t, \tilde{x}(t))$$

for every $t \in I_0$, regardless of the finiteness of $U(t, \tilde{x}(t))$.

From equation (2.15), we also deduce the boundedness of the following integrals:

$$\int_{I_0} U_{\varepsilon_n}(t, x_{\varepsilon_n})\, dt \le \bar{\mathcal{A}}_{\varepsilon_n}(x_{\varepsilon_n}, I_0) < +\infty,$$

and therefore, since the sequence $(U_{\varepsilon_n}(t, x_{\varepsilon_n}))_{\varepsilon_n}$ is positive, by applying Fatou's lemma one deduces that

$$\int_{I_0} U(t, \tilde{x}) \le \liminf_{n \to +\infty} \int_{I_0} U_{\varepsilon_n}(t, x_{\varepsilon_n}) < +\infty.$$

Hence, from the weak semicontinuity of the norm in L^2 (the sequence $(\dot{x}_{\varepsilon_n})_{\varepsilon_n}$ converges weakly in L^2 to $\tilde{x}$), we obtain the inequalities

$$\bar{\mathcal{A}}(\tilde{x}, I_0) \le \liminf_{n \to +\infty} \bar{\mathcal{A}}_{\varepsilon_n}(x_{\varepsilon_n}, I_0) \le \bar{\mathcal{A}}(\bar{x}, I_0)$$

that contradict Proposition 2.9, unless $\tilde{x} = \bar{x}$ and

$$\liminf_{n \to +\infty} \bar{\mathcal{A}}_{\varepsilon_n}(x_{\varepsilon_n}, I_0) = \bar{\mathcal{A}}(\bar{x}, I_0). \tag{2.16}$$

Therefore, we deduce the L^2-convergence of the sequence $(\dot{x}_{\varepsilon_n})_{\varepsilon_n}$ and its convergence almost everywhere to $\dot{\bar{x}}$, up to subsequences. From equation (2.16), it also follows that

$$\lim_{\varepsilon_n \to 0} \int_{I_0} U_{\varepsilon_n}(t, x_{\varepsilon_n}) = \int_{I_0} U(t, \bar{x}). \tag{2.17}$$

From the convergence almost everywhere of $(U_{\varepsilon_n}(t, x_{\varepsilon_n}))_{\varepsilon_n}$, together with equation (2.17) and Egorov's theorem, we conclude its convergence in L^1 to $U(t, \bar{x})$.

We now turn to the convergence of the sequence $(\varphi_n(t))_{\varepsilon_n} = (\frac{\partial U_{\varepsilon_n}}{\partial t}(t, x_{\varepsilon_n}))_{\varepsilon_n}$. To this aim, we observe that condition (U1) together with equation (2.11) imply the following chain of inequalities:

$$\begin{aligned}\left|\frac{\partial U_{\varepsilon_n}}{\partial t}(t, x_{\varepsilon_n}(t))\right| &= \dot\eta_{\varepsilon_n}(U(t, x_{\varepsilon_n}(t)))\left|\frac{\partial U}{\partial t}(t, x_{\varepsilon_n}(t))\right| \\ &\le C_1 \dot\eta_{\varepsilon_n}\big(U\big(t, x_{\varepsilon_n}(t)\big)\big)\big(U\big(t, x_{\varepsilon_n}(t)\big)+1\big) \\ &\le C_1\big(\eta_{\varepsilon_n}\big(U\big(t, x_{\varepsilon_n}(t)\big)\big)+1\big) \\ &= C_1\big(U_{\varepsilon_n}\big(t, x_{\varepsilon_n}(t)\big)+1\big).\end{aligned}$$

This implies the convergence of $(\varphi_n(t))_{\varepsilon_n}$ for all t such that $U(t, \bar x(t))$ is finite, and $x_{\varepsilon_n}(t)$ converges, that is almost everywhere in I_0. In order to prove convergence in $L^1(I_0)$, we apply Vitali's necessary and sufficient condition for convergence, in both logical directions. Indeed, we already know that $U_{\varepsilon_n}(t, x_{\varepsilon_n}(t))$ converges in $L^1(I_0)$: hence this sequence is uniformly integrable, and thus the same is true for the sequence $(\varphi_n(t))_{\varepsilon_n}$. Applying again Vitali's theorem, as, by Egorov's theorem, the almost everywhere convergence implies convergence in measure, we finally obtain the convergence of $(\varphi_n(t))_{\varepsilon_n}$ in L^1. ■

2.3 Conservation laws for locally minimizing solutions

In this section, we use the sequence of solutions to the regularized problems in order to prove boundedness of the energy for locally minimal solutions.

Proposition 2.11. Let $\bar x$ and I_0 be given by Proposition 2.9. Then the energy associated to $\bar x$,

$$h\colon I_0 \to \mathbb{R}, \qquad h(t) := K(\dot{\bar x}(t)) - U(t, \bar x(t)), \tag{2.18}$$

is of class $W^{1,1}$ on I_0 and its weak derivative is

$$\dot h(t) = -\frac{\partial U}{\partial t}(t, \bar x).$$

□

Proof. Let $(x_\varepsilon)_\varepsilon$ be the sequence, converging to $\bar x$, of global minimizers for the corresponding functionals $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ whose existence is proved in Proposition 2.10. Let h_ε be the energy associated to x_ε, that is,

$$h_\varepsilon\colon I_0 \to \mathbb{R}, \qquad h_\varepsilon(t) := K(\dot x_\varepsilon(t)) - U_\varepsilon(t, x_\varepsilon(t)) + \tfrac{1}{2}|\bar x(t) - x_\varepsilon(t)|^2. \tag{2.19}$$

From Proposition 2.10, we immediately deduce that the sequence $(h_\varepsilon)_\varepsilon$ converges almost everywhere and $L^1(I_0)$ to $h(\bar{x})$. Now we compute the weak derivative of h. To this end, let us consider a test function $\varphi \in \mathcal{C}_0^\infty(I_0)$; we can write

$$\begin{aligned}\int_{I_0} h(t)\dot{\varphi}(t)\,dt &= \lim_{\varepsilon\to 0}\int_{I_0} h_\varepsilon(t)\dot{\varphi}(t)\,dt\\ &= \lim_{\varepsilon\to 0}\int_{I_0} \frac{\partial U_\varepsilon}{\partial t}(t, x_\varepsilon(t))\varphi(t)\,dt.\end{aligned}$$

In consequence of Proposition 2.10, the sequence $(\frac{\partial U_\varepsilon}{\partial t}(t, x_\varepsilon(t)))_\varepsilon$ converges to $\frac{\partial U}{\partial t}(t, \bar{x}(t))$ in L^1; then

$$\lim_{\varepsilon\to 0}\int_{I_0} \frac{\partial U_\varepsilon}{\partial t}(t, x_\varepsilon(t))\varphi(t)\,dt = \int_{I_0} \frac{\partial U}{\partial t}(t, \bar{x}(t))\varphi(t)\,dt, \qquad \forall\varphi \in \mathcal{C}_0^\infty(I_0), \tag{2.20}$$

and hence

$$\int_{I_0} h(t)\dot{\varphi}(t)\,dt = \int_{I_0} \frac{\partial U}{\partial t}(t, \bar{x})\varphi(t)\,dt, \qquad \forall\varphi \in \mathcal{C}_0^\infty(I_0),$$

which means that $\frac{\partial U}{\partial t}(t, \bar{x})$ is the weak derivative of $h(\bar{x})$. ∎

Let us remark that Proposition 2.11 implies that the sequence h_ε of the energies of the approximating global minimizers converges to h, the energy of the locally minimal solution, on compact intervals. The next corollary follows straightforwardly.

Corollary 2.12. The energy associated with a locally minimal solution on the interval (a, b) is in the Sobolev space $W^{1,1}_{\mathrm{loc}}((a, b), \mathbb{R})$. Moreover,

$$\dot{h}(t) = -\frac{\partial U}{\partial t}(t, \bar{x}).$$

□

We now investigate the behavior of the moment of inertia of a locally minimal solution when a singularity occurs (see Definition 2.7). The results contained in Proposition 2.13 and Corollary (2.14) are the natural extensions of the classical Lagrange–Jacobi inequality to locally minimal solutions (see [52]).

Proposition 2.13. Let $\bar{x}$ be a locally minimal solution and I_0 be given by Proposition 2.9. Then

$$\frac{1}{2}\int_{I_0} I(\bar{x}(t))\ddot{\varphi}(t)\,dt \geq \int_{I_0} [2h(\bar{x}(t)) + (2-\tilde{\alpha})U(t,\bar{x}(t)) - C_2]\varphi(t)\,dt \tag{2.21}$$

for every $\varphi \in \mathcal{C}_0^\infty(I_0,\mathbb{R})$, $\varphi(t) \geq 0$. □

Proof. Let $(x_\varepsilon)_\varepsilon$ be the sequence of global minimizers for the corresponding functionals $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ convergent to $\bar{x}$ whose existence is proved in Proposition 2.10. When we compute the second derivative of the moment of inertia of x_ε, we obtain

$$\begin{aligned}
\tfrac{1}{2}\ddot{I}(x_\varepsilon(t)) &= |\dot{x}_\varepsilon(t)|^2 + \ddot{x}_\varepsilon(t)\cdot x_\varepsilon(t)\\
&= 2h_\varepsilon(t) + 2U_\varepsilon(t,x_\varepsilon(t)) - |\bar{x}(t) - x_\varepsilon(t)|^2 + [\nabla U_\varepsilon(t,x_\varepsilon(t)) + (x_\varepsilon(t) - \bar{x}(t))]\cdot x_\varepsilon(t)\\
&= 2h_\varepsilon(t) + 2U_\varepsilon(t,x_\varepsilon(t)) + \bar{x}(t)\cdot(x_\varepsilon(t) - \bar{x}(t)) + \dot{\eta}_\varepsilon(U(t,x_\varepsilon))\nabla U(t,x_\varepsilon(t))\cdot x_\varepsilon(t),
\end{aligned}$$

hence, by assumption (U2) on the potential U and inequality (2.11), it follows that

$$\begin{aligned}
\tfrac{1}{2}\ddot{I}(x_\varepsilon(t)) &\geq 2h_\varepsilon(t) + 2U_\varepsilon(t,x_\varepsilon(t)) + \bar{x}(t)\cdot(x_\varepsilon(t) - \bar{x}(t)) - \dot{\eta}_\varepsilon(U(t,x_\varepsilon))[\alpha U(t,x_\varepsilon) + C_2]\\
&\geq 2h_\varepsilon(t) + (2-\tilde{\alpha})U_\varepsilon(t,x_\varepsilon(t)) + \bar{x}(t)\cdot(x_\varepsilon(t) - \bar{x}(t)) - C_2
\end{aligned} \tag{2.22}$$

for some $\tilde{\alpha} \in (0,2)$ and $C_2 > 0$. Therefore, since $x_\varepsilon \in \mathcal{C}^2(I_0)$, for every $\varphi \in \mathcal{C}_0^\infty(I_0,\mathbb{R})$, $\varphi(t) \geq 0$

$$\frac{1}{2}\int_{I_0} I(x_\varepsilon(t))\ddot{\varphi}(t)\,dt\,\frac{1}{2}\int_{I_0} \ddot{I}(x_\varepsilon(t))\varphi(t)\,dt$$

and, from equation (2.22),

$$\frac{1}{2}\int_{I_0} I(x_\varepsilon(t))\ddot{\varphi}(t)\,dt \geq \int_{I_0} [2h_\varepsilon(t) + (2-\tilde{\alpha})U_\varepsilon(t,x_\varepsilon(t)) + \bar{x}(t)\cdot(x_\varepsilon(t) - \bar{x}(t)) - C_2]\varphi(t)\,dt.$$

We conclude by passing to the limit as $\varepsilon \to 0$ in equation (2.22) and using the L^1-convergences proved in Propositions 2.10 and 2.11. ■

The corollaries presented ahead follow directly.

Corollary 2.14 (Lagrange–Jacobi inequality). Let $\bar{x}$ be a locally minimal solution on the interval (a, b). Then the following inequality holds in the distributional sense:

$$\tfrac{1}{2}\ddot{I}(\bar{x}(t)) \geq 2h(t) + (2 - \tilde{\alpha})U(t, \bar{x}(t)) - C_2.$$

□

Remark 2.15. As a consequence, the function $\dot{I}$ is not only (locally) in L^2, but it is also of (local) bounded variation in (a, b), for it is the sum of an increasing function and a $W^{1,1}_{\mathrm{loc}}$ one. Therefore, it has at most a countable number of discontinuities. In addition, we have, for any pair of regular points $a \leq t_1 < t_2 \leq b$,

$$\dot{I}(\bar{x}(t_2)) - \dot{I}(\bar{x}(t_2)) = \int_{t_1}^{t_2} d\dot{I} \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \ddot{I}(\bar{x}(t))\,dt \geq \int_{t_1}^{t_2} [4h(t) + 2(2 - \tilde{\alpha})U(t, \bar{x}(t)) - 2C_2]\,dt,$$

where the first is a Lebesgue–Stieltjes integral, the second has to be intended in the sense of measures, while the last one is a Lebesgue integral. □

Corollary 2.16. Let $\bar{x}$ be a locally minimal solution and I_0 be given by Proposition 2.9. Then its moment of inertia is a convex function on the interval I_0 whenever $\bar{x}$ has a singularity in t_0 and δ_0 is small enough. □

Proof. Whenever ε and δ_0 are sufficiently small, the right-hand side of inequality (2.22) is strictly positive, indeed $h_\varepsilon(t)$ is bounded, x_ε converges to $\bar{x}$ uniformly and $U_\varepsilon(t, x_\varepsilon(t))$ diverges to $+\infty$. Whenever ε is small enough, we conclude that

$$\ddot{I}(x_\varepsilon(t)) > 0,$$

and hence $I(x_\varepsilon)$ are strictly convex functions in a neighborhood of t_0. Since the sequence $I(x_\varepsilon)$ uniformly converges to $I(\bar{x})$, we conclude that also $I(\bar{x})$ is convex on the interval I_0. ■

Now we extend the above corollary to the case of boundary singularities. This will rule the occurrence of accumulation of a sequence of singularities at the right boundary of the finite interval (a, b); from now on, we will suppose that $b < +\infty$.

Lemma 2.17. Let $\bar{x}$ be a locally minimizing solution in (a,b), let h be its energy (defined in equation (2.18)) and fix $\tau \in (a,b)$. Then there exists a constant $K > 0$ such that

$$\left|\int_\tau^t h(s)ds\right| \leq C_1(b-\tau)\int_\tau^t U(\xi,\bar{x}(\xi))\,d\xi + K, \qquad \forall t \in (\tau,b). \tag{2.23}$$

□

Proof. Since h is absolutely continuous on every interval $[\tau,t] \subset (a,b)$ (Corollary 2.12), we have

$$|h(t)| \leq |h(\tau)| + \int_\tau^t |\dot{h}(\xi)|\,d\xi, \qquad \forall t \in (\tau,b).$$

From Proposition 2.11 and assumption (U1), we obtain

$$\begin{aligned}|h(t)| &\leq |h(\tau)| + \int_\tau^t \left|\frac{\partial U}{\partial \xi}(\xi,\bar{x}(\xi))\right|\,d\xi \\ &\leq |h(\tau)| + C_1\int_\tau^t (U(\xi,\bar{x}(\xi))+1)\,d\xi,\end{aligned} \tag{2.24}$$

and, integrating both sides of the inequality on the interval $[\tau,t]$,

$$\int_\tau^t |h(s)|ds \leq |h(\tau)|(t-\tau) + C_1\frac{(t-\tau)^2}{2} + C_1\int_\tau^t ds\int_\tau^s U(\xi,\bar{x}(\xi))\,d\xi.$$

Since U is positive, the integral $\int_\tau^s U(\xi,\bar{x}(\xi))\,d\xi$ increases in the variable s, hence we conclude that

$$\left|\int_\tau^t h(s)ds\right| \leq \int_\tau^t |h(s)|ds \leq |h(\tau)|(t-\tau) + C_1\frac{(t-\tau)^2}{2} + C_1(b-\tau)\int_\tau^t U(\xi,\bar{x}(\xi))\,d\xi,$$

which proves the assertion with $K = |h(\tau)|(b-\tau) + C_1(b-\tau)^2/2$. ■

Next, we show that the potential integral is finite on a left neighborhood of $b < +\infty$. This fact will imply boundedness of the whole action and smoothness of the total energy up to b. This is a remarkable fact when b is a boundary singularity for the locally minimal solution.

Lemma 2.18. Let $\bar{x}$ be a locally minimal solution on a finite interval (a,b). Suppose that

$$\liminf_{t\to b^-} \dot{I}(\bar{x}(t)) = C < +\infty \tag{2.25}$$

(here the lower limit is meant up to zero-measure sets, namely as $\lim_{t\to b^-} \operatorname{ess\,inf}_{[t,b)} \dot{I}$) and let $\delta > 0$ be such that

$$\lambda := \frac{2-\tilde{\alpha}}{2} - C_1\delta > 0. \tag{2.26}$$

Then there exists $\tau \in (b-\delta, b)$ such that

$$\int_\tau^b U(t,\bar{x}(t))\,dt < +\infty.$$

□

Proof. It follows from equation (2.25) that there exists an increasing sequence of instants $(t_n)_n$ of continuity of $\dot{I}$ such that

$$t_n \to b \text{ as } n \to +\infty \quad \text{and} \quad \lim_{n\to\infty} \dot{I}(\bar{x}(t_n)) = C.$$

Now let the constant λ defined in equation (2.26) be strictly positive and let $\tau \in (b-\delta, b)$ be a point of continuity and finiteness for $\dot{I}(\bar{x})$. Hence, for every integer n sufficiently large,

$$C + 1 - \dot{I}(\bar{x}(\tau)) \geq \dot{I}(\bar{x}(t_n)) - \dot{I}(\bar{x}(\tau)) = \int_\tau^{t_n} d\,\dot{I}.$$

Therefore, Remark 2.15 implies the estimate

$$C + 1 - \dot{I}(\bar{x}(\tau)) \geq 4\int_\tau^{t_n} h(t)\,dt + 2(2-\tilde{\alpha})\int_\tau^{t_n} U(t,\bar{x}(t))\,dt - 2C_2(t_n-\tau).$$

We now apply Lemma 2.17 to deduce that

$$4\left(\frac{2-\tilde{\alpha}}{2} - C_1(b-\tau)\right)\int_\tau^{t_n} U(t,\bar{x}(t))\,dt \leq 2C_2(t_n-\tau) + C + 1 + 4K - \dot{I}(\bar{x}(\tau))$$

and therefore,

$$4\lambda\int_\tau^{t_n} U(t,\bar{x}(t))\,dt \leq 2C_2(t_n-\tau) + C + 1 + 4K - \dot{I}(\bar{x}(\tau)).$$

Since $\lambda > 0$ is fixed, as $n \to +\infty$ the proof is completed. Notice that, inserting the value of the constant K, the following estimate holds:

$$4\lambda \int_\tau^b U(t,\bar{x}(t))\,dt \le 2C_2(b-\tau) + C + 1 + 4\left[|h(\tau)|(b-\tau) + C_1\frac{(b-\tau)^2}{2}\right] - \dot{I}(\bar{x}(\tau)). \tag{2.27}$$

■

Corollary 2.19. Let $\bar{x}$ be a locally minimal solution on a finite interval (a,b). Suppose that b is a singularity such that

$$\liminf_{t\to b^-} \dot{I}(\bar{x}(t)) < +\infty,$$

where, as above, the lower limit is meant up to zero-measure sets, namely as $\lim\limits_{t\to b^-} \operatorname*{ess\,inf}\limits_{[t,b)} \dot{I}$. Then, if $a < \tau < b < +\infty$, there hold

(i) $\displaystyle\int_\tau^b U(t,\bar{x}(t))\,dt < +\infty$;

(ii) $\displaystyle\left|\int_\tau^b h(t)\,dt\right| < +\infty$;

(iii) $\displaystyle\int_\tau^b K(\dot{\bar{x}}(t))\,dt < +\infty$;

(iv) $\|h\|_\infty < +\infty$ on $[\tau,b)$.

(v) $\lim\limits_{t\to b^-} \bar{x}(t)$ exists.

(vi) $\dot{I}$ has bounded variation on $[\tau,b)$ and the following inequality holds in the distributional sense:

$$\tfrac{1}{2}\ddot{I}(\bar{x}(t)) \ge 2h(t) + (2-\tilde{\alpha})U(t,\bar{x}(t)) - C_2.$$

(vii) I is a strictly convex function in a neighborhood $[b-\delta,b)$ of b. □

Proof. The boundedness of the first integral follows from the assumption of local boundedness of the action functional on a locally minimal trajectory, assumption (2.8), and from Lemma 2.18. Concerning the second one, we use Corollary 2.12 and inequality (2.23); (iii) follows straightforwardly from (i), (ii), and the definition of the energy h. The boundedness of $\|h\|_\infty$ on (a,b) follows from Corollary 2.12 and inequality (2.24). To deduce (v), it is sufficient to remark that, from (iii), $\bar{x}$ is Hölder continuous on (a,b). Assertions (vi) and (vii) just follow from the Lagrange–Jacoby inequality (2.14), together with the boundedness of the energy and assertion (i). ■

3 Conservation Laws for Generalized Solutions

In order to extend the estimates of the previous section to the case of generalized solutions, we take a sequence x_n of minimal solutions as in Definition 2.5, that is converging to $\bar{x}$ on compact subsets of (a, b), and such that, for almost all $t \in (a, b)$, the associated total energy $h_n(t) := K(\dot{x}_n(t)) - U(t, x_n(t))$ converges. The next Lemma will be a crucial result in order to improve the regularity of a generalized solution.

