Sia X un insieme

Un gruppo abeliano <u>libero</u> generato da X è un gruppo di forma

$$\left\{ \sum_i^{<\infty} x_i - \sum_j^{<\infty} y_j \;\middle|\; \begin{array}{c} \forall x_i, y_j \in X \\ \forall i, j \end{array} \right\}$$

dove $\sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j = 0 \Rightarrow \begin{array}{l} n = m, \exists \sigma \in S_n \\ \text{t.c. } x_i = y_{\sigma(i)} \\ \forall i = 1, \dots, n \end{array}$

Sia A un anello, M un A-mod destro e N un A-mod. sinistro

$L :=$ gruppo abeliano libero generato da $M \times N$.

$$K = \left\langle \begin{array}{ll} (x_1 + x_2, y) - (x_1, y) - (x_2, y), & \forall x_1, x_2 \in M, \\ & y \in N. \\ (x, y_1 + y_2) - (x, y_1) - (x, y_2), & y_1, y_2 \in N \\ (xa, y) - (x, ay) & x \in M \\ & \forall a \in A \end{array} \right\rangle$$

$$M \otimes_A N := L/K \xleftarrow{\pi} L$$

$$x \otimes y =: \pi(x,y) \longleftarrow\!\!| \ (x,y)$$

$$M \otimes_A N = \left\{ \sum_i^{<\infty} x_i \otimes y_i \ \middle| \ \begin{array}{l} \forall x_i \in M \\ \forall y_i \in N \end{array} \right\}.$$

Prop: $\forall x_1, x_2, x \in M$, $y_1, y_2, y \in N$, $\forall a \in A$, :

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y.$$

$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$

$$x \cdot a \otimes y = x \otimes ay.$$

(ex).

e.g.: M un A-mod sinistro

$$A \otimes_A M \cong M.$$

Siano A e B anelli.

Sia M un A-mod sinistro e B-mod destro

Sia N un B-mod sinistro.

$M \otimes_B N$ è un A-mod Sinistro

definito:

$\forall a \in A,\ x \in M,\ y \in N,$

coeff "$\cdot$": $a \cdot (x \otimes y) = ax \otimes y.$

Sia A un anello Comm.

N, M_1 e M_2 due A-moduli

Una funz: $f : M_1 \times M_2 \longrightarrow N$ è

<u>bilineare</u>:

$\forall\ x_1, x_2, x \in M_1, \quad \forall\ a \in A:$

$y_1, y_2, y \in M_2,$

$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y),$

$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$

$f(ax, y) = a f(x, y).$

$f(x, ay) = a f(x, y).$

Def: Siano M_1 e M_2 A-mod.

Un A-mod T è bilinearmente universale per M_1 e M_2:

Se 1): $\exists\ \tau : M_1 \times M_2 \to T$ funz. bilin.

2) se N un A-mod, e se $\exists\ \varphi : M_1 \times M_2 \to N$ bilin.

allora $\exists!\ f: T \to N$ morf. di A-mod. tale che:

$$\begin{tikzcd} M_1 \times M_2 \arrow[r, "\tau"] \arrow[dr, "\varphi"'] & T \arrow[d, dashed, "\exists!\, f"] \\ & N \end{tikzcd}$$

Prop: ~~Dati~~ A-mod. M_1 e M_2 allora tale T esiste, ed è unico a meno isomorf. che isomorfo a.

$M_1 \otimes_A M_2 \cong T$. $\boxed{\text{III}}$.

e.g. V uno spaz. vett. a coeff. in F.

$$V \otimes_F V \cong B(V) := \{\text{funzion. bilin. su } V\}$$

$B(V)$, "$+$"

$\forall f, g \in B(V)$, $f+g : V \times V \to F$
$(u,v) \mapsto f(u,v) + g(u,v)$

coeff. "$\cdot$" $\forall a \in F$,
$\forall v, u \in V$

$$(a \cdot f)(u,v) = a f(u,v).$$

<u>eser</u>: Dimostrazione dell'isomorfismo per $\dim_F V = n$.

$V \cong \mathrm{Hom}_F(V,F) = V^*$

$$\begin{tikzcd} V \otimes_F V \arrow[d, "\wr"] \arrow[dr, dashed] & \\ V^* \otimes_F V^* \arrow[r, "\sigma"] & B(V) \end{tikzcd}$$

$f \otimes g \longmapsto \sigma(f \otimes g) : (x,y) \to f(x) \cdot g(y)$, $V \times V \to F$

$V/\mathbb{R}$ $\quad (-,-)$ prod. scalare :

$\forall u, v \in V, \quad (u,v) = \|u\| \, \|v\| \cdot \cos\theta$.

