Sia G un gp. F campo

Una funz. di classe su G

è

$$f : G \longrightarrow F$$

t.c.

$\forall g, h \in G$,

$f(g) = f(h) \Leftarrow g$ e h

sono coniugati

$xgx^{-1} = h$, per

q.c. $x \in G$.

$X(G) = \{$funz. di classi di $G\}$.

$\exists$ "+" e "coeff. in F" in $X(G)$

$\forall a \in F$, $f \in X(G)$, $g \in G$.

$$(a \cdot f)(g) = a \cdot f(g)$$

$\Rightarrow X(G)$ è uno spaz. vett / F.

$$\dim_F X(G) \underset{\text{eser}}{=} |\{\text{classi coniugati di } G\}|$$

Def: Sia $\rho: G \to GL(V)$ una rapp. di gp. G, V/F, $\dim_F V < \infty$.

La funz. traccia:

$$tr: G \longrightarrow F$$
$$\forall g \in G, \quad g \longmapsto tr(\rho(g))$$

Si chiama un carattere di G ass. a ρ,

$$\chi_\rho: G \to F$$
$$g \mapsto \chi_\rho(g) = tr(\rho(g)).$$

*: $X, Y \in M_n(F)$, $tr(XY) = tr(Y \cdot X)$

$\forall T \in M_n(F)$, $X \in GL_n(F)$.

$$tr(X\underbrace{TX^{-1}}) = tr(\underbrace{TX^{-1}}X) = tr(T)$$

$\chi_\rho \in X(G)$:

Prop: 1) Siano $\rho : G \to GL(V)$

$\rho' : G \to GL(W)$

$V, W / F$

Se ρ e ρ' sono equiv.

$\Rightarrow \quad \chi_\rho = \chi_{\rho'}$

Dim: $\rho \approx \rho' \Leftrightarrow \exists\, V \cong W$, t.c.

$$\forall g \in G, \quad \begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \sigma \downarrow & & \downarrow \sigma \\ W & \xrightarrow{\rho'(g)} & W \end{array}$$

$\Leftrightarrow \rho(g) = \sigma^{-1} \rho'(g) \sigma$

$\chi_\rho(g) = tr(\rho(g)) = tr(\sigma^{-1} \rho'(g) \sigma)$

$= tr(\rho'(g)) = \chi_{\rho'}(g)$

Quest. $\chi_\rho = \chi_{\rho'} \overset{?}{\Rightarrow} \rho \approx \rho'$ (d)

2) $\chi_\rho \in X(G)$.

3) $\operatorname{char} F = 0 \Rightarrow \chi_\rho(1_G) = \underset{\substack{\| \\ \text{grado di } \rho}}{\dim_F V}$

4) $\rho \oplus \rho' : G \to GL(V \oplus W) : g \mapsto \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho'(g) \end{pmatrix}$

$\forall g \in G, \ \chi_{\rho \oplus \rho'}(g) = \chi_\rho(g) + \chi_{\rho'}(g)$

5). Sia V un $F[G]$-mod., i.e.

$\exists \rho : G \to GL(V)$ in corrisp.

$\chi_V := \chi_\rho$

Sia $U \subseteq V$ un ~~sotto~~mod. di V.

$\Rightarrow V/U$ è $F[G]$-mod.

$\forall g \in G$,

$\chi_V(g) = \chi_U(g) + \chi_{V/U}(g)$

Dim: Sia $\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$

una base di V, dove

$\{v_1, \dots, v_r\}$ base di U.

risp. a quelle:

$$\forall g \in G \,.\, GL_n(F) \ni \rho(g) = \begin{array}{cc} \begin{pmatrix} (a_{ij}) & * \\ 0 & (b_{kl}) \end{pmatrix} & \begin{array}{l} \}\, r \\ \}\, n-r \times n-r \end{array} \end{array}$$

dove $(a_{ij})_{r\times r} = \rho(g)\big|_U$

$$(b_{kl}) = \rho(g)\big|_{V/U}$$

$$\Rightarrow \chi_\rho(g) = \chi_U(g) + \chi_{V/U}(g) \qquad (II).$$

6): $\rho: G \longrightarrow GL_n(F)$ una rapp.

$\rho^*: G \longrightarrow GL_n(F)$.

$$\forall g \in G, \quad g \mapsto \rho^*(g) := \left[\rho(g)^t\right]^{-1}$$

ρ^* si chiama rapp. contragrediente di ρ.

e.g. $|G| = m$, $F = \mathbb{C}$.

$$\rho : G \to GL(V).$$

per $g \in G$, $\rho(g) = \begin{pmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{pmatrix}$

$$\left.\begin{aligned} \rho(g)^m &= \rho(g^m) \\ &= \rho(1_G) \end{aligned}\right\} \Rightarrow \begin{aligned} &w_i^m = 1 \\ &\Downarrow \\ &w_i^{-1} = \overline{w_i} \end{aligned}$$

w_i si depend da g.