Lemma 3.1. Let $\bar{x}$ be a generalized solution, let $\tau \notin \bar{x}^{-1}(\Delta)$ be a point of convergence of the energy sequence, and let $0 < \delta < b - \tau$ be such that equation (2.26) holds. Then

(i) $\displaystyle\limsup_{n\to+\infty} \int_\tau^{\tau+\delta/2} U(t, x_n(t))\, dt < +\infty;$

(ii) $\displaystyle\limsup_{n\to+\infty} \int_\tau^{\tau+\delta/2} |\dot{h}_n|\, dt < +\infty;$

(iii) $\displaystyle\limsup_{n\to+\infty} \int_\tau^{\tau+\delta/2} K(\dot{x}_n(t))\, dt < +\infty.$

Equivalent estimates hold in a left neighborhood of τ. □

Proof. Since $\tau \notin \bar{x}^{-1}(\Delta)$, the kinetic and the potential energy sequences converges separately. Since $\bar{x}$ is continuous, we can change τ into a close value $\tau' < \tau$ such that these facts still hold true, and moreover, the derivatives $\dot{I}(x_n(t)) := \dot{I}_n(t)$ are continuous in τ' for every n. On the other hand, as the I_n's are positive and absolutely continuous, there holds, for every n,

$$I_n(\tau + \delta) \geq I_n(\tau + \delta) - I_n\left(\tau + \frac{\delta}{2}\right) \geq \int_{\tau+\delta/2}^{\tau+\delta} \dot{I}_n(t)\, dt \geq \frac{\delta}{2} \operatorname*{essinf}_{[\tau+\delta/2, \tau+\delta]} \dot{I}_n.$$

Since $I_n(\tau + \delta)$ converges to $I(x(\tau + \delta)) := C\delta/2$, for any n sufficiently large, there are points $\tau_n'' \in [\tau + \delta/2, \tau + \delta]$ where $\dot{I}_n$ is continuous and such that $\dot{I}_n(\tau_n'') \leq C + 1$. Now we argue similarly to the proof of Lemma 2.18. From the inequality

$$C + 1 - \dot{I}_n(\tau') \geq \dot{I}_n(\tau_n'') - \dot{I}_n(\tau') = \int_{\tau'}^{\tau_n''} d\, \dot{I}_n$$

and Remark 2.15, we obtain the estimate

$$C + 1 - \dot{I}_n(\tau') \geq 4 \int_{\tau'}^{\tau_n''} h_n(t)\, dt + 2(2 - \tilde{\alpha}) \int_{\tau'}^{\tau_n''} U(t, x_n(t))\, dt - 2C_2(\tau_n'' - \tau').$$

Let the constant λ defined in equation (2.26) be strictly positive; for every sufficiently large integer n, we deduce, using Lemma 2.17, the estimate (cf. equation (2.27))

$$4\lambda\int_{\tau}^{\tau+\delta/2} U(t,x_n(t))\,dt \le 4\lambda\int_{\tau'}^{\tau''_n} U(t,x_n(t))\,dt \le 2C_2\delta + C + 1 + 4|h_n(\tau')|\delta + 2C_1\delta^2 - \dot I_n(\tau').$$

By our choice of τ', both the kinetic energy $K_n(\tau')$ and the total energy $h_n(\tau')$ converge: therefore, since $|\dot I_n(\tau')| \le \sqrt{I_n(\tau')K_n(\tau')}$, we obtain part (i) of the assertion. The second point of the assertion follows directly from the first, taking into account Corollary 2.12 and assumption (U1). As the sequence of the energies converges at the point τ, this implies in particular the boundedness of the energies in the L^∞ norm. Finally, by integration of the energy sequence, point (iii) easily follows. ■

By repeatedly applying this lemma, we can cover every compact subinterval of (a,b) and then, passing to the limit as n tends to infinity, we easily obtain the next result, which states that generalized solutions are actually much more regular, together with their energies. Moreover, the set of collision instants has zero measure, and the energy is bounded. Furthermore, we can pass into the limit in Lagrange–Jacobi distributional inequality (2.21), and, by applying Fatou's lemma on its right-hand side, we obtain the validity of the Lagrange–Jacobi inequality also for the generalized solutions.

Corollary 3.2. Let $\bar x$ be a generalized solution on a finite interval (a,b). Then

(i) $U(t,\bar x(t)) \in L^1_{\mathrm{loc}}(a,b)$;
(ii) $h \in BV_{\mathrm{loc}}(a,b)$;
(iii) $\bar x \in H^1_{\mathrm{loc}}(a,b)$;
(iv) $\dot I(\bar x) \in BV_{\mathrm{loc}}(a,b)$ and the following inequality holds in the distributional sense:

$$\tfrac{1}{2}\ddot I(\bar x(t)) \ge 2h(t) + (2-\tilde\alpha)U(t,\bar x(t)) - C_2.$$

□

As a consequence, we can extend to generalized solutions all the analysis developed in the previous section about the asymptotics at the bounday points of the finite interval (a,b).

Corollary 3.3. Let $\bar{x}$ be a generalized solution on a finite interval (a, b). Suppose that b is a singularity such that

$$\liminf_{t \to b^-} \dot{I}(\bar{x}(t)) < +\infty,$$

where the lower limit is meant up to zero-measure sets, namely as $\lim_{t \to b^-} \operatorname*{ess\,inf}_{[t,b)} \dot{I}$. Then the same conclusions of Corollary 2.19 hold. Namely,

(i) $\int_\tau^b U(t, \bar{x}(t))\, dt < +\infty$;
(ii) $\left|\int_\tau^b h(t)\, dt\right| < +\infty$;
(iii) $\int_\tau^b K(\dot{\bar{x}}(t))\, dt < +\infty$;
(iv) $\|h\|_\infty < +\infty$ on $[\tau, b)$.
(v) $\lim_{t \to b^-} \bar{x}(t)$ exists.
(vi) $\dot{I}$ has bounded variation on $[\tau, b)$ and the following inequality holds in the distributional sense:

$$\tfrac{1}{2}\ddot{I}(\bar{x}(t)) \geq 2h(t) + (2 - \tilde{\alpha})U(t, \bar{x}(t)) - C_2;$$

(vii) I is a strictly convex function in a neighborhood $[b - \delta, b)$ of b. □

A key remark is that, as one can easily check with a slight modification of the argument used in proving Lemma 3.1, the conclusions of Corollary 2.19 hold indeed, uniformly along the approximating sequence as shown in the following corollary.

Corollary 3.4. Let $\bar{x}$ be a generalized solution, as in Corollary 3.3 and let x_n be the approximating sequence of locally minimal solutions. Then we have

(i) $\lim_{s \to b} \limsup_{n \to +\infty} \displaystyle\int_\tau^s U(t, x_n(t))\, dt < +\infty$;
(ii) $\lim_{s \to b} \limsup_{n \to +\infty} \displaystyle\int_\tau^s |\dot{h}_n| + |h_n|\, dt < +\infty$;
(iii) $\lim_{s \to b} \limsup_{n \to +\infty} \displaystyle\int_\tau^s K(\dot{x}_n(t))\, dt < +\infty$;
(iv) $\limsup_{n \to +\infty} \|h_n\| < +\infty$. □

Remark 3.5. In Corollary 3.3 (v), Von Zeipel's theorem is proved, for generalized solutions, under the additional assumption (2.25). The proof will be completed in Section 5. □

4 Asymptotic Estimates at Total Collisions

The purpose of this section is to deepen the analysis of the asymptotics of generalized solutions as they approach a total collision at the origin. We recall that $\bar{x}$ has a *total collision at the origin* at $t = t^*$ if $\lim_{t\to t^*} \bar{x}(t) = 0$. Since, by our assumptions, 0 belongs to the singular set Δ of the potential, assumption (U0) implies in particular that a total collision instant is a singularity for $\bar{x}$. We will perform here all the analysis in a left neighborhood of the collision instant: this allows us to treat at the same time the case of interior or boundary collision instants. Needless to say, the analysis concerning right neighborhoods is the exact analogue.

The analysis carried in the previous sections has some relevant implications in the case of total collisions at the origin; indeed, as

$$\lim_{t\to(t^*)^-} I(t) = 0,$$

and I is always non-negative and convex, we immediately deduce that

$$\liminf_{t\to(t^*)^-} \dot{I}(t) \le 0.$$

Hence, Corollary 3.3 applies and we obtain the following result.

Corollary 4.1. Let $\bar{x}$ be a generalized solution. If $\lim_{t\to(t^*)^-} |\bar{x}(t)| = 0$, then there exists $\delta > 0$ such that $I(\bar{x})$ is continuous on $I_0 = (t^* - \delta, t^*)$, it admits weak derivative almost everywhere, the function $\dot{I}(\bar{x})$ is monotone increasing and $\dot{I}(\bar{x}) \in BV(I_0)$. Furthermore, the following inequalities hold:

$$\begin{aligned} &\ddot{I}(\bar{x}(t)) > 0 && \text{in the distributional sense in } I_0, \\ &\dot{I}(\bar{x}(t)) < 0, && \text{almost everywhere in } I_0. \end{aligned}$$

□

Of course, the symmetric result holds in a left neighborhood of a total collision. As a straightforward consequence, we deduce that whenever a total collision occurs at $t = t^*$, no other total collisions take place in its neighborhood, regardless of the fact whether the total collision lies in the interior or at the boundary of a finite interval (a, b). We summarize these remarks in the next theorem.

Theorem 4.2. Let $\bar{x}$ be a generalized solution for the dynamical system (2.2) in a bounded interval (a,b). Denote $r(t) = |\bar{x}(t)|$ and suppose that there exists $t^* \in [a,b]$ such that $\lim_{t\to t^*} r(t) = 0$. Then there exists $\delta > 0$ such that, for every $t \in (t^* - \delta, t^* + \delta) \cap (a,b)$, we have

$$\begin{aligned} &r(t) \neq 0, \qquad \dot{r}(t) < 0, \qquad \text{almost everywhere in } (t^* - \delta, t^*) \cap (a,b), \\ &r(t) \neq 0, \qquad \dot{r}(t) > 0, \qquad \text{almost everywhere in } (t^*, t^* + \delta) \cap (a,b). \end{aligned} \tag{4.1}$$

□

It is worthwhile noticing that the isolatedness of total collisions does not prevent, at this stage, the occurrence of infinitely many other singularities in a neighborhood of a total collision at the origin. In this section, we introduce a suitable hypothesis on the potential U that will prevent accumulation of partial collisions and will imply a regular behavior of both the radial and the angular components of the motion.

To proceed with the analysis of the asymptotic behavior near total collisions at the origin, we need some stronger conditions on the potential U when the radial variable r tends to 0. These additional conditions include quasi-homogeneous potential and logarithmic ones, in the following analysis, however, we will separately treat the two different cases.

4.1 Quasi-homogeneous potentials

In this section, we shall consider some stronger assumptions on the behavior of the potential when $r = |x|$ is small. The following conditions are trivially satisfied by α-homogeneous potentials and mimic the behavior of combination of such homogeneous potentials:

$(\mathrm{U2})_\mathrm{h}$ There exist $\alpha \in (0,2)$, $\gamma > 0$ and $C_2 \geq 0$ such that

$$\nabla U(t,x) \cdot x + \alpha U(t,x) \geq -C_2 |x|^\gamma U(t,x),$$

whenever $|x|$ is small.

Remark 4.3. $(\mathrm{U2})_\mathrm{h}$ implies (U2) (for small values of $|x|$); in fact, by taking any $\tilde{\alpha} \in (\alpha, 2)$, one obtains

$$\nabla U(t,x) \cdot x + \tilde{\alpha} U(t,x) \nabla U(t,x) \cdot x + \alpha U(t,x) + (\tilde{\alpha} - \alpha) U(t,x) \geq -C_2 |x|^\gamma U(t,x) + (\tilde{\alpha} - \alpha) U(t,x),$$

and the last term remains bounded below as $|x| \to 0$, since $\tilde{\alpha} - \alpha > 0$. □

Furthermore, we suppose the existence of a function $\tilde{U}$ defined and of class $\mathcal{C}^1$ on $(a,b)\times(\mathcal{E}\setminus\Delta)$ such that

$$\inf_{(a,b)\times(\mathcal{E}\setminus\Delta)} \tilde{U}(t,s)>0 \quad \text{and} \quad \lim_{s\to\mathcal{E}\cap\Delta} \tilde{U}(t,s)=+\infty \qquad \text{uniformly in } t. \tag{4.2}$$

The potential U is then supposed to verify the following condition uniformly in the variables s on $\mathcal{E}\setminus\Delta$ and in t on the compact subsets of (a,b):

(U3)$_\mathrm{h}$ $\lim_{r\to 0} r^\alpha U(t,x)=\tilde{U}(t,s)$.

Remark 4.4. In (U2)$_\mathrm{h}$ and (U3)$_\mathrm{h}$, the value of α must be the same. We shall refer to potentials satisfying such assumptions as quasi-homogeneous (cf. [18]). □

We rewrite the (bounded) energy function using the polar coordinates as

$$h(t)=\tfrac{1}{2}(\dot{r}^2+r^2|\dot{s}|^2)-U(t,rs). \tag{4.3}$$

We shall be concerned with functions of type:

$$\Gamma_{\alpha'}(t):=r^{\alpha'}\left(\tfrac{1}{2}r^2|\dot{s}|^2-U(t,rs)\right), \quad \alpha'\in[\alpha,2).$$

When $\alpha'=\alpha$ and the potential is exactly homogenueous, the function Γ_α differs from the classical Sundman function of the quantity $r^\alpha c^2$, where c^2 is the squared norm of the total angular momentum. Similar to the classical framework, a main tool in proving the regularity of the motion will be a monotonicity formula for the family of functions $\Gamma_{\alpha'}(t)$. Replacing in equation (4.3), we have

$$\Gamma_{\alpha'}(t)=h(t)r^{\alpha'}-\tfrac{1}{2}\dot{r}^2r^{\alpha'}\le h(t)r^{\alpha'};$$

since h is bounded (see Corollary 3.3, (iv)) and r tends to 0, we conclude that the function $\Gamma_{\alpha'}$ is bounded above on the interval $[t^*-\delta,t^*]$. Moreover, it is easy to see that, as $\dot{I}=2r\dot{r}$ has bounded variation in $[t^*-\delta,t^*]$, $\Gamma_{\alpha'}$ is of bounded variation locally in $(t^*-\delta,t^*)$. Our aim is to estimate its total variation in $[t^*-\delta,t^*]$. This will require several steps. At first, by assumption (U2)$_\mathrm{h}$, since $\gamma>0$ and $r\to 0$ as $t\to t^*$, we can always assume, by taking a smaller δ if necessary, that

$$r^\gamma\le\frac{\alpha'-\alpha}{2C_2}, \qquad \forall t\in(t^*-\delta,t^*). \tag{4.4}$$

With this we can prove our first a priori estimate.

Proposition 4.5. Let $\bar{x}$ be a locally minimizing solution, let $t^* \in (a, b]$ be a total collision instant, and let δ be given in Theorem 4.2. Let $\alpha' \in (\alpha, 2)$, where $\alpha \in (0, 2)$ is the constant fixed in assumption $(\mathrm{U2})_{\mathrm{h}}$, and assume that equation (4.4) holds. Then in the interval $(t^* - \delta, t^*)$ there holds, in the sense of distributions,

$$\frac{d}{dt}\Gamma_{\alpha'}(t) \geq -\frac{\alpha' - \alpha}{2} r^{\alpha'} \frac{\dot{r}}{r} U(t, rs) - C_1 r^{\alpha'}(U(t, rs) + 1). \tag{4.5}$$

□

Proof. To prove the assertion, we exploit the usual approximation procedure, by taking an interval $I_0 = [t_0 - \delta_0, t_0 + \delta_0] \subset (t^* - \delta, t^*)$ as in Proposition 2.10 and the associated sequence $(x_\varepsilon)_\varepsilon$ of global minimizers for the regularized functional $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ converging to the locally minimal collision solution $\bar{x}$. Let us define, for every ε,

$$r_\varepsilon := |x_\varepsilon| \in \mathbb{R} \qquad \text{and} \qquad s := \frac{x_\varepsilon}{|x_\varepsilon|} \in \mathcal{E},$$

and let us write the energy in equation (2.19) as

$$h_\varepsilon(t) = \tfrac{1}{2}\left(\dot{r}_\varepsilon^2 + r_\varepsilon^2 |\dot{s}_\varepsilon|^2\right) - U_\varepsilon(t, r_\varepsilon s_\varepsilon) + \tfrac{1}{2}|rs - r_\varepsilon s_\varepsilon|^2.$$

The approximating action functional and the corresponding Euler–Lagrange equations in the (r, s)-variables are, respectively,

$$\bar{\mathcal{A}}_\varepsilon(r_\varepsilon s_\varepsilon, I_0) : \int_{I_0} \frac{1}{2}\left(\dot{r}_\varepsilon^2 + r_\varepsilon^2 |\dot{s}_\varepsilon|^2\right) + U_\varepsilon(t, r_\varepsilon s_\varepsilon) + \frac{1}{2}|rs - r_\varepsilon s_\varepsilon|^2 \, dt$$

and

$$-\ddot{r}_\varepsilon + r_\varepsilon |\dot{s}_\varepsilon|^2 + \nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon) \cdot s_\varepsilon - (rs - r_\varepsilon s_\varepsilon) \cdot s_\varepsilon = 0 \tag{4.6}$$

$$-2 r_\varepsilon \dot{r}_\varepsilon \dot{s}_\varepsilon - r_\varepsilon^2 \ddot{s}_\varepsilon + r_\varepsilon \nabla_T U_\varepsilon(t, r_\varepsilon s_\varepsilon) - r_\varepsilon (rs - r_\varepsilon s_\varepsilon) = \mu_\varepsilon s_\varepsilon, \tag{4.7}$$

where $\mu_\varepsilon = r_\varepsilon^2 |\dot{s}_\varepsilon|^2 - r_\varepsilon (rs - r_\varepsilon s_\varepsilon) \cdot s_\varepsilon$ is the Lagrange multiplier due to the presence of the constraint $|s_\varepsilon|^2 = 1$ and the vector $\nabla_T U_\varepsilon(t, r_\varepsilon s_\varepsilon)$ is the tangent components to the ellipsoid $\mathcal{E}$ of the gradient $\nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)$.

