follow that $\mathfrak{M}$ is the envelope of holomorphy of some domain $\Omega_0 \subset \Omega$? Of course, this problem is nontrivial only when $m > 1$.

A final question concerns the bound given in the Main Theorem for the multiplicity of the map Φ. Clearly, we could improve this bound somewhat by paying closer attention to the geometric details of our constructions, but it is not clear that any striking improvement is possible. Conceivably the bound could be lowered to Cm, with C a constant independent of the dimension of the manifold.

PRINCETON UNIVERSITY.
UNIVERSITY OF WASHINGTON.

REFERENCES.

[1] T. Carleman, "Sur un théorème de Weierstrass," *Ark. Mat. Astron. Fys.* **20B**, No. 4 (1927).
[2] R. Carmignani, "Envelopes of holomorphy and holomorphic convexity," *Trans. Am. Math. Soc.* **179** (1973), 415–431.
[3] H. Grauert, "On Levi's problem and the embedding of real-analytic manifolds," *Ann. Math. (2)* **68** (1958), 460–472.
[4] P. A. Griffiths, "Complex analytic properties of certain Zariski open subsets of algebraic varieties," *Ann. Math. (2)* **94** (1971), 21–51.
[5] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.
[6] R. Harvey and R. O. Wells, "Holomorphic approximation and hyperfunction theory on a C^1 totally real submanifold of a complex manifold," *Math. Ann.* **197** (1972), 287–318.
[7] L. V. Hörmander, *An Introduction to Complex Analysis in Several Complex Variables*, Van Nostrand, Princeton, N.J., 1966.
[8] E. Kallin, "Polynomial convexity: the three spheres problem," in *Proceedings of the Conference on Complex Analysis, Minneapolis, 1964*, (A. Appeli, E. Calabi, and H. Röhrl, Eds.), New York, 1965.
[9] C. B. Morrey, "The analytic embedding of abstract real-analytic manifolds," *Ann. Math. (2)* **68** (1958), 159–201.
[10] K. Nagumi, *Dimension Theory*, Academic Press, New York, 1970.
[11] R. Narasimhan, "The Levi problem for complex spaces, II," *Math. Ann.* **146** (1962), 195–216.
[12] R. Narasimhan, *Analysis on Real and Complex Manifolds*, Masson and North Holland, Paris and Amsterdam, 1968.
[13] E. L. Stout, *The Theory of Uniform Algebras*, Bogden and Quigley, Tarrytown-on-Hudson, N.Y., 1971.
[14] P. Thullen, "Zur Theorie der Singularitäten der Funktionen zweier komplexen Veränderlichen. Die Regularitätshullen," *Math. Ann.* **106** (1932), 64–76.
[15] R. O. Wells, "Real-analytic subvarieties and holomorphic approximation," *Math. Ann.* **179** (1969), 130–141.
[16] H. Whitney, "Analytic coordinate systems and arcs in a manifold," *Ann. Math. (2)* **38** (1937), 809–818.

2

A MINIMIZING PROPERTY OF KEPLERIAN ORBITS.

By WILLIAM B. GORDON.

Abstract. It is shown that the periodic elliptical solutions to the Kepler problem minimize the action integral. Generalizations of this theorem are obtained for other types of conservative dynamical systems involving potentials which have infinitely deep wells.

Contents.

I. Introduction.

1A. Statement of the Main Result. In this paper it will be shown that the periodic elliptical solutions to the Kepler problem actually minimize the action integral of Hamilton. More specifically, we shall prove the following theorem.