$$\chi_\rho(g) = \sum_{i=1}^{n} w_i$$

$$\rho^*(g) = \left[\begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{pmatrix}^t \right]^{-1}$$

$$= \begin{pmatrix} w_1^{-1} & & \\ & \ddots & \\ & & w_n^{-1} \end{pmatrix}$$

$$\chi_{\rho^*}(g) = \sum_{i=1}^{n} w_i^{-1} = \sum_{i=1}^{n} \overline{w_i} = \overline{\chi_\rho(g)}$$

i.e. $\chi_\rho = \overline{\chi_{\rho^*}}$

Sia G gp. finito.

$$X = \{\chi_\rho \mid \forall \rho : G \to GL(V) \text{ rapp } /\mathbb{C}\}_{\dim V < \infty} \subseteq X(G) \subseteq \mathbb{C}[G]$$

Prop (Artin) $\dot{X} = \{\chi_\rho \mid \rho \text{ rapp. di } G \text{ irr}\}$ è un ins. linearmente indipendenti.

Dim. Siano $\{S_1, \dots, S_r\}$ $\mathbb{C}[G]$-mod. semplice distinti.

Tesi: $\chi_{S_1}, \chi_{S_2}, \dots, \chi_{S_r}$ lin. indip.

(1) $\chi_{S_i} : G \to F$

estendere: $\mathbb{C}[G] \to F$

$$\sum_{\forall g \in G} a_g g \mapsto \sum_{\forall g \in G} a_g \chi_{S_i}(g)$$

(2) $S_i \subseteq \mathbb{C}[G] \Rightarrow S_i = \mathbb{C}[G] e_i$

dove $e_i \in \mathbb{C}[G]$ e $e_i^2 = e_i$

$e_i \in S_i$

$$\chi_{S_i}(e_i) = \dim_{\mathbb{C}} S_i$$

$$\chi_{S_i}(e_j) = 0, \quad \forall\, j \neq i$$
$$j \in \{1, \dots, r\}$$

Se $\exists \displaystyle\sum_{i=1}^{r} a_i \chi_{S_i} = 0, \quad a_i \in \mathbb{C}$

$$\Rightarrow \quad \sum a_i \chi_{S_i}(e_j) = a_j \cdot \underbrace{\dim_{\mathbb{C}} S_j}_{\neq 0}$$
$$\|$$
$$0$$

$$\Rightarrow a_j = 0, \quad \forall\, j = 1, \dots r.$$

$$\Rightarrow \chi_{S_1}, \dots, \chi_{S_r} \text{ lin. indip.}$$

<u>Teo</u>: Sia G un gp. finito

Siano $\rho_1, \ldots, \rho_r$ rapp. irr

di G a coeff. in $\mathbb{C}$ non-equival.

Siano $\chi_1, \ldots, \chi_r$ caratt. corrispond.

Allora ogni caratte di G

è di forma

① $\sum_{i=1}^{r} m_i \chi_i$, dove $m_i \in \mathbb{N}$.

e due rapp: $\rho : G \to GL(V)$ sono
$\rho' : G \to GL(W)$.

equivalenti

② $\Longleftrightarrow \chi_\rho = \chi_{\rho'}$.

Dim: ① Siano $S_1, \dots, S_r$ $\mathbb{C}[G]$-mod semplici non-isomorf.

Sia M un $\mathbb{C}[G]$-mod, da Teo Maschke

M semi-semplice $\Rightarrow$

$$M \cong \underbrace{S_1 \oplus \cdots \oplus S_1}_{m_1} \oplus \underbrace{S_2 \oplus \cdots \oplus S_2}_{m_2} \oplus \cdots \oplus \underbrace{S_r \oplus \cdots \oplus S_r}_{m_r}$$

$$\Rightarrow \chi_M = \chi_{\bigoplus_{i=1}^{r} m_i S_i} = \sum_{i=1}^{r} m_i \chi_{S_i} \quad \checkmark$$

② "$\Rightarrow$" ✓

"$\Leftarrow$" V e W sono $\mathbb{C}[G]$-mod.

$$\text{semi-semplicità} \Rightarrow V \cong \underbrace{S_1 \oplus \cdots \oplus S_1}_{m_1} \oplus \cdots \oplus \underbrace{S_r \oplus \cdots \oplus S_r}_{m_r}$$

$$W \cong \underbrace{S_1 \oplus \cdots \oplus S_1}_{n_1} \oplus \cdots \oplus \underbrace{S_r \oplus \cdots \oplus S_r}_{n_r}.$$

$$\chi_V = \sum_{i=1}^{r} m_i \chi_{S_i}$$

$$\chi_W = \sum_{i=1}^{r} n_i \chi_{S_i}$$

$$\chi_V - \chi_W = \sum_{i=1}^{n} (m_i - n_i) \chi_{S_i}$$

Per l'ipotesi: $\chi_V - \chi_W = 0$

Per prop. Artin $\{\chi_{S_i} \mid i=1\dots r\}$

lin. indip. $\Rightarrow m_i - n_i = 0$

$\forall i = 1, \dots, r.$

$$\Rightarrow V \cong \bigoplus_{i=1}^{r} \underset{\substack{\shortparallel \\ m_i}}{n_i} S_i \cong W.$$

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