Next, we consider the corresponding functions (still bounded above) for the approximating problems

$$\begin{aligned} \Gamma_{\alpha', \varepsilon}(t) &= r_\varepsilon^{\alpha'} \left(\tfrac{1}{2} r_\varepsilon^2 |\dot{s}_\varepsilon|^2 - U_\varepsilon(t, r_\varepsilon s_\varepsilon) + \tfrac{1}{2}|rs - r_\varepsilon s_\varepsilon|^2\right) \\ &= h_\varepsilon(t) r_\varepsilon^{\alpha'} - \tfrac{1}{2}\dot{r}_\varepsilon^2 r_\varepsilon^{\alpha'} \leq h_\varepsilon(t) r_\varepsilon^{\alpha'}, \end{aligned}$$

and we observe that the sequence $(\Gamma_{\alpha',\varepsilon})_\varepsilon$ converges almost everywhere and $L^1(I_0)$ to $\Gamma_{\alpha'}$, as $\varepsilon \to 0$. We compute the derivative of $\Gamma_{\alpha',\varepsilon}(t)$ with respect to time as

$$\begin{aligned}\frac{d}{dt}\Gamma_{\alpha',\varepsilon}(t) = \frac{2+\alpha'}{2} r_\varepsilon^{1+\alpha'}\dot r_\varepsilon |\dot s_\varepsilon|^2 + r_\varepsilon^{2+\alpha'}\dot s_\varepsilon\cdot\ddot s_\varepsilon + \alpha' r_\varepsilon^{\alpha'-1}\dot r_\varepsilon\left[\frac{1}{2}|rs - r_\varepsilon s_\varepsilon|^2 - U_\varepsilon(t, r_\varepsilon s_\varepsilon)\right] \\ - r_\varepsilon^{\alpha'}\left[\frac{\partial U_\varepsilon}{\partial t}(t, r_\varepsilon s_\varepsilon) + \nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)(\dot r_\varepsilon s_\varepsilon + r_\varepsilon \dot s_\varepsilon)\right] + r_\varepsilon^{\alpha'}(rs - r_\varepsilon s_\varepsilon)\frac{d}{dt}(rs - r_\varepsilon s_\varepsilon).\end{aligned} \tag{4.8}$$

Now we multiply the angular Euler–Lagrange equation (4.7) by $\dot s_\varepsilon$ to obtain (we recall that $\nabla_T U_\varepsilon(t, r_\varepsilon s_\varepsilon)\cdot\dot s_\varepsilon = \nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)\cdot \dot s_\varepsilon$, since s_ε and $\dot s_\varepsilon$ are orthogonal)

$$r_\varepsilon^2 \ddot s_\varepsilon \cdot \dot s_\varepsilon = -2 r_\varepsilon \dot r_\varepsilon |\dot s_\varepsilon|^2 + r_\varepsilon \nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)\cdot \dot s_\varepsilon - r_\varepsilon rs\cdot \dot s_\varepsilon. \tag{4.9}$$

Replacing equation (4.9) in equation (4.8), we have

$$\begin{aligned}\frac{d}{dt}\Gamma_{\alpha',\varepsilon}(t) = &-\frac{2-\alpha'}{2} r^{1+\alpha'}\dot r_\varepsilon |\dot s_\varepsilon|^2 - \alpha' r_\varepsilon^{\alpha'-1}\dot r_\varepsilon U_\varepsilon(t, r_\varepsilon s_\varepsilon) - r_\varepsilon^{\alpha'}\frac{\partial U_\varepsilon}{\partial t}(t, r_\varepsilon s_\varepsilon) \\ &- r_\varepsilon^{\alpha'-1}\dot r_\varepsilon \nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)\cdot (r_\varepsilon s_\varepsilon) - r_\varepsilon^{\alpha'+1} rs\cdot \dot s_\varepsilon \\ &+ \frac{\alpha'}{2} r_\varepsilon^{\alpha'-1}\dot r_\varepsilon |rs - r_\varepsilon s_\varepsilon|^2 + r_\varepsilon^{\alpha'}(rs - r_\varepsilon s_\varepsilon)\frac{d}{dt}(rs - r_\varepsilon s_\varepsilon).\end{aligned} \tag{4.10}$$

We now combine assumptions (U1), $(\mathrm{U2})_{\mathrm{h}}$, and equation (2.11) to obtain the following inequalities:

$$\begin{aligned} -r_\varepsilon^{\alpha'}\frac{\partial U_\varepsilon}{\partial t}(t, r_\varepsilon s_\varepsilon) &= -r_\varepsilon^{\alpha'}\dot\eta_\varepsilon(U(t, r_\varepsilon s_\varepsilon))\frac{\partial U}{\partial t}(t, r_\varepsilon s_\varepsilon) \\ &\geq -C_1 r_\varepsilon^{\alpha'}\dot\eta_\varepsilon(U(t, r_\varepsilon s_\varepsilon))(U_\varepsilon(t, r_\varepsilon s_\varepsilon) + 1) \\ &\geq -C_1 r_\varepsilon^{\alpha'}(\eta_\varepsilon(U(t, r_\varepsilon s_\varepsilon)) + 1) \\ &\geq -C_1 r_\varepsilon^{\alpha'}(U_\varepsilon(t, r_\varepsilon s_\varepsilon) + 1), \\ -r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}\nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)\cdot (r_\varepsilon s_\varepsilon) &- r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}\dot\eta_\varepsilon(U(t, r_\varepsilon s_\varepsilon))\nabla U(t, r_\varepsilon s_\varepsilon)\cdot (r_\varepsilon s_\varepsilon) \\ &\geq -r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}\dot\eta_\varepsilon(U(t, r_\varepsilon s_\varepsilon))U(t, r_\varepsilon s_\varepsilon)\big[-\alpha - C_2 r_\varepsilon^{\gamma}\big] \\ &\geq r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}U_\varepsilon(t, r_\varepsilon s_\varepsilon)\big[\alpha + C_2 r_\varepsilon^{\gamma}\big].\end{aligned}$$

Finally, by replacing in equation (4.10), we obtain

$$\frac{d}{dt}\Gamma_{\alpha',\varepsilon}(t) \geq \Psi_{\alpha',\varepsilon}(t),$$

where

$$\Psi_{\alpha',\varepsilon}(t) = -\frac{2-\alpha'}{2} r_\varepsilon^{1+\alpha'} \dot r_\varepsilon |\dot s_\varepsilon|^2 - (\alpha'-\alpha) r_\varepsilon^{\alpha'} \frac{\dot r_\varepsilon}{r_\varepsilon} U_\varepsilon(t, r_\varepsilon s_\varepsilon) - C_1 r_\varepsilon^{\alpha'} (U_\varepsilon(t, r_\varepsilon s_\varepsilon) + 1)$$
$$+ C_2 r_\varepsilon^{\alpha'+\gamma} \frac{\dot r_\varepsilon}{r_\varepsilon} U_\varepsilon(t, r_\varepsilon s_\varepsilon) - r_\varepsilon^{\alpha'+1} rs \cdot \dot s_\varepsilon$$
$$+ \frac{\alpha'}{2} r_\varepsilon^{\alpha'-1} \dot r_\varepsilon |rs - r_\varepsilon s_\varepsilon|^2 + r_\varepsilon^{\alpha'} (rs - r_\varepsilon s_\varepsilon) \frac{d}{dt}(rs - r_\varepsilon s_\varepsilon).$$

Now, using equation (4.4), since $r_\varepsilon \to r$ uniformly in I_0, for small $\varepsilon > 0$, we can find positive $\lambda_\varepsilon \to (\alpha' - \alpha)/2$ such that

$$C_2 r_\varepsilon^{\alpha'+\gamma} U_\varepsilon(t, r_\varepsilon s_\varepsilon) \le \lambda_\varepsilon r_\varepsilon^{\alpha'} U_\varepsilon(t, r_\varepsilon s_\varepsilon)$$

on I_0; furthermore, since $-\frac{2-\alpha'}{2} r_\varepsilon^{1+\alpha'} \dot r_\varepsilon |\dot s_\varepsilon|^2$ is positive, we have

$$\Psi_{\alpha',\varepsilon}(t) \ge -(\alpha' - \alpha - \lambda_\varepsilon) r_\varepsilon^{\alpha'} \frac{\dot r_\varepsilon}{r_\varepsilon} U_\varepsilon(t, r_\varepsilon s_\varepsilon) - C_1 r_\varepsilon^{\alpha'} (U_\varepsilon(t, r_\varepsilon s_\varepsilon) + 1)$$
$$- r_\varepsilon^{\alpha'+1} rs \cdot \dot s_\varepsilon + \frac{\alpha'}{2} r_\varepsilon^{\alpha'-1} \dot r_\varepsilon |rs - r_\varepsilon s_\varepsilon|^2 + r_\varepsilon^{\alpha'} (rs - r_\varepsilon s_\varepsilon) \frac{d}{dt}(rs - r_\varepsilon s_\varepsilon).$$

We can then conclude that, for every ε,

$$\frac{d}{dt}\Gamma_{\alpha',\varepsilon} \ge -(\alpha' - \alpha - \lambda_\varepsilon) r_\varepsilon^{\alpha'} \frac{\dot r_\varepsilon}{r_\varepsilon} U_\varepsilon(t, r_\varepsilon s_\varepsilon) - C_1 r_\varepsilon^{\alpha'} (U_\varepsilon(t, r_\varepsilon s_\varepsilon) + 1)$$
$$- r_\varepsilon^{\alpha'+1} rs \cdot \dot s_\varepsilon + \frac{\alpha'}{2} r_\varepsilon^{\alpha'-1} \dot r_\varepsilon |rs - r_\varepsilon s_\varepsilon| + r_\varepsilon^{\alpha'} (rs - r_\varepsilon s_\varepsilon) \frac{d}{dt}(rs - r_\varepsilon s_\varepsilon).$$

Now we remark that the right-hand side of the inequality converges strongly in $L^1(I_0)$; this follows from Proposition 2.10, taking into account that $r_\varepsilon \to r > 0$, the $\dot r_\varepsilon$'s are bounded on I_0, and $\lambda_\varepsilon \to (\alpha' - \alpha)/2$. Therefore, by testing with any $\varphi \in C_0^\infty(I_0)$ and passing to the limit as $\varepsilon \to 0$, we easily obtain that the inequality

$$\frac{d}{dt}\Gamma_{\alpha'} \ge -\frac{\alpha' - \alpha}{2} r^{\alpha'} \frac{\dot r}{r} U(t, rs) - C_1 r^{\alpha'} (U(t, rs) + 1)$$

holds in the sense of distributions. ∎

Proposition 4.6. Let $\bar x$ be a locally minimal solution, let $t^* \in (a, b]$ be a total collision instant, and let δ be given in Theorem 4.2 such that equation (4.4) holds. Then for every

$\alpha' \in (\alpha, 2)$,

$$-\frac{\alpha'-\alpha}{2}\int_{t^*-\delta}^{t^*} r^{\alpha'}\frac{\dot r}{r}U(t,rs) \le \int_{t^*-\delta}^{t^*} C_1 r^{\alpha'}(U(t,rs)+1)\,dt - \Gamma_{\alpha'}(t^*-\delta) < +\infty,$$

where $\alpha \in (0,2)$ is the constant fixed in assumption (U2)$_\mathrm{h}$. □

Proof. By integrating in Proposition 4.5, we obtain

$$\begin{aligned}-\frac{\alpha'-\alpha}{2}\int_{t^*-\delta}^{t^*} r^{\alpha'}\frac{\dot r}{r}U(t,rs) &\le \int_{t^*-\delta}^{t^*} C_1 r^{\alpha'}(U(t,rs)+1)\,dt + \Gamma_{\alpha'}(t^*) - \Gamma_{\alpha'}(t^*-\delta)\\ &\le \int_{t^*-\delta}^{t^*} C_1 r^{\alpha'}(U(t,rs)+1)\,dt - \Gamma_{\alpha'}(t^*-\delta);\end{aligned}$$

this proves the assertion because of the L^1 bound on the potential given in Corollary 2.19 if we chose $t^*-\delta$ to be a point of boundedness for $\Gamma_{\alpha'}$. ■

Let us consider a sequence of locally minimizing solutions approximating a generalized solution; we can take advantage of the above inequality, together with the uniform bounds given by Corollary 3.4 and the uniform boundedness above of the approximating functions $\Gamma_{\alpha'}$ in order to obtain the following estimate.

Lemma 4.7. Let $\bar x$ be a generalized solution, let $t^* \in (a,b]$ be a total collision instant, and let δ be given in Theorem 4.2 such that equation (4.4) holds. Then for every $\alpha' \in (\alpha, 2)$, we have

$$\int_{t^*-\delta}^{t^*} -r^{\alpha'}\frac{\dot r}{r}U(t,rs)\,dt < +\infty,$$

where $\alpha \in (0,2)$ is the constant fixed in assumption (U2)$_\mathrm{h}$. □

Remark 4.8. A word of caution must be entered at this point. Generally speaking, there is no need that the approximating sequence presents a total collision. However, going back to the proof of the results of this section, the assumption that t^* is a total collision instant is used only in order to ensure that $\liminf_{t\to t^*}\dot I(t) < +\infty$. On the other hand, one can easily prove, by using tha Lagrange–Jacobi inequality, that this last condition also holds for the approximating sequence. □

Now we consider the limiting case $\alpha' = \alpha$, corrsesponding to the function

$$\Gamma_\alpha(t) := r^\alpha \left[\tfrac{1}{2} r^2 |\dot{s}|^2 - U(t, rs)\right]. \tag{4.11}$$

Proposition 4.9 (Monotonicity formula). Let $\bar{x}$ be a locally minimizing solution, let $t^* \in (a, b]$ be a total collision instant, and let $\delta > 0$ be the constant obtained in Theorem 4.2. Then the function Γ_α is of bounded variation on $[t^* - \delta, t^*]$ and

$$\frac{d}{dt}\Gamma_\alpha(t) \geq -\frac{2-\alpha}{2} r^{1+\alpha}\dot{r}|\dot{s}|^2 - C_1 r^\alpha (U(t, rs) + 1) + C_2 r^{\alpha+\gamma}\frac{\dot{r}}{r} U(t, rs), \tag{4.12}$$

where $t \in [t^* - \delta, t^*]$. Moreover,

$$-\frac{2-\alpha}{2}\int_{t^*-\delta}^{t^*} r^{1+\alpha}\dot{r}|\dot{s}|^2 \leq \int_{t^*-\delta}^{t^*}\left[C_1 r^\alpha (U(t, rs) + 1) - C_2 r^{\alpha+\gamma}\frac{\dot{r}}{r} U(t, rs)\right] dt - \Gamma_\alpha(t^* - \delta) < +\infty. \tag{4.13}$$

□

Proof. Replacing in equation (4.3) the expression of the function Γ_α, we have

$$\Gamma_\alpha(t) = h(t) r^\alpha - \dot{r}^2 r^\alpha \leq h(t) r^\alpha;$$

since h is bounded (see Corollary 3.3) and r tends to 0, we conclude that the function Γ_α is bounded above. Using the same approximation arguments described in Proposition 4.5, we obtain equation (4.12). From the application of Lemma 4.7 with $\alpha' = \alpha + \gamma$, we deduce the integrability of the negative function $r^{\alpha+\gamma}\frac{\dot{r}}{r}U(t, rs)$. Hence, since $-\frac{2-\alpha}{2} r^{1+\alpha}\dot{r}|\dot{s}|^2$ is positive and both $r^\alpha U(t, rs)$ and $r^{\alpha+\gamma}\frac{\dot{r}}{r}U(t, rs)$ are integrable (Lemma 4.7), the boundedness below of the function Γ_α follows from equation (4.12). Inequality (4.13) follows from the boundedness above of the function Γ_α and inequality (4.12), since the terms $-C_1\int_{t^*-\delta}^{t^*} r^\alpha (U(t, rs) + 1) dt$ and $C_2 \int_{t^*-\delta}^{t^*} r^{\alpha+\gamma}\frac{\dot{r}}{r}U(t, rs)\, dt$ are negative. ■

Lemma 4.10 (Monotonicity formula). Let $\bar{x}$ be a generalized solution, let $t^* \in (a, b]$ be a total collision instant, and let $\delta > 0$ be the constant obtained in Theorem 4.2. Then the function $\Gamma_\alpha(t)$ is bounded and has bounded variation on $[t^* - \delta, t^*)$. In addition,

$$-\frac{2-\alpha}{2}\int_{t^*-\delta}^{t^*} r^{1+\alpha}\dot{r}|\dot{s}|^2 < +\infty. \tag{4.14}$$

□

Proof. Replacing in equation (4.3) the expression of the function Γ_α, we have

$$\Gamma_\alpha(t) = h(t)r^\alpha - \tfrac{1}{2}\dot{r}^2 r^\alpha \leq h(t)r^\alpha;$$

since h is bounded (see Corollary 3.3) and r tends to 0, we conclude that the function Γ_α is bounded above. aNow we take advantage of the usual sequence of approximating locally minimal solutions and we obtain from equation (4.12) the uniform boundedness of total variation of the sequence. From this, the boundedness of the total variation of the uniform limit easily follows. In order to prove equation (4.14), we simply pass to the limit in equation (4.13). ■

As a straightfoward consequence, we have the existence of the following limits.

Corollary 4.11. There exists $b \geq 0$ such that

$$\lim_{t\to(t^*)^-} \Gamma_\alpha(t) = -b \qquad \text{and} \qquad \lim_{t\to(t^*)^-} \dot{r}^2 r^\alpha = 2b.$$

□

Proof. Since Γ_α has bounded variation, it admits a limit when t tends to t^* from the right. We call this limit $-b \in \mathbb{R}$. Furthermore, since $\Gamma_\alpha(r,s) = h(t)r^\alpha - \frac{1}{2}\dot{r}^2 r^\alpha$, the energy h is bounded and r tends to 0 as $t \to t^*$, we conclude that $\dot{r}^2 r^\alpha$ converges to $2b$. ■

The next step consists in proving that the limit b is non-zero.

Lemma 4.12. Let $\varphi(t) := -\dot{r}(t)r^{\alpha/2}(t)$, $t \in [t^* - \delta, t^*]$. Then there exist two constants depending on α, $c_{1,\alpha} \leq c_{2,\alpha}$, such that for all $t \in [t^* - \delta, t^*]$,

$$c_{1,\alpha} \leq \varphi(t) \leq c_{2,\alpha}.$$

□

Proof. We write

$$\tfrac{1}{2}\varphi^2(t) = r^\alpha h(t) - \Gamma_\alpha(t).$$

Since the energy function h is bounded (see Corollary 3.3, (iv)) and we assume that r tends to 0 as t tends to t^*, the function $r^\alpha h(t)$ is also bounded. Moreover, by Lemma 4.10 also $\Gamma_\alpha(t)$ is bounded. ■

Remark 4.13. The same inequality proved in Lemma 4.12 holds when we replace the function φ with $\varphi_n(t) = -\dot{r}_n(t)r_n^{\alpha/2}(t)$, and $x_n = r_n s_n$ is an approximating sequence of locally minimal solutions for the generalized solution $\bar{x}$. $\square$

Corollary 4.14. In the same setting of Lemma 4.10, we have $\lim_{t\to t^*}\int_{t^*-\delta}^{t}\frac{1}{r^{\alpha/2+1}} = +\infty$. $\square$

Proof. We can write the boundedness above of the function φ (proved in Lemma 4.12) as

$$-\frac{\dot{r}}{r} \leq \frac{c_{2,\alpha}}{r^{\alpha/2+1}}, \quad t \in [t^* - \delta, t^*]. \tag{4.15}$$

Integrating inequality (4.15) on the interval $[t^* - \delta, t]$ when $t \to t^*$, we obtain

$$\lim_{t\to t^*} c_{2,\alpha}\int_{t^*-\delta}^{t}\frac{d\xi}{r^{\alpha/2+1}} \geq \lim_{t\to t^*}\int_{t^*-\delta}^{t} -\frac{\dot{r}}{r}d\xi = \log r(t^* - \delta) - \lim_{t\to t^*}\log r(t) = +\infty,$$

since r tends to 0 as $t \to t^*$. $\blacksquare$

Lemma 4.15. The lower bound $c_{1,\alpha}$ of the function φ defined in Lemma 4.12 can be chosen strictly positive, that is, $c_{1,\alpha} > 0$. $\square$

Proof. We start proving an estimate of the derivative of the function φ, in the case of locally minimal solutions. With this purpose we consider, as usual, the approximating sequence $(\varphi_\varepsilon)_\varepsilon$, where

$$\varphi_\varepsilon(t) = -\dot{r}_\varepsilon(t)r_\varepsilon^{\alpha/2}(t)$$

in the interval I_0 and, for every $\varepsilon > 0$, we compute the first derivative of the smooth function φ_ε and we use the Euler–Lagrange equation (4.6) for the approximating problem to obtain

$$\begin{aligned}\dot{\varphi}_\varepsilon(t) &= -\frac{\alpha}{2}r_\varepsilon^{\alpha/2-1}\dot{r}_\varepsilon^2 - r_\varepsilon^{\alpha/2}\ddot{r}_\varepsilon \\ &= -\frac{\alpha}{2}r_\varepsilon^{\alpha/2-1}\dot{r}_\varepsilon^2 - r_\varepsilon^{\alpha/2+1}|\dot{s}_\varepsilon|^2 - r_\varepsilon^{\alpha/2-1}\nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)\cdot(r_\varepsilon s_\varepsilon) + r_\varepsilon^{\alpha/2}(rs - r_\varepsilon s_\varepsilon)\cdot s_\varepsilon.\end{aligned}$$

Similar to the proof of Proposition 4.5, we use assumption $(U2)_h$ and equation (2.11) to deduce

$$\begin{aligned}\dot{\varphi}_\varepsilon(t) &\leq r_\varepsilon^{\alpha/2-1}\left[-\frac{\alpha}{2}\dot{r}_\varepsilon^2 - r_\varepsilon^2|\dot{s}_\varepsilon|^2 + (\alpha + C_2 r_\varepsilon^\gamma)U_\varepsilon(t, r_\varepsilon s_\varepsilon) + (rs - r_\varepsilon s_\varepsilon)\cdot(r_\varepsilon s_\varepsilon)\right]\\ &= \frac{1}{r_\varepsilon^{\alpha/2+1}}\left[\frac{2-\alpha}{2}\varphi_\varepsilon^2(t) - 2r_\varepsilon^\alpha h_\varepsilon(t) - (2-\alpha)r_\varepsilon^\alpha U_\varepsilon(t, r_\varepsilon s_\varepsilon)\right.\\ &\qquad \left. + C_2 r_\varepsilon^{\alpha+\gamma}U_\varepsilon(t, r_\varepsilon s_\varepsilon) + r_\varepsilon^\alpha(rs - r_\varepsilon s_\varepsilon)\cdot(rs)\right].\end{aligned}$$

As $\varepsilon \to 0$, from Proposition 2.10 and Remark 4.13 we obtain the validity of the inequality, in the sense of distributions,

$$\dot{\varphi}(t) \leq \frac{1}{r^{\alpha/2+1}}\left[\frac{2-\alpha}{2}\varphi^2(t) - 2r^\alpha h(t) - (2-\alpha)r^\alpha U(t, rs) + C_2 r^{\alpha+\gamma}U(t, rs)\right].$$

We can use equation (4.4) to estimate

$$\dot{\varphi}(t) \leq \frac{1}{r^{\alpha/2+1}}\left[\frac{2-\alpha}{2}\varphi^2(t) - 2r^\alpha h(t) - \frac{2-\alpha}{2}r^\alpha U(t, rs)\right].$$

The last inequality holds also for generalized solutions: indeed, using an approximating sequence of locally minimizing solutions, one can pass to the limit using Fatou's lemma, finding, for almost every $t^* - \delta < t_0 < t < t^*$,

$$\varphi(t) - \varphi(t_0) \leq \int_{t_0}^t \frac{1}{r^{\alpha/2+1}}\left[\frac{2-\alpha}{2}\varphi^2(\xi) - 2r^\alpha h(\xi) - \frac{2-\alpha}{2}r^\alpha U(\xi, rs)\right]d\xi.$$

Now, since $|2r^\alpha h(t)|$ converges uniformly to zero as $t \to t^*$ on $[t^* - \delta, t^*]$, by condition $(U3)_h$ we obtain that, denoting by $\tilde{U}_0$ the minimal value assumed by $\tilde{U}$ on the ellipsoid $\mathcal{E}$, there exist two positive constants $k_1, k_2 > 0$ such that

$$\varphi(t) \leq \varphi(t_0) + \int_{t_0}^t \frac{k_1}{r^{\alpha/2+1}}(\varphi^2(\xi) - k_2\tilde{U}_0)d\xi$$

whenever $t^* - t_0$ is sufficiently small. We will conclude showing that necessarily $\varphi^2(t) \geq k_2\tilde{U}_0$ and then choosing $c_{1,\alpha} := \sqrt{k_2\tilde{U}_0} > 0$.