THEOREM 1.1. *Let P be a fixed but arbitrary positive number, and let $\Sigma(P)$ be the set of all P-periodic "cycles" $x = x(t)$ in the plane $\mathbf{E}^2$ which are absolutely continuous, have L^2 derivatives defined almost everywhere, and which wind around but do not intersect the origin. (See Section 2B for more details.) Let $\mathcal{Q}$ be the action integral*

$$\mathcal{Q}(x) = \int_0^P \left\{ \tfrac{1}{2}|\dot{x}(t)|^2 + \frac{1}{|x(t)|} \right\} dt \tag{1.1}$$

Manuscript received January 15, 1975; revised December 11, 1975.
American Journal of Mathematics, Vol. 99, No. 5, pp. 961–971

corresponding to the potential $V(x) = -1/|x|$. *Then* $\mathcal{Q}|\Sigma(P)$ *actually attains its minimum value at those elliptical P-periodic solutions to the Keplerian equations of motion,*

$$\ddot{x} = -\nabla\left(\frac{-1}{|x|}\right) = -\frac{x}{|x|^3}, \tag{1.2}$$

for which P is the minimum period.

Remark. Let x be a periodic orbit with minimum period P, and let y be an orbit whose minimum period in P/n (n an integer). Then the P-periodic orbit obtained by repeating y n times has its action integral equal to $n\mathcal{Q}(y)$, and as a consequence of Eq. (2.2) below it turns out that $n\mathcal{Q}(y) = n^{2/3}\mathcal{Q}(x)$. Hence the action integral is minimized at only those P-periodic solutions for which P is the *minimum* period.

1B. Strong vs. Weak Force Systems. In a previous paper [2] we have considered the problem of minimizing action integrals of the type

$$\mathcal{Q}(x) = \int_0^P \left\{ \tfrac{1}{2}|\dot{x}(t)|^2 - V[x(t)] \right\} dt, \tag{1.3}$$

where $V = V(x)$ is a "potential" corresponding to the conservative dynamical system

$$\ddot{x} = -\nabla V(x). \tag{1.4}$$

It was shown that action integrals of the type (1.3) (and others) satisfy Condition C of Palais and Smale, and hence attain minimum values at solutions to (1.4), *provided* that the potential V satisfies a certain "strong force" (SF) condition; viz., it was assumed that the potential V has singularities at points S at which V has infinitely deep wells [$V(x) \to -\infty$ as $x \to$ S], and also that there exists a function U with infinitely deep wells at S, such that

$$-V(x) \geqslant |\nabla U(x)|^2 \qquad \text{in a neighborhood of S.} \tag{SF}$$

For action integrals of the particular type (1.3) it is also required that V be bounded above. The dimension of the configuration space may be higher than 2, but it is required that any homotopy class of cycles on which $\mathcal{Q}$ is to be minimized must be "tied" to S in the sense that no cycle in the class can be continuously moved off to infinity without either crossing S or having its arc length become infinite. (Similar results are also obtained concerning the problem of minimizing $\mathcal{Q}$ on a class of paths joining two fixed points with a given time of transit.)

The SF condition arises in the following way: Let $\sigma \to x^\sigma$, $0 \leqslant \sigma < \infty$, be a continuous family of cycles which is constructed in such a manner that $\mathcal{Q}(x^\sigma) \downarrow \inf \mathcal{Q}$, the infimum being taken over the set of cycles which are homotopic to the initial cycle x^0 (in the configuration space with the set of singularities of V removed). The SF condition guarantees that the family x^σ is *bounded away* from the set of singularities S. The functional $\mathcal{Q}$ also dominates arc length, so that if x^0 is "tied" to S, the family x^σ will remain in some compact subset of the domain on which V is regular. Hence by passing to the limit as $\sigma \to \infty$, one could expect to obtain a minimizing cycle x^∞ homotopic to x^0.

spiegaz di (SF)

utile x 1° art.

It is easily shown that the SF condition excludes the gravitational case $V(x) = -|x|^{-1}$. (Cf. [2, Section 1A]. A function U satisfying (SF) can behave no worse than $|x|^{1/2}$ in a neighborhood of the origin, contradicting the requirement that $U(x) \to \infty$ as $x \to$ S.) And for gravitational and other *weak force* systems one cannot expect a minimizing family x^σ to be bounded away from S. Hence in passing to the limit one might obtain a solution x^∞ which intersects S. We shall call such solutions *continued* solutions, to distinguish them from the *regular* solutions which lie in the domain on which V is regular. (This terminology has nothing to do with "regularizing variables.")