θ = angolo tra u e v.

$$\|u\| = \sqrt{(u,u)}$$

$$\|v\| = \sqrt{(v,v)}$$

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e.g. F campo,

$$M_n(F) \times M_n(F) \to F$$
$$(X, Y) \mapsto tr(X \cdot Y)$$

è funz. bilin.

e.g: V spaz. vett $/F$.

$\rho_1, \rho_2 \in V^*$,

$$V \times V \to F$$

$\forall u, v \in V$, $\quad (u, v) \mapsto \rho_1(u) \cdot \rho_2(v)$.

<u>osser</u>. $\exists$ ~~forma~~ bilin. $B \in \mathcal{B}(V)$.

t.c. $B(u,v) \neq B(v,u)$

Se per $u, v \in V$

$$B(u,v)=0 \not\Rightarrow B(v,u)=0$$

Sia V spaz. vett. $/F$, $\dim_F V = n$

Sia $B : V \times V \to F$ f. bilin.

$$\{v_1, \dots, v_n\} \underset{\text{base}}{\subset} V$$

$$V \ni v = \sum_{i=1}^{n} a_i v_i \,, \quad u = \sum_{j=1}^{n} b_j v_j \,, \quad a_i, b_j \in F.$$

$$B(v,u) = B\left(\sum_{i=1}^{n} a_i v_i \,, \sum_{j=1}^{n} b_j v_j\right).$$

$$= \sum_{1 \le i,j \le n} a_i b_j \, B(v_i; v_j).$$

$$= (a_1 \cdots a_n) \begin{pmatrix} B(v_i, v_j) \end{pmatrix}_{n \times n} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} B(v_i, v_j) \end{pmatrix}_{n \times n} \in M_n(F)$$

matrice di B risp. alla base $\{v_1 \dots v_n\}$

Prop: $\exists$ corrisp. 1-1 tra:

$$\{\text{forme bilin. su } V\} \overset{1-1}{\longleftrightarrow} M_n(F).$$

Dim: ~~Dil~~ eser: la funz.

$$\sigma : \mathcal{B}(V) \longrightarrow M_n(F)$$

fissando una base $\{v_1 \dots v_n\}$. $B \longmapsto (B(v_i, v_j))$.

trovate σ^{-1}. $\boxed{///}$.

Def: Dato V spaz. vett. $/F$. $\dim_F V = n$. una forma bilin. $B \in \mathcal{B}(V)$ è

__simmetrica__ se

$\forall u, v \in V$, $B(u, v) = B(v, u)$;

è __antisimmetrica__ (alternate) se

$$B(u, v) = -B(v, u)$$

dove $\operatorname{char} F \neq 2$.

$B(V)_s = \{$forme bilin. simm. su $V\}$.

$B(V)_a = \{$ - - - - - - - antisimm. - - - $\}$,

Prop: $\exists$ corrisp. 1-1 tra.

$$B(V)_s \xleftrightarrow{1-1} \{X \in M_n(F) \mid X^t = X\}$$

$$B(V)_a \xleftrightarrow{1-1} \{ - - - - - \mid X^t = -X\}.$$

$B(V)_s$ e $B(V)_a$ sono sottospazi di $B(V)$.

Prop: $B(V) = B(V)_a \oplus B(V)_s$.

Dim: Per $X \in M_n(F)$, char$F \neq 2$

$$X = \underbrace{\frac{X+X^t}{2}}_{\in B(V)_s} + \underbrace{\frac{X-X^t}{2}}_{\in B(V)_a}.$$

$\Rightarrow M_n(F) = \{$Matrici simm$\} + \{$Matr. antisimm$\}$

$$\{\text{Matr. simm}\} \cap \{\text{Mat. antisimm}\} = \{0\}$$

$$\Rightarrow M_n(F) = \{\text{Matrici simm.}\} \oplus \{\text{Matr. antisimm}\}$$

$$\begin{array}{ccccc} \updownarrow & & \updownarrow & & \updownarrow \\ B(V) & = & B(V)_s & \oplus & B(V)_a \end{array}$$

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