For the sake of contradiction, we suppose the existence of $\hat{t}$ such that $\varphi^2(\hat{t}) < k_2\tilde{U}_0$; then $\varphi^2 - k_2\tilde{U}_0 < 0$ in a neighborhood of $\hat{t}$ and

$$\varphi(t) \leq \varphi(\hat{t}) + \int_{\hat{t}}^{t} \frac{k_1}{r^{\alpha/2+1}}(\varphi^2(\xi) - k_2\tilde{U}_0)d\xi < \varphi(\hat{t})$$

for every $t \in (\hat{t}, t^*)$. We deduce the existence of a strictly positive constant $\hat{k}$ such that, for every $t \in (\hat{t}, t^*)$,

$$\varphi(t) - \varphi(\hat{t}) \leq -\hat{k}\int_{\hat{t}}^{t} \frac{d\xi}{r^{\alpha/2+1}}.$$

Since the right-hand side tends to $-\infty$ as t approaches t^* (see Corollary 4.14), the last inequality contradicts the boundedness of the function φ. ∎

Corollary 4.16. There exist two strictly positive constants $0 < k_{1,\alpha} \leq k_{2,\alpha}$ such that

$$k_{1,\alpha}(t^* - t)^{\frac{2}{\alpha+2}} \leq r(t) \leq k_{2,\alpha}(t^* - t)^{\frac{2}{\alpha+2}}$$

whenever $t \in [t^* - \delta, t^*]$. □

Proof. The statement follows from Lemmata 4.12 and 4.15 with $k_{i,\alpha} := (\frac{\alpha+2}{2}c_{i,\alpha})^{\frac{2}{\alpha+2}}$, $i = 1, 2$. ∎

Joining this with Corollary (4.11), we easily obtain the next result.

Corollary 4.17. There exists $b > 0$ such that

$$\lim_{t\to(t^*)^-} \Gamma_\alpha(t) = -b \qquad \text{and} \qquad \lim_{t\to(t^*)^-} \dot{r}^2 r^\alpha = 2b.$$

□

Theorem 4.18. Let $\bar{x}$ be a generalized solution for the dynamical system (2.2), let $t^* \in (a, b)$ (if $b < +\infty$ t^* can coincide with b) be a total collision instant, and let $\delta > 0$ be the constant obtained in Theorem 4.2. Let r, s be the new variables defined in equation (2.1); if the potential U satisfies assumptions (U0), (U1), $(\mathrm{U2})_\mathrm{h}$, $(\mathrm{U3})_\mathrm{h}$, then the following assertions hold:

(a) $\lim_{t\to(t^*)^-} r^\alpha U(t, rs) = b$, where b is the strictly positive constant introduced in Corollary 4.17;

(b) there is a positive constant K such that, as t tends to t^*,

$$r(t) \sim [K(t^* - t)]^{\frac{2}{2+\alpha}}$$
$$\dot{r}(t) \sim -\frac{2K}{2+\alpha}[K(t^* - t)]^{\frac{-\alpha}{2+\alpha}};$$

(c) $\lim_{t \to (t^*)^-} |\dot{s}(t)|(t^* - t) = 0;$

(d) for every real positive sequence $(\lambda_n)_n$ such that $\lambda_n \to 0$ as $n \to +\infty$, we have

$$\lim_{n \to +\infty} |s(t^* - \lambda_n) - s(t^* - \lambda_n t)| = 0, \quad \forall t > 0.$$

□

Remark 4.19. Condition (a) of Theorem 4.18, together with assumptions $(U3)_h$ on U and equation (4.2) on $\tilde{U}$ imply that, if $\bar{x}$ is a generalized solution with a total collision at the origin at time $t = t^*$, then there exists $\delta > 0$ such that, for every $t \in (t^* - \delta, t^*)$ $\bar{x}(t) \notin \Delta$, i.e. in a (left) neighborhood of the total collision instant, no other collision is allowed: neither total nor partial. As a consequence, in such a neighborhood, the generalized solution $\bar{x}$ satisfies the dynamical system (2.2) and the corresponding variables (r, s) verify the Euler–Lagrange equations (2.6). □

Proof (Proof of Theorem 4.18)**.** We begin by proving statement (a). At first we recall that, as $r^\alpha U$ is continuous an extended valued function, it makes sense to compute its upper and lower limits as $t \to (t^*)^-$. We already know from equation (4.11) the integrability of the function $-r^{\alpha+1}\dot{r}|\dot{s}|^2$ on the interval $[t^* - \delta, t^*]$. Furthermore, since the integral of $-\dot{r}/r$ on the same interval diverges to $+\infty$, we conclude that

$$\liminf_{t \to (t^*)^-} r^{\alpha+2}|\dot{s}|^2 = 0$$

and from the definition of Γ_α in equation (4.11) together with Corollary 4.17, we infer that

$$\liminf_{t \to (t^*)^-} r^\alpha U(t, rs) = b. \tag{4.16}$$

It remains to prove that also $\limsup_{t \to (t^*)^-} r^\alpha U(t, rs) b$. Suppose, for the sake of contradiction, the existence of a strictly positive ε such that

$$\limsup_{t \to (t^*)^-} r^\alpha U(t, rs) \geq b + 3\varepsilon. \tag{4.17}$$

Using assumption (U3)$_{\mathrm{h}}$, we have that equations (4.16) and (4.17) are, respectively, equivalent to

$$\liminf_{t\to(t^*)^-} \tilde{U}(t,s) = b \quad \text{and} \quad \limsup_{t\to(t^*)^-} \tilde{U}(t,s) \geq b + 3\varepsilon,$$

and Corollary 4.17 implies the existence of t_ε such that $\Gamma_\alpha(t) \geq -b - \varepsilon/2$ whenever $t \in (t_\varepsilon, t^*]$. We can then define the set

$$\mathcal{U} := \{t \in (t_\varepsilon, t^*) : \tilde{U}(t, s(t)) \geq b + \varepsilon\}.$$

We define two nonempty subsets of the ellipsoid $\mathcal{E}$ as

$$A := \left\{s : \tilde{U}(t^*, s) \leq b + \frac{5\varepsilon}{4}\right\} \quad \text{and} \quad B := \left\{s : \tilde{U}(t^*, s) \geq b + \frac{7\varepsilon}{4}\right\};$$

since $\varepsilon > 0$, the quantity

$$d := \mathrm{dist}(A, B) = \inf_{s_1\in A, s_2\in B} |s_1 - s_2|$$

is strictly positive and there exists a sequence $(t_n)_{n\geq 0} \subset [t^* - \delta, t^*]$, such that

$$\begin{gathered} t_n \to t^* \text{ as } n \to +\infty \\ s(t_{2k}) \in \partial A \quad \text{and} \quad s(t_{2k+1}) \in \partial B, \quad \text{for every } k \in \mathbb{N} \\ b + \varepsilon \leq \tilde{U}(t, s(t)) \leq b + 2\varepsilon, \quad \text{for every } t \in (t_{2k}, t_{2k+1}) \text{ and } k \in \mathbb{N}. \end{gathered}$$

Hence $(t_{2k}, t_{2k+1}) \subset \mathcal{U}$, for every k, and from the definition of the function Γ_α in equation (4.11), we have that

$$r^{\alpha+2}|\dot{s}|^2 \geq \varepsilon \text{ in the intervals } (t_{2k}, t_{2k+1}). \tag{4.18}$$

We now estimate the integral on (t_{2k}, t_{2k+1}) of the integrable (on $[t^* - \delta, t^*]$) function $r^{\alpha+1}\dot{r}|\dot{s}|^2$ using equation (4.18) and Corollary 4.16,

$$\begin{aligned} \int_{t_{2k}}^{t_{2k+1}} -\frac{\dot{r}}{r} r^{\alpha+2}|\dot{s}|^2 dt &\geq \varepsilon \int_{t_{2k}}^{t_{2k+1}} -\frac{\dot{r}}{r} dt = \varepsilon \log \frac{r(t_{2k})}{r(t_{2k+1})} \\ &\geq \frac{2\varepsilon}{2+\alpha} \log \frac{c_{1,\alpha}(t^* - t_{2k})}{c_{2,\alpha}(t^* - t_{2k+1})}. \end{aligned} \tag{4.19}$$

On the other hand, using Hölder inequality we have

$$d^2 \le |s(t_{2k+1}) - s(t_{2k})| \le \left(\int_{t_{2k}}^{t_{2k+1}} |\dot{s}| dt \right)^2 \le \int_{t_{2k}}^{t_{2k+1}} -r^{\alpha+2} \frac{\dot{r}}{r} |\dot{s}|^2 dt \int_{t_{2k}}^{t_{2k+1}} \frac{dt}{-r^{\alpha+1}\dot{r}} \tag{4.20}$$

and from Lemma 4.12 and Corollary 4.16, we obtain

$$\int_{t_{2k}}^{t_{2k+1}} \frac{dt}{-r^{\alpha+1}\dot{r}} \int_{t_{2k}}^{t_{2k+1}} \frac{1}{-r^{\alpha/2}\dot{r}} \frac{1}{r^{\alpha/2+1}} dt \le \frac{2}{2+\alpha} \frac{1}{c_{1,\alpha}^2} \int_{t_{2k}}^{t_{2k+1}} \frac{dt}{t^*-t} = \frac{2}{2+\alpha} \frac{1}{c_{1,\alpha}^2} \log \frac{t^*-t_{2k}}{t^*-t_{2k+1}}. \tag{4.21}$$

Combining equations (4.20) and (4.21) we obtain

$$\int_{t_{2k}}^{t_{2k+1}} -r^{\alpha+2} \frac{\dot{r}}{r} |\dot{s}|^2 dt \ge \frac{2+\alpha}{2} d^2 c_{1,\alpha}^2 \left[\log \frac{t^*-t_{2k}}{t^*-t_{2k+1}} \right]^{-1}. \tag{4.22}$$

From estimates (4.19) and (4.22), we deduce

$$\int_{t_{2k}}^{t_{2k+1}} -r^{\alpha+2} \frac{\dot{r}}{r} |\dot{s}|^2 dt \ge \frac{\varepsilon}{2+\alpha} \log \frac{c_{1,\alpha}(t^*-t_{2k})}{c_{2,\alpha}(t^*-t_{2k+1})} + \frac{2+\alpha}{4} d^2 c_{1,\alpha}^2 \left[\log \frac{t^*-t_{2k}}{t^*-t_{2k+1}} \right]^{-1}.$$

Summing on the index k and recalling that the positive function $-\dot{r}r^{\alpha+1}|\dot{s}|^2$ has a finite integral on $[t^*-\delta, t^*]$ (equation (4.14)), we have

$$\begin{aligned} +\infty > \int_{t^*-\delta}^{t^*} -\dot{r}r^{\alpha+1}|\dot{s}|^2 dt &> \sum_{k\ge 0} \int_{t_{2k}}^{t_{2k+1}} -\dot{r}r^{\alpha+1}|\dot{s}|^2 dt \\ &\ge \frac{\varepsilon}{2+\alpha} \sum_{k\ge 0} \log \frac{c_{1,\alpha}(t^*-t_{2k})}{c_{2,\alpha}(t^*-t_{2k+1})} + \frac{2+\alpha}{4} d^2 c_{1,\alpha}^2 \sum_{k\ge 0} \left[\log \frac{t^*-t_{2k}}{t^*-t_{2k+1}} \right]^{-1}. \end{aligned} \tag{4.23}$$

Since $c_{2,\alpha}/c_{1,\alpha}$ is bounded (see Lemma 4.15), for the last term in equation (4.23) to be finite it is necessary that

$$\lim_{k\to+\infty} \frac{t^*-t_{2k}}{t^*-t_{2k+1}} = \frac{c_{2,\alpha}}{c_{1,\alpha}} \quad \text{and} \quad \lim_{k\to+\infty} \frac{t^*-t_{2k}}{t^*-t_{2k+1}} = +\infty. \tag{4.24}$$

This is a contradiction, hence we conclude that

$$\limsup_{t\to t^*} r^\alpha U(t, rs) = b$$

and, after replacing the value in equation (4.11),

$$\lim_{t\to t^*} r^{\alpha+2}|\dot{s}|^2 = 0. \tag{4.25}$$

To prove (b), from Corollary 4.17 we obtain

$$\lim_{t\to(t^*)^-} \frac{r(t)^{\alpha/2+1}}{(\alpha/2+1)(t^*-t)} \lim_{t\to(t^*)^-} -r(t)^{\alpha/2}\dot{r}(t) = \sqrt{2b};$$

we then conclude by defining $K := \dfrac{2+\alpha}{2}\sqrt{2b}$. The second estimate follows directly.

Part (c) directly follows from equation (4.25) and (b).

We conclude by proving statement (d). If $t = 1$, there is nothing to prove. Suppose $t > 0$, $t \neq 1$, and consider a sequence $(\lambda_n)_n$, $\lambda_n \to 0$; let N be such that $\lambda_n < \delta/\max(1,t)$, $\forall n \geq N$. Whenever $t > 1$, for every $n \geq N$, we have

$$t^* - \delta < t^* - \lambda_n t < t^* - \lambda_n < t^*$$

and

$$\begin{aligned} |s(t^*-\lambda_n) - s(t^*-\lambda_n t)| &\leq \int_{t^*-\lambda_n t}^{t^*-\lambda_n} |\dot{s}| du \\ &\leq \left(\int_{t^*-\lambda_n t}^{t^*-\lambda_n} r^{1+\alpha/2}|\dot{s}|^2 du\right)^{1/2} \left(\int_{t^*-\lambda_n t}^{t^*-\lambda_n} \frac{du}{r^{1+\alpha/2}}\right)^{1/2}. \end{aligned}$$

It is not restrictive to suppose $t > 1$: indeed, when $t \in (0,1)$, we obtain an equivalent estimate by permuting the integration bounds. From equation (4.14) and Lemmata 4.12 and 4.15, we obtain

$$+\infty > \int_{t^*-\delta}^{t^*} r^{1+\alpha}\dot{r}|\dot{s}|^2 du \geq \int_{t^*-\delta}^{t^*} c_{1,\alpha} r^{1+\alpha/2}|\dot{s}|^2 du.$$

Then, since the constant $c_{1,\alpha}$ is strictly positive, we have

$$\lim_{n\to+\infty} \int_{t^*-\lambda_n t}^{t^*-\lambda_n} r^{1+\alpha/2}|\dot{s}|^2 du = 0.$$

Moreover, as n tends to $+\infty$, the second integral $\int_{t^*-\lambda_n t}^{t^*-\lambda_n} r^{-(1+\alpha/2)} < +\infty$; indeed, both integration bounds tend to t^* and the asymptotic estimate proved in (b) holds. Hence as $\lambda_n \to 0$,

$$\lim_{n\to+\infty}\int_{t^*-\lambda_n t}^{t^*-\lambda_n}\frac{du}{r^{1+\alpha/2}}\lim_{n\to+\infty}\left[\int_{t^*-\lambda_n t}^{t^*-\lambda_n} C\,\frac{du}{(t^*-u)}+o(1)\right]$$
$$= C\lim_{n\to+\infty}[\log(\lambda_n)-\log(\lambda_n t)+o(1)] = -C\log t,$$

that is bounded, since t is fixed and $C = [\frac{\sqrt{2b}(\alpha+2)}{2}]^{-(\alpha+2)/2}$. ■

Theorem 4.20. In the same setting of Theorem 4.18, assume that the potential U verifies further the assumption uniformly on compact subsets of $(a,b)\times(\mathcal{E}\setminus\Delta)$,

(U4)$_\mathrm{h}$ $\lim_{r\to 0} r^{\alpha+1}\nabla_T U(t,x) = \nabla_T\tilde{U}(t,s)$.

Then

$$\lim_{t\to t^*}\operatorname{dist}(\mathcal{C}^b,s(t))\lim_{t\to t^*}\inf_{\bar{s}\in\mathcal{C}^b}|s(t)-\bar{s}| = 0,$$

where $\mathcal{C}^b$ is the set of central configurations for $\tilde{U}$ at level b, namely the subset of critical points of the restriction of $\tilde{U}$ to the ellipsoid $\mathcal{E}$,

$$\mathcal{C}^b := \{s : \tilde{U}(t^*,s) = b, \nabla_T\tilde{U}(t^*,s) = 0\}. \tag{4.26}$$

□

Remark 4.21. When U is homogeneous, as in the classical Keplerian potential, then $\tilde{U}$ is simply the restriction of U on $\mathcal{E}$ and Theorem 4.20 asserts that the angular component s of the motion tends to a set of central configurations. □

Proof. Since in (a) of Theorem 4.18 we have already proved that $\lim_{t\to t^*}\tilde{U}(t,s(t)) = b$, it remains to show that

$$\lim_{t\to(t^*)^-}|\nabla_T\tilde{U}(t,s(t))| = 0,$$

that, using condition (U4)$_\mathrm{h}$, is equivalent to

$$\lim_{t\to(t^*)^-} r^{\alpha+1}|\nabla_T U(t,rs)| = 0.$$

We now consider the Euler–Lagrange equation (2.6)$_2$ multiplied by r^α,

$$-2r^{\alpha+1}\dot{r}\dot{s} - r^{\alpha+2}\ddot{s} + r^{\alpha+1}\nabla_T U(t, rs) = r^{\alpha+2}|\dot{s}|^2 s;$$

since $r^{\alpha+1}\dot{r}\dot{s} = r^{\alpha/2+1}\dot{r}r^{\alpha/2}\dot{s}$ is the product of a bounded term with an infinitesimal one (see equation (4.25) and Lemma 4.12), while $|r^{\alpha+2}|\dot{s}|^2 s| = r^{\alpha+2}|\dot{s}|^2$ tends to 0 for equation (4.25), we claim that

$$\lim_{t\to(t^*)^-} r^{\alpha+2}\ddot{s} = 0. \tag{4.27}$$

We perform the time rescaling (cf. McGehee's change of coordinates in 7.1)

$$\tau = \int_{t^*-\delta}^{t} \frac{d\xi}{r^{\alpha/2+1}}, \tag{4.28}$$

which maps the interval $[t^* - \delta, t^*)$ into $[0, +\infty)$ (see Corollary 4.14). If the prime $'$ denotes the derivative with respect to the new variable τ, then equation (4.27) is equivalent to

$$\lim_{\tau\to+\infty} s''(\tau) = 0 \tag{4.29}$$

and the limit (4.25) reads simply

$$\lim_{\tau\to+\infty} |s'(\tau)|^2 = 0. \tag{4.30}$$

Suppose now, for the sake of contradiction, that there exists a sequence $(\tau_n)_n$ such that $\tau_n \to +\infty$ as $n \to +\infty$, and

$$\lim_{n\to+\infty} \nabla_T \tilde{U}(\tau_n, s(\tau_n)) = \lim_{n\to+\infty} s''(\tau_n) = \sigma$$

for some $\sigma \neq 0$. Since the ellipsoid $\mathcal{E}$ is compact, up to subsequences, $(s(\tau_n))_n$ converges to some $\bar{s}$. Furthermore, from Theorem 4.18 we know that $\tilde{U}(\tau_n, s(\tau_n))$ tends to the finite limit b as $n \to +\infty$, hence $(t^*, \bar{s})$ is a regular point both for $\tilde{U}$ and for $\nabla_T \tilde{U}$. We moreover remark that, since the limit (4.30) holds, for every fixed positive constant $h > 0$, there holds

$$s(\tau) \to \bar{s}, \qquad \text{uniformly on } [\tau_n, \tau_n + h], \text{ for every } n$$

and also

$$\sup_{\tau\in[\tau_n,\tau_n+h]} |\nabla_T \tilde{U}(\tau, s(\tau)) - \sigma| \to 0 \qquad \text{as } n \to +\infty.$$

We can then compute

$$\begin{aligned} s'(\tau_n+h) - s'(\tau_n) &= \int_{\tau_n}^{\tau_n+h} s''(\tau)d\tau \\ &= \int_{\tau_n}^{\tau_n+h} \nabla_T \tilde{U}(\tau, s(\tau))d\tau + o(1) \\ &= h\sigma + o(1) \qquad \text{as } n \to +\infty. \end{aligned}$$

We obtain the contradiction

$$0 = \lim_{n\to+\infty} |s'(\tau_n+h) - s'(\tau_n)| = h|\sigma| \neq 0.$$

∎

4.2 Logarithmic potentials

The aim of this section is to extend the asymptotic estimates of Theorem 4.18 to potentials with logarithmic singularities. We follow the same scheme and we still work in a left neighborhood of a total collision instant t^*, $(t^* - \delta, t^*)$. The main differences concern the monotonicity formulæ (Lemmata 4.24 and 4.25).