In general, if V is any negative potential with infinitely deep wells S, then any P-periodic cycle $x^0 = x^0(t)$ tied to S can be continuously deformed into a minimizing cycle x^∞ which satisfies (1.4) almost everywhere, but which may be a continued solution of (1.4) if the system is of the weak force type. If x^∞ is a continued solution, $\mathcal{Q}(x^\infty)$ exists as a Lebesgue integral even though the kinetic and potential energies become infinite when $x^\infty(t)$ crosses S. This proposition will be formalized in Theorem 4.1 of Part IV, where generalizations of Theorem 1.1 will be discussed.

For the Kepler problem the set of all P-periodic continued solutions can be classified according to a simple scheme, and their action integrals are easily computed. The action integrals of the regular solutions are also known, and it is easily shown that the action integral of each P-periodic continued solution is equal to or greater than those of the P-periodic regular solutions. Hence, using this special property of the Kepler problem, to establish the existence of minimizing *regular* orbits one only has to establish the existence of minimizing orbits which are possibly *continued*.

2A. The Action Integral for Continued Solutions. It is well known that a regular solution to (1.2) is periodic if and only if it has negative total energy H ($= \frac{1}{2}|\dot{x}|^2 - [1/|x|]$), and that the energy H, period P and action $\mathcal{Q}$ are functionally related. (For generalizations see [1].) In fact, for solutions with minimum period P we have

$$P=(2^{-1/2}\pi)\cdot(-H)^{-3/2}, \tag{2.1}$$

$$\mathcal{Q}=(3\pi)(2\pi)^{1/3}\cdot P^{1/3}. \tag{2.2}$$

(These relations are easily obtained by integrating the equations for *circular* motion.)

FIGURE 1

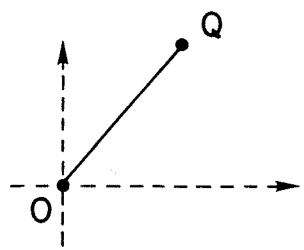
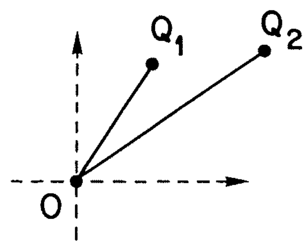


FIGURE 2



We now wish to describe the continued periodic solutions to (1.2) and compute their action integrals. The simplest type of continued periodic solution is the straight line solution indicated in Figure 1. Here a particle initially at rest at the point Q falls towards the origin, reaching it in time $t=P/2$. The particle then reverses course and moves up the line OQ until it reaches the point Q, where it again has zero velocity. We shall call such a solution a *leg of period P*. It has a total energy $H=-(\overline{OQ})^{-1}$, and by a straightforward integration of the equation $\frac{1}{2}\dot{r}^2-(1/r)=H$ (=constant) one can easily show that the relations between H, P and $\mathcal{Q}$ are again given by (2.1) and (2.2).

Figure 2 shows a continued solution which consists of two legs of periods P_1 and P_2, the total period of the entire trajectory being $P=P_1+P_2$. Starting at rest at point Q_1, the particle falls towards the origin, where it emerges at an arbitrary angle as it moves up to Q_2, where it attains zero velocity. At Q_2 the course is reversed, and the trajectory is terminated at Q_1.

More generally, a continued P-periodic solution may consist of a finite or a countably infinite number of legs of period P_i, the only requirement on the P_i being that $P=\Sigma P_i<\infty$.

LEMMA 2.1. *Among all the continued solutions of period P, those which consist of a single leg have the least action* $\mathcal{Q}$.