In this setting, we suppose the existence of a continuous function

$$M\colon (a, b) \to [M_1, M_2] \subset (0, +\infty) \quad \text{such that } \dot{M}(t) \text{ is bounded on } (t^* - \delta, t^*) \tag{4.31}$$

and we replace conditions $(U2)_h$ and $(U3)_h$ with the following.

$(U2)_l$ There exist $\gamma > 0$ and $C_2 \geq 0$ such that

$$\nabla U(t, x) \cdot x + M(t) \geq -C_2|x|^{\gamma} U(t, x),$$

whenever $|x|$ is small.

$(U3)_l$ $\lim_{|x|\to 0} [U(t, x) + M(t) \log |x|] = \tilde{U}(t, s)$, uniformly for $s \in \mathcal{E} \setminus \Delta$ and for t in compact subsets of (a, b),

where $\tilde{U}$, as in the quasi-homogeneous case, is of class $\mathcal{C}^1$ on $(a, b) \times (\mathcal{E} \setminus \Delta)$ and verifies equation (4.2).

Remark 4.22. $(U2)_l$ implies (U2) (for small value of $|x|$) for every $\tilde{\alpha} \in (0, 2)$. □

Remark 4.23. From Corollary 3.3 and assumption $(U3)_1$, it follows that the positive function $-M(t)\log|x| + \tilde{U}(t,s)$ is integrable in a neighborhood of a total collision at the origin. Furthermore, thanks to the positivity of $\tilde{U}$ (cf. equation (4.2)), we have that also $\log|x|$ is integrable in such a neighborhood. □

We now prove the analogue of Lemmata 4.7 and 4.10 in the setting of logarithmic-type potentials.

Lemma 4.24. Let $\bar{x}$ be a generalized solution, let $t^* \in (a,b]$ be a total collision instant, and let δ be given in Theorem 4.2. Let γ be the positive exponent appearing in $(U2)_1$, then

$$\int_{t^*-\delta}^{t^*} -r^{\gamma}\frac{\dot{r}}{r}U(t,rs)dt < +\infty. \tag{4.32}$$

□

Proof. We define the functions

$$\Gamma_{\log}(r,s) := \tfrac{1}{2}r^2|\dot{s}|^2 - [U(t,rs) + M(t)\log r] \tag{4.33}$$

and

$$\tilde{\Gamma}_{\log}(r,s) := r^{\gamma}\Gamma_{\log}; \tag{4.34}$$

since

$$\tilde{\Gamma}_{\log}(r,s) = r^{\gamma}\left[h(t) - \tfrac{1}{2}\dot{r}^2 - M(t)\log r\right] \le r^{\gamma}h(t) - r^{\gamma}M(t)\log r,$$

then $\tilde{\Gamma}_{\log}$ is bounded above, indeed h is bounded, M is continuous and, since $\gamma > 0$, $\lim_{r\to 0} r^{\gamma}\log r = 0$. We now proceed exactly as in the proof of Lemma 4.7: we omit here the approximation arguments and we formally compute the time derivative of $\tilde{\Gamma}_{\log}$,

$$\frac{d}{dt}\tilde{\Gamma}_{\log}(r,s) = \gamma r^{\gamma-1}\dot{r}\Gamma_{\log}(r,s) + r^{\gamma}\frac{d}{dt}\Gamma_{\log}(r,s).$$

Using the Euler–Lagrange equation $(2.6)_2$, we obtain

$$\frac{d}{dt}\Gamma_{\log}(r,s) = -r\dot{r}|\dot{s}|^2 - \frac{\partial U}{\partial t}(t,rs) - \frac{\dot{r}}{r}\nabla U(t,rs)\cdot(rs) - \dot{M}(t)\log r - M(t)\frac{\dot{r}}{r}.$$

From assumptions (U1) and $(U2)_1$, we deduce that

$$\frac{d}{dt}\Gamma_{\log}(r,s) \geq -r\dot{r}|\dot{s}|^2 - C_1(U(t,rs)+1) + C_2 r^{\gamma}\frac{\dot{r}}{r}U(t,rs) - \dot{M}(t)\log r, \tag{4.35}$$

and then

$$\begin{aligned}\frac{d}{dt}\tilde{\Gamma}_{\log}(r,s) \geq & -\frac{2-\gamma}{2}r^{\gamma+1}\dot{r}|\dot{s}|^2 - \gamma r^{\gamma}\frac{\dot{r}}{r}U(t,rs) - \gamma r^{\gamma}\frac{\dot{r}}{r}M(t)\log r \\ & - C_1 r^{\gamma}U(t,rs) - C_1 r^{\gamma} + C_2 r^{2\gamma}\frac{\dot{r}}{r}U(t,rs) - \dot{M}(t)r^{\gamma}\log r. \end{aligned}\tag{4.36}$$

The first term in equation (4.36) is positive, since equation (4.1) holds; moreover, since r tends to 0 as t approaches t^*, there exist $\varepsilon \in (0,\gamma)$ and $\delta_0 \in (0,\delta]$ such that

$$-\gamma r^{\gamma}\frac{\dot{r}}{r}U(t,rs) + C_2 r^{2\gamma}\frac{\dot{r}}{r}U(t,rs) \geq -(\gamma-\varepsilon)r^{\gamma}\frac{\dot{r}}{r}U(t,rs) \geq 0 \tag{4.37}$$

on $(t^*-\delta_0, t^*)$. The remaining terms in equation (4.36) are integrable functions, indeed the last term $\dot{M}(t)r^{\gamma}\log r$ is bounded as r tends to 0 (see equation (4.31)), $r^{\gamma}U \leq U$, and U is integrable and we have the following estimate:

$$-\gamma r^{\gamma}\frac{\dot{r}}{r}M(t)\log r \geq -\gamma r^{\gamma-1}\dot{r}\log r \max_{t\in[t^*-\delta,t^*]} M(t)$$

and

$$\int_{t^*-\delta}^{t^*} \gamma r^{\gamma-1}\dot{r}\log r\,dt = -r_0^{\gamma}\log r_0 + \int_{t^*-\delta}^{t^*} r^{\gamma-1}\dot{r}\,dt = r_0^{\gamma}\left(-\log r_0 + \frac{1}{\gamma}\right) < +\infty,$$

where $r_0 = r(t^*-\delta)$. Hence, the right-hand side of equation (4.36) is the sum of an integrable function with a positive one; since the $\tilde{\Gamma}_{\log}(r,s)$ is bounded above from equation (4.37), we have the estimate in equation (4.32). ■

Lemma 4.25 (Monotonicity formula). The function $\Gamma_{\log}$ defined in equation (4.33) is bounded on $[t^*-\delta, t^*]$. □

Proof. We consider the expression of the derivative of $\Gamma_{\log}$ with respect to the time variable computed in equation (4.35). Using Lemma 4.24, the integrability of the function U, and Remark 4.23 we deduce the boundedness below (in a left neighborhood of t^*) of the function $\Gamma_{\log}$ being the right-hand side of equation (4.35) the sum of a positive function with an integrable one.

To prove the boundedness above of $\Gamma_{\log}$ we cannot use the boundedness of the energy function, indeed in this case we can just estimate $\Gamma_{\log}(r,s) + M(t)\log r = h(t) - \frac{1}{2}\dot{r}^2$. For the sake of contradiction, suppose that $\Gamma_{\log}$ diverges to $+\infty$ as t tends to t^*; since $U(t, rs) + M(t)\log r$ converges uniformly to $\tilde{U}(t,s)$ as t tends to t^* and $\tilde{U}(t,s)$ is a positive function, if $\Gamma_{\log}$ diverges to $+\infty$,

$$\exists t_1 \in (t^* - \delta, t^*) \text{ such that } \forall\, t \in (t_1, t^*), \quad r^2|\dot{s}|^2 > \max_{t\in[t^*-\delta,t^*]} M(t). \tag{4.38}$$

From assumption (4.38) we have

$$\begin{aligned}\int_{t^*-\delta}^{t^*} -\frac{\dot{r}}{r}(r^2|\dot{s}|^2 - M(t))dt \int_{t^*-\delta}^{t_1} -\frac{\dot{r}}{r}(r^2|\dot{s}|^2 - M(t))dt + \int_{t_1}^{t^*} -\frac{\dot{r}}{r}(r^2|\dot{s}|^2 - M(t))dt \\ \geq \text{constant} - \lim_{t\to t^*} \log r(t) = +\infty.\end{aligned} \tag{4.39}$$

We now define the function

$$\Omega_{\log}(r,s) := \Gamma_{\log}(r,s) + M(t)\log r = h(t) - \tfrac{1}{2}\dot{r}^2$$

that is bounded above. When we compute its derivative with respect to the time variable, we obtain the sum of a positive function with an integrable one (we use equation (4.38) and Lemma 4.24), indeed, as $\dot{M}$ is bounded and, asymptotically, $U(t,rs) \geq -M(t)\log r$, we have

$$\begin{aligned}\frac{d}{dt}\Omega_{\log}(r,s) &= \frac{d}{dt}\Gamma_{\log}(r,s) + \dot{M}(t)\log r + M(t)\frac{\dot{r}}{r} \\ &\geq -\frac{\dot{r}}{r}[r^2|\dot{s}|^2 - M(t)] - C_1(U(t,rs)+1) + C_2 r^{\gamma}\frac{\dot{r}}{r}U(t,rs).\end{aligned}$$

We can then conclude the boundedness of $\Omega_{\log}$ on the interval $[t^*-\delta, t^*]$ and from the estimate on its derivative, we have

$$\int_{t^*-\delta}^{t^*} -\frac{\dot{r}}{r}(r^2|\dot{s}|^2 - M(t))dt < +\infty$$

that contradicts equation (4.39). We conclude that the function $\Gamma_{\log}$ is also bounded above. ■

Corollary 4.26. As t tends to t^*, the limit of the function $\Gamma_{\log}$ exists finite and

$$\lim_{t\to(t^*)^+} -\frac{\dot r^2}{2\log r} = M_0,$$

where $M_0 := M(t^*)$. $\square$

Proof. We argue as in the proof of Corollary 4.17 to show that the function $\Gamma_{\log}$ has a finite limit as t tends to t^*. Since $\Gamma_{\log} = h(t) - \frac{1}{2}\dot r^2 - M(t)\log r$, we conclude dividing by $\log r$ using the boundedness of the function h. $\blacksquare$

Theorem 4.27. Let $\bar x$ be a generalized solution for the dynamical system (2.2) and let $t^* \in (a,b)$ (in the case $b < +\infty$, t^* can coincide with b) be a total collision instant. Let r, s be the new variables defined in equation (2.1); if the potential U satisfies assumptions (U0), (U1), $(U2)_1$, $(U3)_1$, then the following assertions hold:

(a) $\lim_{t\to t^{*-}} [U(t, rs) + M(t)\log r] = -\lim_{t\to t^{*-}} \Gamma_{\log}(r,s) = b$;
(b) as t tends to t^*,

$$r(t) \sim (t^*-t)\sqrt{-2M_0\log(t^*-t)}$$
$$\dot r(t) \sim -\sqrt{-2M_0\log(t^*-t)};$$

(c) $\lim_{t\to t^{*-}} |\dot s(t)|(t^*-t)\sqrt{-2M_0\log(t^*-t)} = 0$;
(d) for every real positive sequence $(\lambda)_n$, such that $\lambda_n \to 0$ as $n\to+\infty$, we have

$$\lim_{n\to+\infty} |s(t^*-\lambda_n) - s(t^*-\lambda_n t)| = 0, \quad \forall t > 0.$$

$\square$

Proof. (b) From Corollary 4.26 we deduce that

$$\dot r(t) \sim -\sqrt{-2M_0\log r(t)} \quad \text{as } t \text{ tends to } t^*.$$

We define $R(t) := (t^*-t)\sqrt{-2M_0\log(t^*-t)}$ and we remark that, as t tends to t^*,

$$-\log R(t) = -\log(t^*-t) - \log(\sqrt{-2M_0\log(t^*-t)}) \sim -\log(t^*-t)$$

and

$$\dot{R}(t) = -\sqrt{-2M_0 \log(t^* - t)} + \frac{M_0}{\sqrt{-2M_0 \log(t^* - t)}} \sim -\sqrt{-2M_0 \log R(t)}.$$

Our aim is then to prove that the function $r(t)$ is asymptotic to $R(t)$ as t tends to t^*. We define the following functions:

$$f(\xi) := -\sqrt{-2M_0 \log \xi} \quad \text{and} \quad \Phi(\xi) := \int_0^\xi \frac{d\eta}{f(\eta)}, \quad \xi \in (0, 1],$$

and we remark that $\Phi(0) = 0$ and Φ is a strictly decreasing function on $[0, 1]$. Moreover,

$$\dot{r}(t) \sim f(r(t)), \quad \dot{R}(t) \sim f(R(t)) \text{ as } t \text{ tends to } t^*,$$

or equivalently,

$$\lim_{t \to t^*} \frac{d}{dt} \Phi(r(t)) = \lim_{t \to t^*} \frac{d}{dt} \Phi(R(t)) = 1.$$

Since the function $\Phi(\xi)$ decreases in ξ and $r(t)$, $R(t)$ decrease in t (for small t's), we have that the compositions $\Phi(r(t))$ and $\Phi(R(t))$ are negative on $(t^* - \delta_0, t^*)$, vanishes at t^* (since $r(t^*) = R(t^*) = 0$) and increasing in the variable t. Furthermore, fixed $\bar{t} < t^*$, the following property holds:

$$\begin{aligned} \frac{d}{dt}\Phi(r(t)) \le 1 \le \frac{d}{dt}\Phi(R(t)),\ \forall t \in (\bar{t}, t^*) \quad &\Rightarrow \quad \Phi(r(t)) \ge \Phi(R(t)),\ \forall t \in (\bar{t}, t^*) \\ &\Rightarrow \quad r(t) \le R(t),\ \forall t \in (\bar{t}, t^*). \end{aligned} \tag{4.40}$$

For every $\epsilon > 0$, we consider the functions

$$\begin{aligned} R_\varepsilon^+(t) &:= (1 + \varepsilon)R(t), \\ R_\varepsilon^-(t) &:= (1 - \varepsilon)R(t). \end{aligned}$$

Since $\dot{R}(t) \sim f(R(t))$, we deduce that in a left neighborhood of t^*,

$$\begin{aligned} \dot{R}_\varepsilon^+(t) &= (1 + \varepsilon)\dot{R}(t) \le \left(1 + \frac{\varepsilon}{2}\right) f(R(t)) \le \left(1 + \frac{\varepsilon}{2}\right) f(R_\varepsilon^+(t)), \\ \dot{R}_\varepsilon^-(t) &= (1 - \varepsilon)\dot{R}(t) \ge \left(1 - \frac{\varepsilon}{2}\right) f(R(t)) \ge \left(1 - \frac{\varepsilon}{2}\right) f(R_\varepsilon^-(t)), \end{aligned} \tag{4.41}$$

indeed

$$f(R(t)) = -\sqrt{-2M_0 \log(R(t))} \le -\sqrt{-2M_0 \log(1+\varepsilon) - 2M_0 \log(R(t))} = f(R_\varepsilon^+(t)),$$

and similarly,

$$f(R(t)) \ge -\sqrt{-2M_0 \log(1-\varepsilon) - 2M_0 \log(R(t))} = f(R_\varepsilon^-(t)).$$

From equation (4.41) we then obtain

$$\frac{d}{dt}\Phi(R_\varepsilon^+(t)) \ge 1 + \frac{\varepsilon}{2} \quad \text{and} \quad \frac{d}{dt}\Phi(R_\varepsilon^-(t)) \le 1 - \frac{\varepsilon}{2}. \tag{4.42}$$

Moreover, since $\dot{r}(t) \sim f(r(t))$, again in a left neighborhood of t^* we have that

$$\left(1+\frac{\varepsilon}{2}\right) f(r(t)) \le \dot{r}(t) \le \left(1-\frac{\varepsilon}{2}\right) f(r(t)) \tag{4.43}$$

and dividing equation (4.43) by the negative function $f(r(t))$ and comparing the resulting inequalities with equation (4.42), we have

$$\frac{d}{dt}\Phi(R_\varepsilon^-(t)) \le \frac{d}{dt}\Phi(r(t)) \le \frac{d}{dt}\Phi(R_\varepsilon^+(t)).$$

From equation (4.40) we deduce that, in a neighborhood of the collision instant t^*, the following chain of inequalities holds:

$$(1-\varepsilon) \le \frac{r(t)}{R(t)} \le (1+\varepsilon).$$

The second estimate follows directly.

(a) Having proved (b), the proof of (a) is essentially the same that the one in Theorem 4.18. Indeed, going back to that proof, we notice that equation (4.19) still holds, since $-\dot{r}/r \sim (t^*-t)^{-1}$. Moreover, since $-r\dot{r} \sim -2M_0(t^*-t)\log(t^*-t) \ge (t^*-t)$, also equation (4.21) is still available.

(c) From the result proved in (a) we have that $\lim_{t\to t^*} r|\dot{s}| = 0$; we conclude using (b).

(d) As in the proof of Theorem 4.18, if $t=1$ there is nothing to prove. We then chose $t>0$, $t \neq 1$, a sequence $(\lambda_n)_n$, $\lambda_n \to 0$ and N, sufficiently large such that $\lambda_n < \delta/\max(1,t)$,

$\forall n \geq N$. We then obtain

$$|s(t^* - \lambda_n) - s(t^* - \lambda_n t)| \leq \int_{t^*-\lambda_n t}^{t^*-\lambda_n} |\dot{s}(u)| du$$
$$\leq \left(\int_{t^*-\lambda_n t}^{t^*-\lambda_n} -r(u)\dot{r}(u)|\dot{s}(u)|^2 du \right)^{\frac{1}{2}} \left(\int_{t^*-\lambda_n t}^{t^*-\lambda_n} -\frac{du}{r(u)\dot{r}(u)} \right)^{\frac{1}{2}}.$$

The boundedness of the $\Gamma_{\log}$ and the estimate on its derivative in equation (4.35) imply the boundedness of the integral $\int_0^{t^*} r\dot{r}|\dot{s}|^2$ and then

$$\lim_{n\to+\infty} \int_{t^*-\lambda_n t}^{t^*-\lambda_n} -r(u)\dot{r}(u)|\dot{s}(u)|^2 du = 0,$$

Moreover, as n tends to $+\infty$, from (b) and (c) we have $r(u)\dot{r}(u) \sim -2M_0(t^* - u)\log(t^* - u)$, hence

$$\lim_{n\to+\infty} \int_{t^*-\lambda_n t}^{t^*-\lambda_n} \frac{du}{r(u)\dot{r}(u)} = \frac{1}{M_0} \lim_{n\to+\infty} \log \frac{\log \lambda_n t}{\log \lambda_n} = 0.$$

The proof is now complete. ■

The behavior of the angular part is conserved also for logarithmic potential and the following result can be proved in analogy with Theorem 4.20.