Proof. In a P-periodic trajectory consisting of legs with periods P_i, each leg of period P_i has an action integral equal to $CP_i^{1/3}$, where C is the same constant as occurs in (2.2). Hence, since $P=\Sigma P_i$, we have

$$\mathcal{Q}=C\Sigma P_i^{1/3}=CP^{1/3}\Sigma(P_i/P)^{1/3}\geqslant CP^{1/3}\Sigma(P_i/P)=CP^{1/3}.$$

Since the action $\mathcal{Q}$ and period P are related in the same way for regular orbits and for continued orbits consisting of a single leg, Lemma 2.1 has the following consequence.

LEMMA 2.2. *In order to prove Theorem 1.1 it is sufficient to show that there exists a possibly continued P-periodic solution* x^0 *with the property that* $\mathcal{Q}(x^0)\leqslant\mathcal{Q}(x)$ *for every x in* $\Sigma(P)$.

2B. Function Theoretic Preliminaries. In this section we shall assemble some well-known facts about Sobolev spaces which are to be used in our proofs. For references see [5, 6, 7, 9].

For C^∞ P-periodic maps $t\to x(t)$ of $\mathbf{R}$ into $\mathbf{E}^2$, let,

$$(x,y)_0=\int_0^P\langle x(t),y(t)\rangle\,dt, \tag{2.3}$$

$$(x,y)_1=(x,y)_0+(\dot{x},\dot{y})_0, \tag{2.4}$$

let $\|\cdot\|_0,\|\cdot\|_1$ be the corresponding norms, and let H^0 and H^1 be the completions of $C^\infty([O,P],\mathbf{E}^2)$ with respect to H^0 and H^1, respectively. Then H^0 is merely the ordinary space of L^2 maps, and H^1 is the Sobolev space of all absolutely continuous maps $t\to x(t)$ for which $\int_O^P|\dot{x}(t)|^2dt<\infty$. It is well known that the weak topology in the Hilbert space H^1 is stronger than the C^0 topology (so that the weak H^1 convergence of maps implies their uniform convergence).

For any functional $\mathcal{F}$ on H^1 and for every real number c, let $\mathcal{F}^c=\{x\in H^1|\mathcal{F}(x)\leqslant c\}$. Recall that $\mathcal{F}$ is lower semi-continuous in some topology if $\mathcal{F}^c$ is closed in that topology for every c, in which case $\mathcal{F}$ is bounded below and attains its infimum on every subset which is compact with respect to that topology. Hence,

LEMMA 2.3. *Let* $\mathcal{F}$ *be a functional which is defined on a subspace X of* H^1, *and suppose that for every real number c,* $\mathcal{F}^c\cap X$ *is a weakly compact subset of* H^1. *Then* $\mathcal{F}$ *is bounded below and attains its infimum on X.*

Proof. Since the weak H^1 topology is Hausdorff, $\mathcal{F}^c \cap X$ is closed in the weak H^1 topology for every c. Hence $\mathcal{F}^c \cap X$ is closed in the (relative) weak topology of X; i.e., $\mathcal{F}|X$ is lower semi-continuous on X. Fix a real number c such that $\mathcal{F}^c \cap X$ is not empty. Then $\mathcal{F}$ attains its infimum on $\mathcal{F}^c \cap X$, since $\mathcal{F}^c \cap X$ is compact in the weak H^1 topology. But the infimum of $\mathcal{F}$ on $\mathcal{F}^c \cap X$ is the infimum of $\mathcal{F}$ on X.

III. Proof of Theorem 1.1.

3A. Let $\Sigma^* = \Sigma^*(P)$ be the space of all P-periodic cycles $x = x(t)$ of class H^1 which wind around or intersect the origin, and for which $\mathcal{Q}(x)$ exists as a Lebesgue integral. We wish to show that $\mathcal{Q}|\Sigma^*$ attains its infimum. According to Lemma 2.3 it suffices to show that $\mathcal{Q}^c \cap \Sigma^*$ is a weakly compact subset of H^1 for every real c; i.e., we have to show that for every real c,

(i) $\mathcal{Q}^c \cap \Sigma^*$ is bounded in H^1 norm,
(ii) $\mathcal{Q}^c \cap \Sigma^*$ is closed in the weak topology of H^1.