Theorem 4.28. In the same setting of Theorem 4.27, assuming furthermore that the potential U verifies

(U4)$_l$ $\lim_{r\to 0} r\nabla_T U(t,x) = \nabla_T \tilde{U}(t,s)$,

then there holds

$$\lim_{t\to t^*} \mathrm{dist}(\mathcal{C}^b, s(t)) \lim_{t\to t^*} \inf_{\bar{s}\in\mathcal{C}^b} |s(t) - \bar{s}| = 0,$$

where $\mathcal{C}^b$ is the central configuration subset defined in equation (4.26). □

Proof. We follow the proof of Theorem 4.20, setting $\alpha = 0$ and we find that

$$-2r\dot{r}\dot{s} - r^2\ddot{s} + r\nabla_T U(t, rs) = r^2|\dot{s}|^2 s.$$

We already know that the right-hand side converges to zero. We claim that

$$\lim_{t \to (t^*)^-} \frac{d}{dt}(r^2 \dot{s}) = 0. \tag{4.44}$$

Again, we perform the time rescaling

$$\tau = \int_{t^*-\delta}^{t} \frac{d\xi}{r}, \tag{4.45}$$

which maps the interval $[t^* - \delta, t^*)$ into $[0, +\infty)$. If the prime $'$ denotes the derivative with respect to the new variable τ, then equation (4.44) is equivalent to

$$\lim_{\tau \to +\infty} \frac{1}{r}(rs')'(\tau) = 0 \tag{4.46}$$

and we know that

$$\lim_{\tau \to +\infty} |s'(\tau)|^2 = 0. \tag{4.47}$$

Arguing as in the proof of Theorem 4.20, suppose now, for the sake of contradiction, that there exists a sequence $(\tau_n)_n$ such that $\tau_n \to +\infty$ as $n \to +\infty$ and

$$\lim_{n \to +\infty} \nabla_T \tilde{U}(\tau_n, s(\tau_n)) = \lim_{n \to +\infty} \frac{1}{r}(rs')'(\tau_n) = \sigma$$

for some $\sigma \neq 0$. We have that, up to subsequences, $(s(\tau_n))_n$ converges to some $\bar{s}$. Furthermore, from Theorem 4.27 we know that $\tilde{U}(\tau_n, s(\tau_n))$ tends to the finite limit b as $n \to +\infty$, hence $(t^*, \bar{s})$ is a regular point both for $\tilde{U}$ and for $\nabla_T \tilde{U}$. We moreover remark that, since the limit (4.47) holds, for every fixed positive constant $h > 0$, there holds

$$s(\tau) \to \bar{s}, \qquad \text{uniformly on } [\tau_n, \tau_n + h], \text{ for every } n$$

and also

$$\sup_{\tau \in [\tau_n, \tau_n + h]} |\nabla_T \tilde{U}(\tau, s(\tau)) - \sigma| \to 0 \qquad \text{as } n \to +\infty.$$

We can then integrate by parts, obtaining a contradiction

$$\begin{aligned} s'(\tau_n + h) - s'(\tau_n) &= \int_{\tau_n}^{\tau_n+h} \frac{1}{r}(rs')'(\tau)d\tau - \int_{\tau_n}^{\tau_n+h} \frac{r'}{r}s'(\tau)d\tau \\ &= \int_{\tau_n}^{\tau_n+h} \nabla_T \tilde{U}(\tau, s(\tau))d\tau + o(1) \\ &= h\sigma + o(1) \qquad \text{as } n \to +\infty. \end{aligned}$$

Indeed, we have

$$0 = \lim_{n\to+\infty} \int_{\tau_n}^{\tau_n+h} \frac{r'}{r}s'(\tau)d\tau = \lim_{n\to+\infty} \int_{t(\tau_n)}^{t(\tau_n+h)} r\dot{r}\dot{s}dt,$$

since this last integral converges. ■

5 Partial Collisions

This section is devoted to the study of the singularities that are not total collisions at the origin. At first we shall prove the existence of a limiting configuration for bounded trajectories, that is, the Von Zeipel's theorem (stated before). This fact allows the reduction from partial to total collisions through a change of coordinates. To carry on the analysis, we shall extend the clustering argument proposed by McGehee in [36] to prove the Von Zeipel's theorem. Before starting, let us remark that, thanks to the energy estimate of Corollary 3.3, one can easily extend Painlevé's theorem to the wider class of locally minimal, or even generalized solutions, using the standard arguments.

Lemma 5.1. Let $\bar{x}$ be a generalized solution for the dynamical system (2.2) on the bounded interval (a, b). Suppose that the potential U satisfies assumptions (U0), (U1), and (U2). Then

$$\limsup_{t\to t^*} U(t, \bar{x}(t)) = +\infty \;\Rightarrow\; \liminf_{t\to t^*} U(t, \bar{x}(t)) = +\infty.$$

□

In order to proceed, we need to introduce some further assumptions on the potential U and its singular set Δ. More precisely, we suppose that

$$\Delta = \bigcup_{\mu\in\mathcal{M}} V_\mu, \tag{5.1}$$

where the V_μ's are distinct linear subspaces of $\mathbb{R}^k$ and $\mathcal{M}$ is a finite set; observe that the set Δ is a cone, as required before. We endow the family of the V_μ's with the inclusion partial ordering and we assume the family to be closed with respect to intersection (thus, we are assuming that $\mathcal{M}$ is a semilattice of linear subspaces of $\mathbb{R}^k$: it is the intersection semilattice generated by the arrangement of maximal subspaces V_μ's). With each $\xi \in \Delta$, we associate

$$\mu(\xi) = \min\{\mu \ : \ \xi \in V_\mu\} \qquad \text{i.e.} \qquad V_{\mu(\xi)} = \bigcap_{\xi \in V_\mu} V_\mu.$$

Fixed $\mu \in \mathcal{M}$, we define the set of collision configurations satisfying

$$\Delta_\mu = \{\xi \in \Delta : \mu(\xi) = \mu\}$$

and we observe that this is an open subset of V_μ and its closure $\overline{\Delta_\mu}$ is V_μ. We also notice that the map $\xi \to \dim(V_{\mu(\xi)})$ is lower semicontinuous.

We denote by p_μ the orthogonal projection onto V_μ and we write

$$x = p_\mu(x) + w_\mu(x),$$

where, of course, $w_\mu = \mathbb{I} - p_\mu$.

We assume that near the collision set the potential depends, roughly, only on the projection orthogonal to the collision set: more precisely, we assume the following.

(U5) For every $\xi \in \Delta$, there is $\varepsilon > 0$ such that

$$U(t,x) - U(t, \xi + w_{\mu(\xi)}(x)) = W(t,x) \in \mathcal{C}^1((a,b) \times B_\varepsilon(\xi)) \cap \in W^{1,\infty}((a,b) \times B_\varepsilon(\xi)),$$

where $B_\varepsilon(\xi) = \{x \ : \ |x - \xi| < \varepsilon\}$.

Theorem 5.2. Let $\bar{x}$ be a generalized solution for the dynamical system (2.2) on the bounded interval (a,b). Suppose that the potential U satisfies assumptions (U0), (U1), (U5), and $(U2)_{\mathrm{h}}$, $(U3)_{\mathrm{h}}$, $(U4)_{\mathrm{h}}$ (or $(U2)_{\mathrm{l}}$, $(U3)_{\mathrm{l}}$, $(U4)_{\mathrm{l}}$).
If $\bar{x}$ is bounded on the whole interval (a,b), then

(a) $\bar{x}$ has a finite number of singularities which are collisions (the Von Zeipel's theorem holds).

(b) Furthermore, if $t^* \in \bar{x}^{-1}(\Delta)$ is a collision instant, x^* the limit configuration of $\bar{x}$ as t tends to t^*, and $\mu^* = \mu(x^*) \in \mathcal{M}$, then $r_{\mu^*} = |w_{\mu^*}(\bar{x})|$, $s_{\mu^*} = w_{\mu^*}(\bar{x})/r_{\mu^*}$, and $U_{\mu^*}U(t, w_{\mu^*}(\bar{x}))$ satisfy the asymptotic estimates given in Theorems 4.18 and 4.20 (or Theorems 4.27 and 4.28 when $(U2)_1$, $(U3)_1$, and $(U4)_1$ hold). □

Proof. Let $\bar{x}$ be a generalized solution with a singularity at $t = t^*$ (see Definition 2.7) and Δ^* its ω-limit set, that is,

$$\Delta^* = \{x^* : \exists (t_n)_n \text{ such that } t_n \to t^* \text{ and } \bar{x}(t_n) \to x^*\}.$$

It is well known that the ω-limit of a bounded trajectory is a compact and connected set. From the Painlevé's theorem, we have the inclusion

$$\Delta^* \subset \Delta.$$

Von Zeipel's theorem asserts that whenever $\bar{x}$ remains bounded as t approaches t^*, then the ω-limit set of $\bar{x}$ contains just one element, that is, $\Delta^* = \{x^*\}$.

In view of Corollary 3.3, where we proved the theorem in the case $\liminf_{t\to t^*} \dot{I}(\bar{x}(t)) < +\infty$, we are left with the case when

$$\lim_{t\to t^*} \dot{I}(\bar{x}(t)) = +\infty.$$

From this and our assumptions it follows that $I(\bar{x}(t))$ is a definitely increasing and bounded function. Hence it admits a limit

$$\lim_{t\to t^*} I(\bar{x}(t)) = I^*. \tag{5.2}$$

We perform the proof of Von Zeipel's theorem in two steps.

Step 1. We suppose that $\mu(\Delta^*) = \{\mu^*\}$ for some $\mu^* \in \mathcal{M}$ and we show that $\Delta^* = \{x^*\}$.

As Δ^* is a compact and connected subset of V_{μ^*}, we have the following inclusions:

$$\Delta^* \subset \Delta_{\mu^*} \subset V_{\mu^*}.$$

We consider the orthogonal projections

$$p(t) = p_{\mu^*}(\bar{x}(t)), \qquad w(t) = w_{\mu^*}(\bar{x}(t)).$$

Since we have assumed that $\mu(\Delta^*) = \{\mu^*\}$, then

$$\lim_{t\to t^*} w(t) = 0, \tag{5.3}$$

our aim is now to prove that

$$\lim_{t\to t^*} p(t) = x^*.$$

Projecting on V_{μ^*} the equations of motion, from (U5) we obtain that the following differential equation holds (in the sense of distributions):

$$\ddot{p} = p_{\mu^*}(\nabla U(t, \bar{x}(t))) = p_{\mu^*}(\nabla W(t, \bar{x}(t))), \tag{5.4}$$

where ∇W is globally bounded as $t \to t^*$. Indeed, fixed $\varepsilon > 0$, there exists $\delta > 0$ such that $\bar{x}(t) \in B_\varepsilon(\Delta^*)$ whenever $t \in (t^* - \delta, t^*)$, and from assumption (U5) and the compactness of $\Delta^* \subset \Delta_{\mu^*}$, we deduce the boundedness of the right-hand side of equation (5.4). From this fact we easily deduce the existence of a limit for $p(t)$ as t tends to t^*. A word of caution must be entered at this point. As $\bar{x}$ is a generalized solution to equation (2.2), the equations of motions are not available, because of the possible occurence of collisions, and therefore they cannot be projected on V_{μ^*}. Nevertheless, exploiting the regularization method exposed in Section 2 and projecting the regularized equations, one can easily obtain the validity of equation (5.4) after passing to the limit almost everywhere.

Step 2. There always exists $\mu^* \in \mathcal{M}$ such that $\mu(\Delta^*) = \{\mu^*\}$.

Let μ^* be the element of $\mu(\Delta^*)$ associated with the subspace V_{μ^*} having *minimal dimension*. Since the function $\xi \to \dim(V_{\mu(\xi)})$ is lower semicontinuous, the minimality of the dimension has as a main implication that $\Delta_{\mu^*} \cap \Delta^*$ is compact. Hence, the function ∇W appearing in (U5) can be thought to be globally bounded in a neighborhood of $\Delta_{\mu^*} \cap \Delta^*$. In other words, when considering the orthogonal projections $p(t) = p_{\mu^*}(\bar{x}(t))$ and $w(t) = w_{\mu^*}(\bar{x}(t))$, as a major consequence of the minimality of the dimension μ^*, we find the following implication:

$$\exists M > 0,\ \exists \varepsilon > 0 : |w(t)|^2 < \varepsilon \Rightarrow |p_{\mu^*}(\nabla W(t, \bar{x}))| \le M. \tag{5.5}$$

We now compute the second derivative (with respect to the time t) of the function $|p(t)|^2$,

$$\frac{d^2}{dt^2}|p(t)|^2 = 2\ddot{p}(t)\cdot p(t) + 2\dot{p}(t)\cdot\dot{p}(t) \ge 2p_{\mu^*}(\nabla W(t, \bar{x}(t)))\cdot p(t).$$

Thus, from the projected motion equation (5.4) and from equation (5.5) we infer

$$\exists K > 0,\ \exists \varepsilon > 0 : |w(t)|^2 < \varepsilon \Rightarrow \frac{d^2}{dt^2}|p(t)|^2 \geq -K. \tag{5.6}$$

We now argue by contradiction, supposing that $\mu(\Delta^*) \neq \{\mu^*\}$. Then

$$0 = \liminf_{t\to t^*} |w(t)|^2 < \limsup_{t\to t^*} |w(t)|^2. \tag{5.7}$$

Since, obviously, the total moment of inertia splits as

$$I(\bar{x}(t)) = |p(t)|^2 + |w(t)|^2,$$

from equations (5.7) and (5.2) we deduce that

$$I^* = \limsup_{t\to t^*} |p(t)|^2 > \liminf_{t\to t^*} |p(t)|^2 \tag{5.8}$$

and from equation (5.8), together with equation (5.6) we have

$$\exists K > 0,\ \exists \varepsilon > 0\ :\ |p(t)|^2 \geq I^* - \varepsilon \Rightarrow \frac{d^2}{dt^2}|p(t)|^2 \geq -K.$$

Let $(t_n^0)_n$ and $(t_n^*)_n$ be two sequences such that, fixed $\varepsilon > 0$,

$$\begin{gathered} t_n^* < t_n^0 < t_{n+1}^* \quad \forall n \\ t_n^0 \to t^* \quad t_n^* \to t^* \text{ as } n \to +\infty \\ |p(t_n^*)|^2 \to I^* \text{ as } n \to +\infty \text{ and } \frac{d}{dt}(|p(t)|^2)|_{t=t_n^*} = 0, \quad \forall n \\ t_n^0 = \inf\{t > t_n^* : |p(t)|^2 \leq I^* - \varepsilon\}, \quad \forall n. \end{gathered}$$

Hence $|p(t_n^0)|^2 - |p(t_n^*)|^2 = \frac{d}{dt^2}|p(\xi)|^2(t_n^0 - t_n^*)^2/2 \geq -K(t_n^0 - t_n^*)^2/2$, and then

$$-\frac{\varepsilon}{2} \geq \frac{-K}{2}\left(t_n^0 - t_n^*\right)^2 \quad \text{or} \quad \left(t_n^0 - t_n^*\right)^2 \geq \frac{\varepsilon}{K}$$

in contradiction with the assumptions that both sequences $(t_n^0)_n$ and $(t_n^*)_n$ tend to the finite limit t^*. This concludes the proof of the Von Zeipel's theorem. Next we prove isolatedness of collision instants.

To this aim, let us select a collision instant t^* such that the dimension of $V_{\mu(\bar{x}(t^*))}$ is minimal among all dimensions of collision configurations $V_{\mu(\bar{x}(t))}$ in $(t^* - \delta, t^* + \delta)$ for some $\delta > 0$. As before, let us split the components of the trajectory $\bar{x}(t) = p(t) + w(t)$ on V_{μ^*} and its orthogonal complement.

Since μ^* is minimal (see equation (5.5)), we already know from the previous discussion that the equations of motion projected on the subspace V_{μ^*} (equation (5.4)) are not singular; on the other hand, by (U5), the trajectories in the orthogonal coordinates w are generalized solutions to a dynamical system of the form

$$-\ddot{w} = \nabla U(t, w) + \nabla W(t, p(t) + w). \tag{5.9}$$

Now, since $w(t)$ has total collisions at the origin at t^*, we can apply the results of Section 4. More precisely, at first we deduce from Theorem 4.2 that t^* is isolated in the set of collisions Δ_{μ^*}; furthermore, from Corollary 3.3 we deduce the boundedness of the action and the energy. Finally, we conclude applying Theorems 4.18, 4.20 (or Theorems 4.27, 4.28 when $(U2)_l$, $(U3)_l$, and $(U4)_l$ hold) to the projection w. In particular, from (a) in Theorem 4.18 (or Theorem 4.27) we obtain that every collision is isolated and hence, whenever the interval (a, b) is finite, the existence of a finite number of collisions. ■

6 Absence of Collisions for Locally Minimal Path

As a matter of fact, solutions to the Newtonian n-body problem which are minimizers for the action are, very likely, free of any collision. This was discovered in [47] for a class of periodic three-body problems and, since then, widely exploited in the literature concerning the variational approach to the periodic n-body problem. In general, the proof goes for the sake of the contradiction and involves the construction of a suitable variation that lowers the action in presence of a collision. A recent breakthrough in this direction comes from the averaging method introduced by C. Marchal in [33]. The method of averaged variations for Newtonian potentials has been exposed in [9], and then fully proved and extended to α-homogeneous potentials and various constrained minimization problems in [26]. This argument can be used in most of the known cases to prove that minimizing trajectories are collisionless. In this section, we prove the absence of collisions for locally minimal solutions when the potentials have quasi-homogeneous or logarithmic singularities.

We consider separately the quasi-homogeneous and the logarithmic cases; indeed, in the first case one can exploit the blow-up technique as developed in Section 7 of

[26]; in Section 6.1 we will just recall the main steps of this argument. On the other hand, when dealing with logarithmic potentials, the blow-up technique is no longer available and we conclude proving directly some averaging estimates that can be used to show the nonminimality of large classes of colliding motions.

6.1 Quasi-homogeneous potentials

Let $\tilde{U}$ be the $\mathcal{C}^1$ function defined on $(a,b) \times (\mathcal{E} \smallsetminus \Delta)$ introduced previously; we extend its definition on the whole $(a,b) \times (\mathbb{R}^k \smallsetminus \Delta)$ in the following way:

$$\tilde{U}(t,x) = |x|^{-\alpha} \tilde{U}(t, x/|x|).$$

Fixed t^* (in this section we will consider a locally minimal trajectory $\bar{x}$ with a collision at t^*) in this section, with an abuse of notation, we denote

$$\tilde{U}(x) = \tilde{U}(t^*, x). \tag{6.1}$$

Of course, the function $\tilde{U}$ is homogeneous of degree $-\alpha$ on $\mathbb{R}^k \smallsetminus \Delta$.

Theorem 6.1. In addition to (U0), (U1), $(\mathrm{U2})_\mathrm{h}$, $(\mathrm{U3})_\mathrm{h}$, $(\mathrm{U4})_\mathrm{h}$, (U5), assume that for a given $\xi \in \Delta$

(U6), there is a 2-dimensional linear subspace of $V^\perp_{\mu(\xi)}$, say W, where $\tilde{U}$ is rotationally invariant;

$(\mathrm{U7})_\mathrm{h}$ for every $x \in \mathbb{R}^k$ and $\delta \in W$ there holds

$$\tilde{U}(x+\delta) \leq \tilde{U}\left(\left(\frac{\tilde{U}(\pi_W(x))}{\tilde{U}(x)}\right)^{1/\alpha} \pi_W(x) + \left(\frac{\tilde{U}(x)}{\tilde{U}(\pi_W(x))}\right)^{1/\alpha} \delta\right),$$

where π_W denotes the orthogonal projection onto W; this is the property depicted in Figure 1;

$(\mathrm{U8})_\mathrm{h}$ for every $y \in W^\perp$ and $\delta \in W \setminus \{0\}$ there holds

$$\tilde{U}(y+\delta) < \tilde{U}(y).$$

Then generalized solutions do not have collisions at the configuration ξ at the time t^*. $\square$

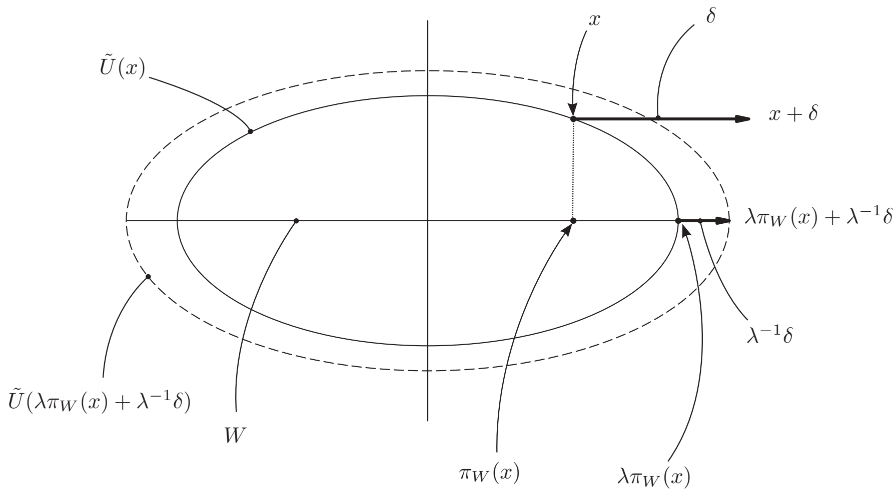

Fig. 1. Potential levels, with $\lambda = (\frac{\tilde U(\pi_W(x))}{\tilde U(x)})^{1/\alpha} > 1$

Remark 6.2. Some comments on assumptions (U6),, (U7)$_\mathrm{h}$, and (U8)$_\mathrm{h}$ are in order. Of course, as our potential $\tilde U$ is homogeneous of degree $-\alpha$, the function

$$\varphi(x) = \tilde U^{-1/\alpha}(x)$$

is a non-negative, homogeneous of degree 1 function, having now Δ as a zero set. In most of our applications, φ will be indeed a quadratic form. Assume that φ^2 splits in the following way

$$\varphi^2(x) = K|\pi_W(x)|^2 + \varphi^2(\pi_{W^\perp}(x))$$

for some positive constant K. Then (U6),, (U7)$_\mathrm{h}$, and (U8)$_\mathrm{h}$ are satisfied. Indeed, denoting $w = \pi_W(x)$ and $z = x - w$ we have, for every $\delta \in W$,

$$\begin{aligned}
\varphi^2(x+\delta) &= K|w+\delta|^2 + \varphi^2(z)\\
&= K\left|\frac{\varphi(x)}{\varphi(w)}w + \frac{\varphi(w)}{\varphi(x)}\delta\right|^2 + K\frac{\varphi^2(z)}{\varphi^2(x)}|\delta|^2\\
&\geq K\left|\frac{\varphi(x)}{\varphi(w)}w + \frac{\varphi(w)}{\varphi(x)}\delta\right|^2\\
&= \varphi^2\left(\frac{\varphi(x)}{\varphi(w)}w + \frac{\varphi(w)}{\varphi(x)}\delta\right),
\end{aligned}$$

which is obviously equivalent to (U7)$_\mathrm{h}$. Therefore, we have the following proposition. □

Proposition 6.3. Assume $\tilde{U}(x) = \mathcal{Q}^{-\alpha/2}(x)$ for some non-negative quadratic form $\mathcal{Q}(x) = \langle Ax, x\rangle$. Then assumptions (U6),, $(U7)_{\mathrm{h}}$, and $(U8)_{\mathrm{h}}$ are satisfied whenever W is included in an eigenspace of A associated with a multiple eigenvalue. □

Remark 6.4. Given two potentials satisfying (U6),, $(U7)_{\mathrm{h}}$, and $(U8)_{\mathrm{h}}$ for a common subspace W, their sum enjoys the same properties. On the other hand, if they do not admit a common subspace W, their sum does not satisfy (U6),, $(U7)_{\mathrm{h}}$, and $(U8)_{\mathrm{h}}$. □

Proof (Proof or Theorem 6.1)**.** Let $\bar{x}(t)$ be a generalized solution with a collision at the time t^*, i.e. $\bar{x}(t^*) = \xi \in \Delta$; up to time translation, we assume that the collision instant is $t^* = 0$. Furthermore, using the same arguments needed in the proof of the Von Zeipel's theorem in Section 5, we can suppose that $\xi = 0$. We consider the case of a boundary collision (interior collisions can be treated in a similar way). Then Theorem 4.18 ensures the existence of $\delta_0 > 0$ such that no other collision occurs in some interval $[0, \delta_0]$.

We consider the family of rescaled generalized solutions

$$\bar{x}^{\lambda_n}(t) := \lambda_n^{-\frac{2}{2+\alpha}} \bar{x}(\lambda_n t), \quad t \in [0, \delta_0/\lambda_n],$$

where $\lambda_n \to 0$ as $n \to +\infty$. From the asymptotic estimates of Theorem 4.20 we know that the angular part $(s(\lambda_n))_n$ converges, up to subsequences, to some central configuration $\bar{s}$, in particular $\bar{s}$ is in the ω-limit of $s(t)$.

For any $\bar{s}$ in the ω-limit of $s(t)$, a (right) blowup of $\bar{x}$ in $t = 0$ is a path defined, for $t \in [0, +\infty)$, as

$$\bar{q}(t) := \zeta t^{\frac{2}{2+\alpha}}, \qquad \zeta = K\bar{s}, \tag{6.2}$$

where the constant $K > 0$ is determined by part (b) of Theorem 4.18. We note that the blowup is a homothetic solution to the dynamical system associated with the homogeneous potential $\tilde{U}$ and that it has zero energy (the blowup is parabolic). If $s(\lambda_n) \to \bar{s}$ as $n \to +\infty$, from Theorem 4.18, we obtain straightforwardly the pointwise convergence of $\bar{x}^{\lambda_n}$ to the blowup $\bar{q}$ and the H^1-boundedness of $\bar{x}^{\lambda_n}$ implies its uniform convergence on compact subsets of $[0, +\infty)$. Furthermore, the convergence holds locally in the $H^1([0, +\infty))$-topology. Finally, also the sequence $\dot{\bar{x}}^{\lambda_n}$ converges uniformly on every interval $[\varepsilon, T]$, with arbitrary $0 < \varepsilon < T$.

The following fact has been proven in [26, Proposition 7.9].

Lemma 6.5. Let $\bar{x}$ be a locally minimizing trajectory with a total collision at $t = 0$ and let $\bar{q}$ be its blowup in $t = 0$. Then $\bar{q}$ is a locally minimizing trajectory for the dynamical system associated with the homogeneous potential $\tilde{U}$ introduced in equation (6.1). □

We will conclude the proof showing that $\bar{q}$ cannot be a locally minimizing trajectory for the dynamical system associated with $\tilde{U}$. Following [26], we now introduce a class of suitable variations as follows.

Definition 6.6. The standard variation associated with $\delta \in \mathbb{R}^k \setminus \{0\}$ is defined as

$$v^\delta(t) = \begin{cases} \delta & \text{if } \ 0 \le |t| \le T - |\delta| \\ (T - t)\frac{\delta}{|\delta|} & \text{if } \ T - |\delta| \le |t| \le T \\ 0 & \text{if } \ |t| \ge T, \end{cases}$$

for some positive T. □

We wish to estimate the action differential corresponding to a standard variation. To this aim, we give the next definition.

Definition 6.7. The displacement potential differential associated with $\delta \in \mathbb{R}^k$ is defined as

$$S(\zeta, \delta) = \int_0^{+\infty} (\tilde{U}(\zeta t^{2/(2+\alpha)} + \delta) - \tilde{U}(\zeta t^{2/(2+\alpha)}))dt,$$

where $\bar{q}(t) = \zeta t^{2/(2+\alpha)}$ is a blowup of $\bar{x}$ in $t = 0$. □

The quantity $S(\zeta, \delta)$ represents the *potential differential* needed for displacing the colliding trajectory originarily traveling along the ζ-direction to the point δ. It has been proven in [26, proposition 9.2], that the function S represents the limiting behavior, as $\delta \to 0$, of the whole action differential

$$\Delta\mathcal{A}^\delta := \int_{-\infty}^{+\infty} [K(\dot{\bar{q}} + \dot{v}^\delta) + \tilde{U}(\bar{q} + v^\delta) - K(\dot{\bar{q}}) - \tilde{U}(\bar{q})]dt.$$

Indeed, the fundamental estimate holds.

Lemma 6.8. Let $\bar{q} = \zeta t^{2/(2+\alpha)}$ be a blowup trajectory and v^δ any standard variation. Then as $\delta \to 0$,

$$\Delta \mathcal{A}^\delta = |\delta|^{1-\alpha/2} S\left(\zeta, \frac{\delta}{|\delta|}\right) + O(|\delta|).$$

$\square$

We observe that, from the homogeneity of $\tilde{U}$ it follows that

$$S(\lambda\xi, \mu\delta) = |\lambda|^{-1-\alpha/2} |\mu|^{1-\alpha/2} S(\xi, \delta) \tag{6.3}$$

(see [26, (8.2)]) and hence, if $\tilde{U}$ is invariant under rotations, the sign of S depends only on the angle between ξ and δ. To deal with the isotropic case (which is not the case here), the following function was introduced in [26]:

$$\Phi_\alpha(\vartheta) = \int_0^{+\infty} \frac{1}{\left(t^{\frac{4}{\alpha+2}} - 2\cos\vartheta\, t^{\frac{2}{\alpha+2}} + 1\right)^{\alpha/2}} - \frac{1}{t^{\frac{2\alpha}{\alpha+2}}} dt.$$

The value of $\Phi_\alpha(\vartheta)$ ranges from positive to negative values, depending on ϑ and α. Nevertheless, it is always negative, when averaged on a circle. Indeed, the following inequality was obtained in [26, Theorem 8.4].

Lemma 6.9. For any $\alpha \in (0, 2)$ there holds

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_\alpha(\vartheta) d\vartheta < 0.$$

$\square$

This inequality will be a key tool in proving the following averaged estimate.

Lemma 6.10. Assume (U6),, $(\mathrm{U7})_{\mathrm{h}}$, and $(\mathrm{U8})_{\mathrm{h}}$, then, if $\mathbb{S}$ is the unitary circle of W, for any $\zeta \in \mathbb{R}^k \setminus \{0\}$ the following inequality holds:

$$\int_{\mathbb{S}} S(\zeta, \delta) d\delta < 0.$$

As a consequence,

$$\forall \zeta \in \mathbb{R}^k \setminus \{0\}\ \exists \delta = \delta(\zeta) \in \mathbb{S}\ :\ S(\zeta, \delta(\zeta)) < 0.$$

$\square$

Proof. As a first obvious application of Lemma 6.9, we obtain the assertion *for any* $\zeta \in W \setminus \{0\}$. Indeed, by equation (6.3) and (U6), we easily obtain

$$\zeta \in W \setminus \{0\} \quad \Longrightarrow \quad S(\zeta, \delta) = K|\zeta|^{-1-\alpha/2}\Phi_\alpha(\vartheta),$$

where K is a positive constant and ϑ denotes the angle between ζ and δ.

Now we prove the assertion for any $\zeta \neq 0$ in the configuration space. When $\pi_W(\zeta) = 0$, the assertion obviously follows from $(\mathrm{U8})_\mathrm{h}$. It follows from the homogeneity of $\tilde{U}$ that, if $\pi_W(\zeta) \neq 0$,

$$\tilde{U}\left(\left(\frac{\tilde{U}(\pi_W(\zeta))}{\tilde{U}(\zeta)}\right)^{1/\alpha} \pi_W(\zeta)\right) = \tilde{U}(\zeta).$$

Hence $(\mathrm{U7})_\mathrm{h}$ implies, for every $\delta \in \mathbb{S}$,

$$S(\zeta, \delta) \leq S\left(\left(\frac{\tilde{U}(\pi_W(\zeta))}{\tilde{U}(\zeta)}\right)^{1/\alpha} \pi_W(\zeta), \left(\frac{\tilde{U}(\zeta)}{\tilde{U}(\pi_W(\zeta))}\right)^{1/\alpha} \delta\right).$$

Hence equation (6.3) implies

$$S(\zeta, \delta) \leq \left(\frac{\tilde{U}(\zeta)}{\tilde{U}(\pi_W(\zeta))}\right)^{2/\alpha} S(\pi_W(\zeta), \delta),$$

and thus

$$\int_{\mathbb{S}} S(\zeta, \delta)\, d\delta \leq \left(\frac{\tilde{U}(\zeta)}{\tilde{U}(\pi_W(\zeta))}\right)^{2/\alpha} \int_{\mathbb{S}} S(\pi_W(\zeta, \delta))\, d\delta < 0.$$

■

End of the Proof of Theorem 6.1. To conclude the proof, according to Lemma 6.10 we chose $\delta = \delta(\zeta) \in W \setminus \{0\}$ with the property that $S(\zeta, \delta(\zeta)/|\delta(\zeta)|) < 0$. As a consequence of Lemma 6.8, we can lower the value of the action of $\bar{q}$ by performing the standard variation $v^{\delta(\zeta)}$, provided the norm of $|\delta(\zeta)|$ is sufficiently small (in order to apply Lemma 6.8). Hence $\bar{q}$ cannot be locally minimizing for the action. ■

As we have already noticed, the class of potentials satisfying (U6), and $(\mathrm{U7})_\mathrm{h}$ is not stable with respect to the sum of potentials. In order to deal with a class of potentials which is closed with respect to the sum, we introduce the following variant of Theorem 6.1.

Theorem 6.11. In addition to (U0), (U1), $(U2)_h$, $(U3)_h$, $(U4)_h$, (U5), assume that $\tilde{U}$ has the form

$$\tilde{U}(x) = \sum_{\nu=1}^{N} \frac{K_\nu}{(\mathrm{dist}(x, V_\nu))^\alpha},$$

where K_ν are positive constants and V_ν is a family of linear subspaces, with $\mathrm{codim}(V_\nu) \geq 2$, for every $\nu = 1, \ldots, N$. Then locally minimizing trajectories do not have collisions at the time t^*. □

Proof. Following the arguments of the proof of Theorem 6.1, the assertion will be proved once we show, as in Lemma 6.10, that for every index ν there holds

$$\int_{\mathbb{S}^{k-1}} S_\nu(\zeta, \delta)\, d\delta < 0,$$

where, of course, we denote

$$S_\nu(\zeta, \delta) = \int_0^{+\infty} \left(\mathrm{dist}(\zeta t^{2/(2+\alpha)} + \delta, V_\nu)^{-\alpha} - \mathrm{dist}(\zeta t^{2/(2+\alpha)}, V_\nu)^{-\alpha}\right) dt$$

and $\mathbb{S}^{k-1}$ is the unit sphere of the configuration space $\mathbb{R}^k$. This is an elementary consequence of Lemma 6.10 and the fact that the function $S_\nu(\zeta, \delta)$ only depends on the projection of ζ orthogonal to V_μ and has rotational invariance on $V_\nu^\perp$. Thus, the integral of S_ν over the sphere is a positive multiple of its integral on any circle $\mathbb{S}$ orthogonal to V_ν. ■

6.2 Logarithmic-type potentials

In this section, we prove the equivalents to Theorems 6.1 and 6.11 suitable for logarithmic-type potentials. Concerning the quasi-homogeneous case, we have seen that a crucial role is played by the construction of a blow-up function which minimizes a limiting problem. Before starting, let us highlight the reasons why, when dealing with logarithmic potentials, a blow-up limit cannot exist. Indeed, the natural scaling should be $\bar{x}^{\lambda_n}(t) := \lambda_n^{-1}\bar{x}(\lambda_n t)$, which does not converge, since

$$\lim_{\lambda_n \to 0} \bar{x}^{\lambda_n}(t) = \lim_{\lambda_n \to 0} \frac{r(\lambda_n t) s(\lambda_n t)}{\lambda_n t\sqrt{-2M(0)\log(\lambda_n t)}} t\sqrt{-2M(0)\log(\lambda_n t)} = +\infty$$

for every $t > 0$. On the other hand looking at equation (6.2), the (right) blowup should be, up to a change of time scale,

$$\bar{q}(t) := t\bar{s}, \quad i \in \mathbf{k}, \tag{6.4}$$

where $\bar{s}$ is a central configuration for the system limit of a sequence $s(\lambda_n)$, where $(\lambda_n)_n$ is such that $\lambda_n \to 0$. The blow-up function defined in equation (6.4) is the pointwise limit of the normalized sequence

$$\bar{x}^{\lambda_n}(t) := \frac{1}{\lambda_n\sqrt{-2M(0)\log\lambda_n}}\bar{x}(\lambda_n t).$$

Unfortunately, the path in equation (6.4) is not locally minimal for the limiting problem, indeed since, the sequence $(\ddot{\bar{x}}^{\lambda_n})_n$ converges to 0 as n tends to $+\infty$, the blowup in equation (6.4) minimizes only the kinetic part of the action functional.

We shall overcome this difficulty by proving the averaged estimate in a direct way from the asymptotic estimates of Theorem 4.27 and assuming equation (6.6) on the potential U. As we have done for the quasi-homogeneous case, we extend the function $\tilde{U}$, introduced in assumption (U3)$_1$, to the whole $(a,b) \times \mathbb{R}^k \setminus \Delta$ in the natural way

$$\tilde{U}(t,x) = \tilde{U}(t,s) - M(t)\log|x|, \tag{6.5}$$

where M has been introduced in equation (4.31).

Theorem 6.12. In addition to (U0), (U1), (U2)$_1$, (U3)$_1$, (U4)$_1$, (U5), assume the potential U to be of the form

$$U(t,x) = \tilde{U}(t,x) + W(t,x), \tag{6.6}$$

where $\tilde{U}$ satisfies equation (6.5) and W is a bounded $\mathcal{C}^1$ function on $(a,b) \times \mathbb{R}^k$. Furthermore, assume that, for a given $\xi \in \Delta$, $\tilde{U}$ satisfies (U6), and

(U7)$_1$ for every $x \in \mathbb{R}^k$ and $t \in (a,b)$ there holds

$$\tilde{U}(t,x) = -\tfrac{1}{2}M(t)\log(|\pi_W x|^2 + \psi^2(\pi_{W^\perp}x)),$$

where π_W and $\pi_{W^\perp}$ denote the orthogonal projections onto W and $W^\perp$, ψ is $\mathcal{C}^1$ and homogeneous of degree 1.

Then locally minimizing trajectories do not have collisions at the configuration ξ at the time t^*. □

Proof. As in the proof of Theorem 6.1, we consider a generalized solution $\bar{x}$ and we first reduce to the case of an isolated total collision at the origin occurring at the time $t = 0$. From Theorem 4.27 we deduce the existence of $\delta_0 > 0$ such that no other collision occurs in $[-\delta_0, \delta_0]$, hence we perform a local variation on the trajectory of $\bar{x}$ that removes the collision and makes the action decrease.

Consider now the standard variation v^δ, defined previously, on the interval $[0, \delta_0]$ (i.e. in Definition 6.6 T is replaced by δ_0). Let $\Delta^\delta\mathcal{A}$ denote the difference

$$\Delta^\delta\mathcal{A} : \mathcal{A}(\bar{x} + v^\delta, [0, \delta_0]) - \mathcal{A}(\bar{x}, [0, \delta_0]);$$

generally speaking, this difference can be positive or negative, depending on the choice of δ. Our goal is to prove that, when *averaging over a suitable set of standard variations*, the action lowers. Hence $\Delta^\delta\mathcal{A}$ must be negative for at least one choice of δ and the path $\bar{x}$ cannot be a local minimizer for the action.

We can write $\Delta^\delta\mathcal{A}$ as the sum of three terms

$$\Delta^\delta\mathcal{A} = \int_0^{\delta_0} \Delta^\delta\mathcal{K}(t)\,dt + \int_0^{\delta_0} \Delta^\delta\mathcal{U}(t)\,dt + \int_0^{\delta_0} \Delta^\delta\mathcal{W}(t)\,dt, \tag{6.7}$$

where $\Delta^\delta\mathcal{K}(t)$, $\Delta^\delta\mathcal{U}(t)$, and $\Delta^\delta\mathcal{W}(t)$ are, respectively, the variations of the kinetic energy, of the singular potential $\tilde{U}$, and of the smooth part of the potential, W. More precisely, since the first derivative of the function v^δ vanishes everywhere on $[0, \delta_0]$, except on $[\delta_0 - |\delta|, \delta_0]$, we compute

$$\Delta^\delta\mathcal{K}(t) : \begin{cases} 0, & \text{if } t \in [0, \delta_0 - |\delta|], \\ \dfrac{1}{2}(|\dot{\bar{x}} - \delta/|\delta||^2 - |\dot{\bar{x}}|^2) & \text{if } t \in [\delta_0 - |\delta|, \delta_0]. \end{cases} \tag{6.8}$$

Similarly,

$$\Delta^\delta\mathcal{U}(t) := \tilde{U}(t, \bar{x} + v^\delta) - \tilde{U}(t, \bar{x}) \qquad \text{and} \qquad \Delta^\delta\mathcal{W}(t) := W(t, \bar{x} + v^\delta) - W(t, \bar{x}).$$ ■

We now evaluate separately the mean values of the tree terms of $\Delta^\delta \mathcal{A}$ over the circle $S^{|\delta|}$ of radius $|\delta|$ in W.

Lemma 6.13. There holds

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0}(\Delta^\delta\mathcal{K}+\Delta^\delta\mathcal{W})\,dt\,d\delta = O(|\delta|). \tag{6.9}$$

□

Proof. From equation (6.8) we obtain

$$\int_0^{\delta_0}\Delta^\delta\mathcal{K}(t)dt\int_{\delta_0-|\delta|}^{\delta_0}\frac{1}{2}(|\dot{\bar{x}}-\delta/|\delta||^2-|\dot{\bar{x}}|^2)dt = \frac{1}{2}\left(|\delta| - 2\int_{\delta_0-|\delta|}^{\delta_0}\dot{\bar{x}}(t)\cdot\frac{\delta}{|\delta|}dt\right),$$

hence

$$\left|\int_0^{\delta_0}\Delta^\delta\mathcal{K}(t)dt\right| \le O(|\delta|),$$

which does not depend on the circle $S^{|\delta|}$ where δ varies. Concerning the variation of the $\mathcal{C}^1$ function W, we have

$$\left|\int_0^{\delta_0}\Delta^\delta\mathcal{W}(t)\,dt\right|\left|\int_0^{\delta_0-|\delta|}\Delta^\delta\mathcal{W}(t)\,dt\right| + \left|\int_{\delta_0-|\delta|}^{\delta_0}\Delta^\delta\mathcal{W}(t)\right|\,dt \le W_1|\delta|(\delta_0-|\delta|)+2W_2|\delta| = O(|\delta|),$$

where W_1 is a bound for $|\frac{\partial W}{\partial x}(t,\bar{x}+\lambda v^\delta)|$, with $\lambda\in[0,1]$ and $t\in[0,\delta_0-|\delta|]$, while W_2 is an upper bound for $|W(t,x)|$. ■

In order to estimate the variation of the potential part, $\Delta^\delta\mathcal{U}(t)$, we prove the next two technical lemmata. Let us start with recalling an equivalent version of the mean value property for the fundamental solution of the planar Laplace equation.