Proof of (i). From the Cauchy-Schwarz inequality, we have

$$\text{arc length}(x) = \int_0^P |\dot{x}(t)|\,dt \leqslant \left[P \int_0^P |\dot{x}(t)|^2 dt \right]^{1/2}.$$

Hence, referring to (1.1), we see that the elements of $\mathcal{Q}^c$ are uniformly bounded in arc length. And since all the elements of Σ^* wind around or are attached to the origin, it follows that $\mathcal{Q}^c \cap \Sigma^*$ is bounded in C^0 norm. But $\mathcal{Q}$ also dominates the second term on the right hand side of the relation [cf. (2.3), (2.4)]

$$\|x\|_1^2 = \|x\|_0^2 + \|\dot{x}\|_0^2,$$

and the C^0 norm (squared) dominates the first term. Hence $\mathcal{Q}^c \cap \Sigma^*$ is bounded in H^1 norm.

Proof of (ii). Let $\{x_n\}$ be a sequence in $\mathcal{Q}^c \cap \Sigma^*$ which converges weakly to some x in H^1. Then $x = x(t)$ winds around or intersects the origin, since weak H^1 convergence implies C^0 convergence. We now have to show that $\mathcal{Q}(x)$ exists and that $\mathcal{Q}(x) \leqslant c$.

For each n let $f_n(t) = |x_n(t)|^{-1}$, and let $f(t) = |x(t)|^{-1}$. Each f_n is of class L^1, since $\mathcal{Q}(x_n) < \infty$. This implies that the set of all t for which $x_n(t) = 0$ has zero measure. Hence $f_n(t) \to f(t)$ almost everywhere. Also, $\int_0^P f_n(t)\,dt < \mathcal{Q}(x_n) \leqslant c$. Hence, from Fatou's lemma [3, p. 113] it follows that f is of class L^1 and that

$$\int_0^P f(t)\,dt = \int_0^P \left[\liminf f_n(t)\right] dt \leqslant \liminf \int_0^P f_n(t)\,dt. \tag{3.1}$$

Now, the weak convergence of the x_n to x in the Hilbert space H^1 implies that $\|x\|_1 \leqslant \limsup \|x_n\|_1$. Hence

$$\|x\|_0^2 + \|\dot{x}\|_0^2 \leqslant \limsup\left\{ \|x_n\|_0^2 + \|\dot{x}_n\|_0^2 \right\},$$

so that the C^0 convergence of x_n to x implies that

$$\|\dot{x}\|_0^2 \leqslant \limsup\left\{ \|\dot{x}_n\|_0^2 \right\}. \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$\mathcal{Q}(x) \leqslant \limsup \mathcal{Q}(x_n) \leqslant c.$$

This completes the proof of (ii).

3B. Let x^* be an element of Σ^* at which $\mathcal{Q}|\Sigma^*$ attains its infimum. If x^* belongs to $\Sigma = \Sigma(P)$ (defined in the statement of the theorem), i.e., if $x^* = x^*(t)$ never intersects the origin, then the standard calculus of variations argument shows that x^* is a solution to (1.2), which are the Euler-Lagrange equations for the functional $\mathcal{Q}$.

Suppose now that $x^* = x^*(t)$ intersects the origin at certain times t. According to Lemma 2.2, to complete the proof in this case it suffices to show that $x^* = x^*(t)$ is a *continued* solution to (1.2).