Lemma 6.14. Fixed $z>0$, for every $y\in\mathbb{R}$ such that $y\ge 2z$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log(y+2z\cos\vartheta)\,d\vartheta\log\frac{y+\sqrt{y^2-4z^2}}{2}.$$

□

Proof. Since $y \geq 2z$, then $\frac{y+\sqrt{y^2-4z^2}}{2} \geq z$. Let $x \in \mathbb{R}^2$ be such that $|x| = \frac{y+\sqrt{y^2-4z^2}}{2}$, then $y = (|x|^2 + z^2)/|x|$ and for every $\delta \in S^z$, where S^z is the circle of radius z, we have

$$\begin{aligned} |x+\delta|^2 &= |x|^2 + z^2 + 2z|x| \cos \vartheta \\ &= |x| \left(\frac{|x|^2 + z^2}{|x|} + 2z \cos \vartheta \right) = |x|(y + 2z \cos \vartheta). \end{aligned}$$

We have, as the logarithm is the fundamental solution to the Laplace equation on the plane,

$$\frac{1}{2\pi z} \int_{S^z} \log |x+\delta|^2 \, d\delta = \max\{\log |x|^2, \log z^2\} = \begin{cases} \log |x|^2, & \text{if } |x| > z \\ \log z^2, & \text{if } |x| \leq z. \end{cases} \tag{6.10}$$

Consequently, when computing

$$\begin{aligned} \int_{S^z} \log |x+\delta|^2 d\delta &= \int_{S^z} \log |x| d\delta + z \int_0^{2\pi} \log(y + 2z \cos \vartheta) d\vartheta \\ &= 2\pi z \log |x| + z \int_0^{2\pi} \log(y + 2z \cos \vartheta) d\vartheta, \end{aligned}$$

we find

$$2\pi z \log |x|^2 = 2\pi z \log |x| + z \int_0^{2\pi} \log(y + 2z \cos \vartheta) d\vartheta.$$

We conclude replacing $|x| = \frac{y+\sqrt{y^2-4z^2}}{2}$. ■

Now we consider the averages of the potential with respect to a circle in W (here we assume implicitly that $d \geq 3$).

Lemma 6.15. Fixed $|\delta| > 0$, for every circle of radius $|\delta|$, $S^{|\delta|} \subset W$, for every $x \in \mathbb{R}^d$, and every $t \in [0, \delta_0]$, there holds

$$\frac{1}{2\pi |\delta|} \int_{S^{|\delta|}} (\tilde{U}(x+\delta) - \tilde{U}(x)) d\delta \leq \begin{cases} 0 \;\; (\text{if } |\pi_W x|^2 + \psi^2(\pi_{W^\perp} x) > |\delta|^2), \\ \dfrac{M(t)}{2} \log(|\pi_W x|^2 + \psi^2(\pi_{W^\perp} x)) - \log(|\delta|^2) \;\; (\text{otherwise}). \end{cases}$$

□

Proof. We consider the orthogonal decomposition of x, $x = \pi_W x + \pi_{W^\perp} x$, and we term $u := |\pi_W x|$ and $\varepsilon := \psi(\pi_{W^\perp} x)$. Since whenever $\delta \in W$, we have

$$|\pi_W(x+\delta)|^2 + \psi^2(\pi_{W^\perp} x) = u^2 + |\delta|^2 + 2u|\delta|\cos\vartheta + \varepsilon^2 \geq 0,$$

when $\cos\vartheta = -1$, we have $\frac{u^2+|\delta|^2+\varepsilon^2}{u|\delta|} \geq 2$ and, using Lemma 6.14 and equation (6.10), we compute

$$\begin{aligned}
&\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}} \log(|\pi_W(x+\delta)|^2 + \psi^2(\pi_{W^\perp} x))\, d\delta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log(u^2+\varepsilon^2+|\delta|^2 + 2u|\delta|\cos\vartheta)\, d\vartheta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left(\frac{u^2+\varepsilon^2+|\delta|^2}{u|\delta|} + 2\cos\vartheta\right) d\vartheta + \log(u|\delta|) \\
&= \log\left(\frac{u^2+\varepsilon^2+|\delta|^2+\sqrt{(u^2+\varepsilon^2+|\delta|^2)^2 - 4u^2|\delta|^2}}{2}\right) \\
&\geq \log\left(\frac{u^2+\varepsilon^2+|\delta|^2+\sqrt{(u^2+\varepsilon^2+|\delta|^2)^2 - 4u^2|\delta|^2 - 4\varepsilon^2|\delta|^2}}{2}\right) \\
&= \log\left(\frac{u^2+\varepsilon^2+|\delta|^2+|u^2+\varepsilon^2-|\delta|^2|}{2}\right) \\
&= \max(\log(|\pi_W x|^2 + \psi^2(\pi_{W^\perp} x), \log(|\delta|^2)),
\end{aligned}$$

and the assertion easily follows. ■

Lemma 6.16. Let S be the circle of radius $|\delta|$ on W; then as $|\delta| \to 0$,

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0} \Delta^\delta \mathcal{U}\, dt\, d\delta < -K|\delta|\sqrt{-\log|\delta|}, \qquad K > 0. \tag{6.11}$$

□

Proof. Let $S^{|\delta|}$ be the circle of radius $|\delta|$ on W, we apply Fubini–Tonelli's theorem and we argue as in the proof of Lemma 6.15 to have

$$\begin{aligned}
\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0} \Delta^\delta \mathcal{U}(t) dt\, d\delta &= \int_0^{\delta_0} \frac{1}{2\pi|\delta|}\int_{S^{|\delta|}} \tilde{U}(\bar{x}+v^\delta) - \tilde{U}(\bar{x})\, d\delta dt \\
&= \frac{M^*}{2}\int_0^{\delta_0} \{-\max[\log(|\pi_W\bar{x}|^2 + \psi^2(\pi_{W^\perp}\bar{x})), \log|v^\delta|^2] \\
&\quad + \log(|\pi_W\bar{x}|^2 + \psi^2(\pi_{W^\perp}\bar{x}))\} dt,
\end{aligned}$$

where $M^* = \max_t |M(t)|$. We then straightforwardly deduce that, for every $S^{|\delta|} \subset W$,

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0} \Delta^\delta \mathcal{U}(t)dt\, d\delta < 0.$$

In order to estimate more precisely this quantity, we observe that

$$\int_0^{\delta_0} \frac{1}{2\pi|v^\delta|}\int_{S^{|v^\delta|}} \tilde{U}(\bar{x}+v^\delta) - \tilde{U}(\bar{x})\, d\delta dt \le \int_A \log\frac{|\pi_W\bar{x}|^2 + \psi^2(\pi_{W^\perp}\bar{x})}{|\delta|^2}\, dt, \tag{6.12}$$

where

$$A := \{t \in [0, \delta_0 - |\delta|] : |\pi_W\bar{x}|^2 + \psi^2(\pi_{W^\perp}\bar{x}) < |\delta|^2\}.$$

Furthermore, there exists a strictly positive constant C such that

$$Cr^2 < |\pi_W x|^2 + \psi^2(\pi_{W^\perp}x) < C^{-1}r^2,$$

where, as usual, we denote $r^2 = |\pi_W x|^2 + |\pi_{W^\perp}x|^2$ the radius of x. The left inequality follows from Theorem 4.27. Indeed, the existence of a finite limit of $\tilde{U}(t, s(t))$ prevents the projection $|\pi_W x|^2$ and the function $\psi^2(\pi_{W^\perp}x)$ to be both infinitesimal with r^2. The right inequality follows from the continuity of ψ. From equation (6.12) and the asymptotic estimates of Theorem 4.27, we conclude that as $|\delta| \to 0$,

$$\begin{aligned}
\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0} \Delta^\delta \mathcal{U}(t)dt\, d\delta &\le \int_{t:r(t)<|\delta|/\sqrt{C}} \log\frac{r^2(t)}{C|\delta|^2}\, dt \\
&\sim \int_0^{|\delta|/\sqrt{C}} 2\frac{\log(r/\sqrt{C}|\delta|)}{-\sqrt{-\log r}} dr \\
&< -2\int_0^{|\delta|/\sqrt{C}} \sqrt{-\log r}dr < -K|\delta|\sqrt{-\log|\delta|}
\end{aligned}$$

for some positive K, since $-\sqrt{-\log r}$ is an increasing function on the interval $[0, |\delta|]$. ■

End of the Proof of Theorem 6.1. Let $S^{|\delta|}$ be a circle in W with radius $|\delta|$ and $\Delta^\delta\mathcal{A}$ the variation of the action functional defined in equation (6.7), then from Lemmata 6.13 and 6.16, we conclude that as $|\delta|$ tends to 0,

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}} \Delta^\delta \mathcal{A} d\delta \le O(|\delta|) - K|\delta|\sqrt{-\log|\delta|} < 0.$$

■

Of course, similar to Theorem 6.11, there holds the following theorem.

Theorem 6.17. In addition to (U0), (U1), $(U2)_l$, $(U3)_l$, $(U4)_l$, (U5), assume $\tilde{U}$ be of the form

$$\tilde{U}(x) = -\sum_{\nu=1}^{N} K_\nu \log(\operatorname{dist}(x, V_\nu)),$$

where K_ν are positive constants and V_ν is a family of linear subspaces, with $\operatorname{codim}(V_\nu) \geq 2$, for every $\nu = 1, \ldots, N$. Then locally minimizing trajectories do not have collisions at the time t^*. □

6.3 Neumann boundary conditions and G-equivariant minimizers

As a final comment of this section, we remark that, in our framework, the analysis allows to prove that minimizers to the fixed-ends (Bolza) problems are free of collisions: indeed, all the variations of our class have compact support. However, other type of boundary conditions (generalized Neumann) can be treated in the same way. Indeed, consider a trajectory which is a (local) minimizer of the action among all paths satisfying the boundary conditions

$$x(0) \in X^0 \qquad x(T) \in X^1,$$

where X^0 and X^1 are two given linear subspaces of the configuration space. Consider a (locally) minimizing path $\bar{x}$: of course it does not have interior collisions. In order to exclude boundary collisions, we have to be sure that the class of variations preserve the boundary condition; this can be achieved by restricting to X^i the points δ appearing in the standard variations. Hence, to complete the averaging argument, one needs assumptions (U6), and $(U7)_h$ or $(U7)_l$ to be fulfiled also by the restriction of the potential to the boundary subspaces X^i.

The analysis of boundary conditions was a key point in the paper [26], were symmetric periodic trajectories were constructed by reflections about given subspaces. By Theorems 6.1 and 6.12 one can obtain the absence of collisions also for G-equivariant (local) minimizers, provided the group G satisfies the *Rotating Circle Property* introduced in [26] (see Example 7.6). Hence, existence of G-equivariant collisionless periodic solutions can be proved for the wide class of symmetry groups described in [4, 25, 26], for a much larger class of interacting potentials, including quasi-homogeneous and logarithmic ones. On the other hand, Theorems 6.11 and 6.17 can be applied to prove that G-equivariant minimizers are collisionless for many relevant symmetry groups violating the rotating circle property, such as the groups of rotations in [24]; indeed, the idea of averaging on spheres having maximal dimension has been borrowed from that paper (cf. Example 7.7).

7 Examples and Further Remarks

We now discuss various examples of classes of potentials which fullfil our assumptions.

Example 7.1 (Homogeneous isotropic potentials). The simplest example of a function satisfying all our assumptions (U0), (U1), $(\mathrm{U2})_{\mathrm{h}}$, $(\mathrm{U3})_{\mathrm{h}}$, $(\mathrm{U4})_{\mathrm{h}}$, (U5), (U6),, $(\mathrm{U7})_{\mathrm{h}}$, and $(\mathrm{U8})_{\mathrm{h}}$ is the α-homogeneous one-center problem

$$U_\alpha(x) = \frac{1}{|x|^\alpha},$$

and its associated n-body problem

$$U_\alpha(x) = \sum_{\substack{i<j\\ i,j=1}}^{n} \frac{m_i m_j}{|x_i - x_j|^\alpha}.$$

Assumptions (U0) and (U1) are trivially satisfied, since U is positive, diverges to $+\infty$ when x approaches $\Delta = \{x \in \mathbb{R}^{nd} : x_i = x_j \text{ for some } i \neq j\}$, and does not depend on time. Furthermore, in both (U2) and $(\mathrm{U2})_{\mathrm{h}}$ the equality is achieved with $\tilde{\alpha} = \alpha$ and $C_2 = 0$. Since U is homogeneous of degree $-\alpha$, in $(\mathrm{U3})_{\mathrm{h}}$ and $(\mathrm{U4})_{\mathrm{h}}$ the function $\tilde{U}$ coincides with U. (U5) and (U6), are trivially satisfied, while $(\mathrm{U7})_{\mathrm{h}}$ and $(\mathrm{U8})_{\mathrm{h}}$ hold by virtue of Proposition 6.3. □

Example 7.2 (Logarithmic potentials). Our results also apply to logarithmic singularities of type

$$U_{\log}(x) = \sum_{\substack{i<j\\ i,j=1}}^{n} m_i m_j \log \frac{1}{|x_i - x_j|};$$

indeed, (U2) is in this case satisfied for every value of $\tilde{\alpha}$ and $(\mathrm{U2})_{\mathrm{l}}$, $(\mathrm{U3})_{\mathrm{l}}$, and $(\mathrm{U4})_{\mathrm{l}}$ are verified with $C_2 = 0$.

Dynamical systems of type (2.2) with logarithmic interactions arise in the study of vortex flows in fluid mechanics, and, precisely, in the analysis of systems of n almost-parallel vortex filaments, under a linearized version of the LIA self-interaction assumption (see [29, 30]). □

Example 7.3 (Anisotropic n-body potentials). Consider the potentials having the form

$$U(t,x) = \sum_{\substack{i>j \\ i,j=1}}^{n} U_{i,j}(t, x_i - x_j),$$

where the interaction potentials $U_{i,j}$ have a singularity at zero, of homogeneous or logarithmic type, but do depend on the angle. Typical examples are the Gutzwiller potentials [28]. Notice that the total potential satisfies assumptions (U0), (U1), and $(\mathrm{U2})_{\mathrm{h}}$, $(\mathrm{U3})_{\mathrm{h}}$, $(\mathrm{U4})_{\mathrm{h}}$ (or $(\mathrm{U2})_{\mathrm{l}}$, $(\mathrm{U3})_{\mathrm{l}}$, $(\mathrm{U4})_{\mathrm{l}}$) provided each of the $U_{i,j}$'s does. It not difficult to see that also equation (5.1) and (U5) hold (in the n-body case), while (U6),, $(\mathrm{U7})_{\mathrm{h}}$, and $(\mathrm{U8})_{\mathrm{h}}$ or $(\mathrm{U7})_{\mathrm{l}}$ do not. Hence, we cannot exclude the presence of collisions for locally minimizing paths, though the results about isolatedness and the asymptotic estimates are still available. More generally, we can deal with potentials of the form

$$U_\alpha(rs) = r^{-\alpha}\tilde{U}(s),$$

where $\tilde{U} : \mathcal{E} \setminus \Delta \to \mathbb{R}$ is positive and admits an arbitrary singular set on the ellipsoid $\mathcal{E} = \{I = 1\}$, provided

$$\lim_{s \to \Delta} \tilde{U}(s) = +\infty.$$

It is worthwhile noticing that as a consequence of Theorem 4.18, a total collision trajectory will not interact, definitively, with the singularities of $\tilde{U}$. □

The class of potentials satisfying our assumptions is clearly stable with respect to the addition of arbitrary perturbations of class $\mathcal{C}^1$. Therefore, we are mainly interested in the analysis of those perturbations that are singular themselves.

Example 7.4 (N-body potentials with time-varying masses). Although the potentials in the previous examples do not depend on time, our assumptions allow an effective time-dependence of the potentials. For instance, we can choose positive and bounded $\mathcal{C}^1$ functions $m_i(t)$, $i = 1, \dots, n$.

Obviously, the simplest example is the class of α-homogeneous n-body problem

$$U_\alpha(t,x) = \sum_{\substack{i>j \\ i,j=1}}^{n} \frac{m_i(t)m_j(t)}{|x_i - x_j|^\alpha}, \qquad 0 < \alpha < 2.$$

Assumptions (U0) and (U1) are trivially satisfied, since U is positive, diverges to $+\infty$ when x approaches $\Delta = \{x \in \mathbb{R}^{nd} : x_i = x_j \text{ for some } i \neq j\}$, and does not depend on time. Furthermore, in both (U2) and $(\mathrm{U2})_{\mathrm{h}}$ the equality is achieved with $\tilde{\alpha} = \alpha$ and $C_2 = 0$. Since U is homogeneous of degree $-\alpha$, in $(\mathrm{U3})_{\mathrm{h}}$ and $(\mathrm{U4})_{\mathrm{h}}$ the function $\tilde{U}$ coincides with U. □

Example 7.5 (Quasi-homogeneous potentials). We can also handle homogeneous perturbations of degree $-\beta$ of the potential U_α,

$$U(x) = U_\alpha(x) + \lambda U_\beta(x) 0 < \beta < \alpha < 2.$$

Indeed, when $\lambda > 0$ the condition $(\mathrm{U2})_{\mathrm{h}}$ is verified (with the strict inequality) with $\gamma = C_2 = 0$, while, when $\lambda < 0$, then (U2) holds, when $|x|$ is sufficiently small, with $C_2 = \alpha - \beta$ and $0 < \gamma < \alpha - \beta$.

As pointed out in [18] (where the case $\beta = 1$ and $\alpha > 1$ was treated), quasi-homogeneous potentials generalize classical potentials, such as Newton, Coulomb, Birkhoff, Manev, and many others. Therefore, the range of physical applications of quasi-homogeneous potentials spans from celestial mechanics and atomic physics to chemistry and crystallography. It is worthwhile noticing that the collision problem for quasi-homogeneous potentials exhibit an interesting and peculiar lack of regularity. Indeed, a classical framework for the study of collisions is given by the McGehee coordinates [35] (here and below we assume, for simplicity of notations, all the masses be equal to 1),

$$\begin{aligned} r &= |x| = I^{1/2} \\ s &= \frac{x}{r} \\ v &= r^{\alpha/2}(y \cdot s) \\ u &= r^{\alpha/2}(y - (y \cdot s)s). \end{aligned}$$

After a reparametrization of the time variable (see (4.28)),

$$d\tau = r^{-1-\alpha/2} dt, \tag{7.1}$$

the equations of motions become (here $'$ denotes differentiation with respect to the new time variable τ)

$$\begin{aligned} r' &= rv, \\ v' &= \frac{\alpha}{2} v^2 + |u|^2 - r^{\alpha-\beta} \lambda U_\beta(s) - \alpha U_\alpha(s), \end{aligned}$$

$$
\begin{aligned}
s' &= u, \\
u' &= \left(\frac{\alpha}{2} - 1\right) vu - |u|^2 s + r^{\alpha-\beta}\lambda(U_\beta(s)s - \nabla U_\beta(s)) + \alpha U_\alpha(s)s + \nabla U_\alpha(s).
\end{aligned}
$$

The field depends on r in a nonsmooth manner, unless $\alpha - \beta \geq 1$ (this last condition was indeed assumed in [18]). Hence, the flow cannot be continuously extended to the total collision manifold $C = \{(r, s, v, u) \,:\, r = 0,\ \frac{1}{2}(|u|^2 + v^2) - 2U_\alpha = 0\}$. Another peculiar feature of this system is that the monotonicity of the variable v cannot be ensured close to the collision manifold. As a consequence, the usual analysis of collision and near collision motions cannot be extended to this case. □

Example 7.6 (N-body potential reduced by a symmetry group satisfying the rotating circle property). The paper [50] deals with minimal trajectories to the spatial $2N$-body problem under the hip-hop symmetry, where the configuration is constrained at all time to form a regular antiprism. This problem has three degrees of freedom, and the reduced potential of a configuration generated by the point of coordinates $(u, \zeta) \in \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$ decomposes as

$$
U(u, \zeta) = \frac{K(N)}{|u|^\alpha} + U_0(u, \zeta),
$$

where

$$
\begin{aligned}
K(N) &= \sum_{k=1}^{N-1} \frac{1}{\sin^\alpha\left(\frac{k\pi}{N}\right)}, \\
U_0(u, \zeta) &= \sum_{k=1}^{N} \frac{1}{\left(\sin^2\left(\frac{(2k-1)\pi}{2N}\right)|u|^2 + \zeta^2\right)^{\frac{\alpha}{2}}}.
\end{aligned}
$$

The first term comes from the interaction among points of the same N-agon and is singular at simultaneous partial collisions on the ζ-axis. The second term, $U_0(u, \zeta)$, comes from the interaction between the the upper and lower N-agons and is singular only at the origin. One easily verifies that all the assumptions are satisfied, including, again by Proposition 6.3, (U6), $(U7)_h$, and $(U8)_h$. □

Example 7.7 (N-body potential reduced by a symmetry group not satisfying the rotating circle property). Consider the symmetry groups generated by rotations introduced in [24], such as the icosahedral group of order 60 of Example 7.1 in [24]: the configuration is, at all time, an orbit of a group Y of rotations about given lines in the

3-dimensional space. When Y is a finite group, the reduced potential takes the form required in Theorem 6.11 and minimizers can be shown to be free of collision. $\square$

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