The set of all t for which $x^*(t) \neq 0$ is open. Let $[a,b]$ be any closed interval in this open set. [This interval is made to be closed so that $x^*(a) \neq 0$ and $x^*(b) \neq 0$.] Let $t \to v(t)$ be a smooth vector valued map from $[a,b]$ into $\mathbf{E}^2$ with $v(a) = v(b) = 0$. Now from the minimizing properties of x^* it follows that the arc $\gamma: \{x^* = x^*(t),\ a \leqslant t \leqslant b\}$ must minimize the action $\mathcal{Q}$ on the space of all paths $y = y(t)$ of class H^1 which join $x^*(a)$ to $x^*(b)$ with time of transit $T = b - a$, and which can be continuously deformed into the arc γ without crossing the origin. Hence

$$\left.\frac{d}{d\epsilon}\mathcal{Q}(x^* + \epsilon v)\right|_{\epsilon=0} = 0,$$

and the usual calculus of variations argument indicates that $x^* = x^*(t)$ satisfies the Euler-Lagrange equations (1.2) on the open interval (a,b). (Actually, by operating in the category of H^1 spaces we introduce certain complications into this classical argument. Namely, the minimizing paths have to be shown to be sufficiently regular before one can establish that they satisfy the Euler-Lagrange equations. See [2, Section 7] for a discussion of this point.) Therefore, $x^* = x^*(t)$ is a continued solution to (1.2) and the proof is complete.

IV. Generalizations

4A. Statement of Results. In this section we shall discuss generalizations to Theorem 1.1 which hold when (i) the configuration space $\mathbf{E}^2$ is replaced with a Euclidean space $\mathbf{E}^N$ of arbitrarily high dimension, (ii) the gravitational potential is replaced with an arbitrary potential which is bounded above and has infinitely deep wells, (iii) the problem of minimizing $\mathcal{Q}$ on a class of P-periodic cycles is replaced with the problem of minimizing $\mathcal{Q}$ on a class of paths which join two given points with a given time of transit.

Throughout the remainder of this section it will always be assumed that:

(1) V is a real valued function which is of class C^2 everywhere on $\mathbf{E}^N$ except on a closed non-empty set of points $\mathbf{S}$ where V has infinitely deep wells; i.e., $V(x) \to -\infty$ as $x \to \mathbf{S}$.

(2) V is bounded above. Since the behavior of a conservative dynamical system is not affected by the addition of an arbitrary constant to its potential, we might just as well assume that

$$V \leqslant 0 \qquad \text{on } \mathbf{E}^N - \mathbf{S}. \tag{4.1}$$

Two P-periodic cycles in $\mathbf{E}^N - \mathbf{S}$ will be said to be *homotopic* if they are homotopic in $\mathbf{E}^N - \mathbf{S}$, i.e., if one of them can be continuously deformed into the other without crossing $\mathbf{S}$. (Similarly, two paths joining two fixed points in $\mathbf{E}^N - \mathbf{S}$ will be said to be homotopic if they are homotopic in $\mathbf{E}^N - \mathbf{S}$.) Recall that a P-periodic cycle γ is said to be *tied to* $\mathbf{S}$ if γ cannot be continuously moved off to infinity without either crossing $\mathbf{S}$ or having its arc length become infinite. [See Section 4B(ii) below for examples.] Strictly speaking, this means that for every $c > 0$ there exists a (possibly empty) compact set K_c which contains every cycle which is homotopic to γ and has arc length $\leqslant c$.

The proofs of the theorems given below will be sketched in Section 4B.

THEOREM 4.1. *Let P be a fixed positive number, and let Σ be a homotopy class of P-periodic cycles in $\mathbf{E}^N - \mathbf{S}$ which are of class H^1 and which are tied to $\mathbf{S}$. Let Σ^* be the intersection of the weak H^1 closure of Σ with the set of cycles x of class H^1 for which $\mathcal{Q}(x)$ exists (as a Lebesgue integral). Then*

(4.1a) *There exists a cycle x^* in Σ^* such that*

$$\mathcal{Q}(x^*) = \inf\{\mathcal{Q}(x) | x \in \Sigma^*\} = \inf\{\mathcal{Q}(x) | x \in \Sigma\}.$$

(4.1b) *x^* is a (possibly continued) solution of* (1.4).

(4.1c) *If the system* (1.4) *is SF, then $\Sigma^* = \Sigma$. Hence there are no continued solutions to* (1.4), *and $\mathcal{Q}|\Sigma$ attains its infimum at some (regular) P-periodic solution to* (1.4).

Remark. The weak and strong closures of Σ in H^1 are the same. If the system is of the weak force type, then Σ^* contains cycles which are attached to $\mathbf{S}$ at one or more points.

THEOREM 4.2. *Let T be a fixed positive number, and let q_1, q_2 be two (not necessarily distinct) points in $\mathbf{E}^N - \mathbf{S}$. Let Ω be a homotopy class of paths in $\mathbf{E}^N - \mathbf{S}$ which are of class H^1 and join q_1 to q_2 with time of transit T, and let Ω^* be the intersection of the weak (= strong) closure of Ω in H^1 with the set of all paths x for which $\mathcal{Q}(x)$ exists. Then:*

(4.2a) *There exists a path x^* in Ω^* such that*

$$\mathcal{Q}(x^*) = \inf\{\mathcal{Q}(x) | x \in \Omega^*\} = \inf\{\mathcal{Q}(x) | x \in \Omega\}.$$

(4.2b) *x^* is a (possibly continued) solution of* (1.4).

(4.2c) *If the system* (1.4) *is SF, then $\Omega^* = \Omega$, and the minimizing paths x^* are regular solutions to* (1.4).

THEOREM 4.3. *Let Σ^* and Ω^* be as above. Then the functionals $\mathcal{Q}|\Sigma^*$ and $\mathcal{Q}|\Omega^*$ satisfy Condition C of Palais and Smale.*

4B. Discussion and Examples.

(i) To review the main features of our proof of Theorem 1.1: In Section 3A the boundedness condition (4.1) on V ensures that $\mathcal{Q}$ dominates arc length. Hence, for planar systems, $\mathcal{Q}$ dominates the C^0 norm on any family of P-periodic cycles which wind around or are attached to the origin. Moreover, the kinetic energy occurs in the integrand of $\mathcal{Q}$, so that $\mathcal{Q}$ also dominates the H^1 norm for such a family. Obviously, this property of $\mathcal{Q}$ applies as well to the spaces Σ^* and Ω^*. The remainder of the argument in Section 3A, which establishes the existence of a minimizing cycle, merely uses generalities about Hilbert spaces and Fatou's lemma. Hence this argument, with very slight modifications, also establishes the truth of (4.1a) and (4.2a).

The proofs of (4.1b) and (4.2b) are pretty much standard, except that, as mentioned in Section 3B, the proof of (4.2b) requires certain regularity results which are discussed in [2, Section 7].

The proofs of (4.1c) and (4.2c) are given in [2], which is almost exclusively devoted to SF systems.

For the Kepler problem we were able to assert the existence of *regular* minimizing P-periodic cycles because of Lemma 2.2, which in turn depended on our being able to show that, in this case, the action of every P-periodic continued solution is equal to or greater than the action of some regular solution. At present we do not know whether or not this property of the Kepler system (1.2) extends to weak force systems in general. In fact, even for the Kepler system we do not know what conditions (if any) might be required on two points q_1, q_2 to guarantee the existence of a regular minimizing path joining these points.

(ii) We shall now give some examples of "cycles tied to S", and show how Theorem 4.2 applies to planar n-body gravitational systems. For further details see [2].

Example 1. (a) For $\mathbf{E}^N = \mathbf{E}^2$ and $\mathrm{S} = \{\text{origin}\}$, a cycle in $\mathbf{E}^2 - \mathrm{S}$ is tied to S if and only if it winds around the origin. (b) For $\mathbf{E}^N = \mathbf{E}^3$ and S = a straight line, no cycle can be tied to S. (Any cycle which winds around the line can "slip up" the line and move off to infinity without changing its arc length.) (c) Let $\mathbf{E}^N = \mathbf{E}^3$, and let S be two intersecting lines. Then a cycle is tied to S if and only if it winds around *both* of the lines.

Example 2. Let $\mathbf{E}^N = \mathbf{E}^3$, and let S be three straight lines which intersect at the origin. As in Example 1(c), a cycle is tied to S if and only if it winds around *at least two* of the lines. Although this example may appear to have little physical interest, it does provide a good pictorial representation for the planar 3-body problem (Example 3 below).

Example 3. Consider a planar 3-body problem, i.e., a system of three mutually attracting particles which are constrained to move in a plane $\mathbf{E}^2$. We assume that $\mathrm{V} = -\infty$ at collisions. The dimension of the configuration space can be reduced from 6 to 4 by fixing the centroid of the system at the origin. The set of singularities (collisions) S can then be identified with the union of three 2-planes in $\mathbf{E}^4$, any two of which intersect at precisely one point (the origin). A cycle is tied to S if and only if it winds around at least two of these planes. From the pictorial representation of this system provided by Example 2 it is easy to "see" that there is a countability infinite number of homotopy classes of cycles tied to S. Hence, if the forces are *strong*, Theorem 4.1(c) guarantees the existence of an infinite number of regular P-periodic solutions-—one for each such homotopy class. On the other hand, if the system is gravitational or some other kind of weak force system, we must allow the possibility that each minimizing cycle is *continued*, and in passing to the limit $\Sigma \to \Sigma^*$ all the homotopy classes collapse to a single class of cycles which wind around or are attached to S. In other words, for gravitational and other weak force systems, Theorem 4.1 only guarantees the existence of *one* minimizing P-periodic solution, possibly continued, for each P.

Similar remarks apply to the existence of minimizing periodic solutions to other *planar* n-body systems. The condition that the system is planar is required

to ensure that the collision hyperplanes have codimension 2, which is necessary for the existence of cycles tied to S.

(iii) *Theorem 4.3.* Condition C of Palais and Smale permits the application of the Morse and Lusternik-Schnirelman theories to infinite dimensional manifolds. Hence Theorem 4.3 is actually a stronger result than Theorems 4.1 and 4.2, since positive functionals satisfying Condition C are known to attain their infima [6, 8]. A proof of Theorem 4.3, which will not be given here, can be obtained by augmenting the arguments in [2] (which hold for SF systems) with some of the results in Section 3A above.

Remark. As noted in [2], for gravitational systems $\mathcal{Q}$ does *not* satisfy Condition C on Σ and Ω, which in this case are *open* submanifolds of H^1. On the other hand, if the system is SF, then $\Sigma = \Sigma^*$ and $\Omega = \Omega^*$.

NAVAL RESEARCH LABORATORY, WASHINGTON, D.C.

REFERENCES.

[1] W. B. Gordon, "On the relation between period and energy in periodic dynamical systems," *J. Math. and Mech.* **19** (1969), pp. 111–114.

[2] ———, "Conservative dynamical systems involving strong forces, *Trans. Am. Math. Soc.*," **204** (1975), pp. 113–135.

[3] P. R. Halmos, *Measure Theory*, Van Nostrand, Princeton, N.J., 1950.

[4] R. S. Palais, "Morse theory on Hilbert manifolds," *Topology* **2** (1963), pp. 299–340.

[5] ———, *Foundations of Non-Linear Global Analysis*, Benjamin, New York, 1968.

[6] ———, "Critical point theory and the minimax principle," *Proc. Symp. Pure Math.* **15** (Am. Math. Soc., Providence, R.I., 1970), pp. 185–212.

[7] R. S. Palais et al., *Seminar on the Atiyah-Singer Index Theorem*, Ann. of Math. Stud. No. 57, Princeton U.P., Princeton, N.J., 1965.

[8] J. T. Schwartz, "The Lusternik-Schnirelman theory of critical points," *Commun. Pure and Appl. Math.* **17** (1964), pp. 307–315.

[9] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Scott, Foresman, Glenview, Ill., 1971.