G gp finito

M uno spaz. vett. $/\mathbb{C}$,

G agisce lin. su M

(1) $$\dim_{\mathbb{C}} M^G = \frac{1}{|G|} \sum_{\forall g \in G} \chi_M(g)$$

(2) U, V $\mathbb{C}[G]$-mod.

$\Rightarrow$ $U \otimes_{\mathbb{C}} V$ $\mathbb{C}[G]$-mod.

$U^* \otimes_{\mathbb{C}} V$ $\mathbb{C}[G]$-mod

$$U^* \otimes_{\mathbb{C}} V \xrightarrow{\varphi} \underbrace{\mathrm{Hom}_{\mathbb{C}}(U, V)}_{\mathbb{C}[G]\text{-mod}}$$

$\forall f \in U^*, v \in V$
$\forall u \in U$, $\quad \varphi(f \otimes v)(u) = f(u)v$.

è morf di $\mathbb{C}[G]$-mod.

(3). $$\mathrm{Hom}_{\mathbb{C}}(U, V)^G = \mathrm{Hom}_{\mathbb{C}[G]}(U, V)$$

(4). $$\chi_{U \otimes_{\mathbb{C}} V} = \chi_U \cdot \chi_U, \quad \chi_{U^* \otimes V} = \overline{\chi_U} \cdot \chi_V$$

G gp. finito.

$X(G) = \{$ funz. di class. coniugati di G in $\mathbb{C}$ $\}$

spaz. vett.

Si def. una forma "bilin." su $X(G)$:

Sia $\{\chi_1, \dots, \chi_r\}$ caratteri irr. distin. di G in $\mathbb{C}$

$\Rightarrow$ è una base di $X(G)$.

$\forall \chi_i, \chi_j$, $\forall i, j$

$$(\chi_i, \chi_j) \overset{def}{=} \frac{1}{|G|} \sum_{\forall g \in G} \overline{\chi_i(g) \cdot \chi_j(g)}$$

$\Rightarrow (-, -)$ forma "bilin." su $X[G]$

Prop: 1) $(\chi_i, \chi_i) \geq 0$,

2) $(\chi_i, \chi_j) = \overline{(\chi_j, \chi_i)}$

3) $\forall z \in \mathbb{C}$.

$$(z\chi_i, \chi_j) = \bar{z}(\chi_i, \chi_j) = (\chi_i, z\chi_j)$$

4) $$(\chi_i + \chi_j, \chi_\ell) = (\chi_i, \chi_\ell) + (\chi_j, \chi_\ell)$$

$$(\chi_i, \chi_j + \chi_\ell) = (\chi_i, \chi_j) + (\chi_i, \chi_\ell)$$

Cond. "1) - ... 4)" forma <u>hermitiano</u>.

<u>Teo</u>: Siano U, V $\mathbb{C}[G]$-mod.

$$(\chi_U, \chi_V) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(U, V).$$

<u>Dim</u>.

$$(\chi_U, \chi_V) = \frac{1}{|G|} \sum_{\forall g \in G} \overline{\chi_U(g)}\, \chi_V(g)$$

$$\overset{④}{=} \frac{1}{|G|} \sum_{g \in G} \chi_{U^* \otimes_{\mathbb{C}} V}(g)$$

$$\overset{②}{=} \frac{1}{|G|} \sum_{\forall g \in G} \chi_{\mathrm{Hom}_{\mathbb{C}}(\overline{U}, V)}(g)$$

$$\overset{①}{=} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\overline{U}, V)^{G}$$

$$\overset{③}{=} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(\overline{U}, V).$$ ////.

<u>Cor</u>: Siano $\{\chi_1, \dots \chi_r\}$ caratteri irr. di G a coeff. in $\mathbb{C}$.

allora
$$(\chi_i, \chi_j) = \delta_{ij}$$

Di conseguenza: una rapp. $\rho : G \to GL(V)$ è irr. $\iff$
$$(\chi_\rho, \chi_\rho) = 1.$$

Dim: Siano $S_1, \dots, S_r$ $\mathbb{C}[G]$-mod. Semplici in corrispond. ad $\chi_1, \dots, \chi_r$.

$$(\chi_i, \chi_j) = (\chi_{S_i}, \chi_{S_j}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(S_i, S_j)$$

$$\mathrm{Hom}_{\mathbb{C}[G]}(S_i, S_j) \overset{\text{Lemma Schur}}{=} \begin{cases} \{0\} & \text{se } i \neq j \\ \mathbb{C} & \text{se } i = j \end{cases}$$

$$\Rightarrow \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(S_i, S_j) = \delta_{ij}$$

$\boxed{///}$

Di Conseguenza:

"$\Rightarrow$" ✓

"$\Leftarrow$" V è un $\mathbb{C}[G]$-mod,

$$\overset{\text{Maschke}}{\Longrightarrow} V \cong \bigoplus_{i=1}^{r} n_i S_i, \quad \chi_V = \chi_{\oplus_i n_i S_i} = \sum_i n_i \chi_{S_i}$$

$$1 = (\chi_V, \chi_V) = \left(\sum n_i \chi_{S_i}, \sum n_i \chi_{S_i}\right)$$

$$= \sum_{i=1}^{r} (\chi_{S_i}, \chi_{S_i}) n_i^2$$

$$\Rightarrow \; = \sum_{i=1}^{r} n_i^2 \Rightarrow r = 1$$

$n_i = 1$ per q.c. $1 \leq i \leq r$

$\Rightarrow \quad V \cong S_i \qquad n_j = 0$, e per $j \neq i$

$\Rightarrow$ irr. di V. $\boxed{////}$

G gp. $\quad \rho : G \longrightarrow GL(V)$, $\quad V$ e V' $\mathbb{C}$-spaz.

G' gp. $\quad \rho' : G' \longrightarrow GL(V')$,

$$\rho \otimes \rho' : \quad G \times G' \longrightarrow GL(V \otimes_{\mathbb{C}} V')$$

$$(g, g') \longmapsto \rho \otimes \rho'(g, g') : V \otimes_{\mathbb{C}} V' \to V \otimes_{\mathbb{C}} V'$$

$\forall v \in V$, $\forall v' \in V'$, $\quad [\rho \otimes \rho'(g, g')](v \otimes v') = \rho(g)v \otimes \rho'(g')v'$

$$\chi_{\rho \otimes \rho'} = \chi_\rho \cdot \chi_{\rho'}$$

Teo: Siano G e G' due gp finiti.

e $\{\rho_1, \dots, \rho_r\}$ rapp. irr. distin. di G a coeff. in $\mathbb{C}$.

$\{\rho'_1, \dots, \rho'_{r'}\}$ - - - - - G' . . . . . $\mathbb{C}$.

allora le rapp. irr. non-equivalenti di $G \times G'$ sono

$$\left\{ \rho_i \otimes \rho'_j \;\middle|\; \begin{array}{l} \forall\, 1 \le i \le r \\ \forall\, 1 \le j \le r' \end{array} \right\}.$$

Dim: (1) $\forall i, \forall j$, $\rho_i \otimes \rho'_j$ è irr.:

cioè: $(\chi_{\rho_i \otimes \rho'_j}, \chi_{\rho_i \otimes \rho'_j}) = 1$:

$$(\chi_{\rho_i \otimes \rho'_j}, \chi_{\rho_i \otimes \rho'_j}) = \frac{1}{|G \times G'|} \sum_{\substack{\forall (g,g') \\ \in G \times G'}} \overline{\chi_{\rho_i \otimes \rho'_j}(g,g')} \cdot \chi_{\rho_i \otimes \rho'_j}(g,g')$$

$$= \frac{1}{|G|\cdot|G'|} \sum_{\substack{\forall g\in G \\ \forall g'\in G'}} \overline{\chi_{\rho_i}(g)\chi_{\rho_j'}(g')} \cdot \chi_{\rho_i}(g)\chi_{\rho_j'}(g')$$

$$= \frac{1}{|G|} \sum_{\forall g\in G} \overline{\chi_{\rho_i}(g)}\chi_{\rho_i}(g) \underbrace{\sum_{\forall g'\in G'} \overline{\chi_{\rho_j'}(g')}\chi_{\rho_j'}(g')}_{\frac{1}{|G'|}}$$

$$= (\chi_{\rho_i}, \chi_{\rho_i})\cdot(\chi_{\rho_j'}, \chi_{\rho_j'}) = 1.$$

② $\rho_i \otimes \rho_j' \cong \rho_k \otimes \rho_l' \iff \begin{cases} i = k \\ j = l \end{cases}$:

"$\Rightarrow$"

$$1 = (\chi_{\rho_i\otimes\rho_j'}, \chi_{\rho_k\otimes\rho_l'}) = (\chi_{\rho_i}, \chi_{\rho_k})(\chi_{\rho_j'}, \chi_{\rho_l'})$$

$$\Rightarrow \begin{cases} (\chi_{\rho_i}, \chi_{\rho_k}) = 1 \\ (\chi_{\rho_j'}, \chi_{\rho_l'}) = 1 \end{cases} \Rightarrow \begin{matrix} \rho_i = \rho_k \\ \rho_j' = \rho_l'. \end{matrix}$$

③. $\#\{$rapp. irr. non-equivalenti di $G \times G'\}$

$\|$

$\#\{$classi coniugati di $G \times G'\}$

$\|$

$\#\,|\{$class. coniugati di $G\}| \cdot |\{$class. coniug. di $G'\}|$

$\|$

$\#\{$rapp. irr. distin. di $G\} \cdot \#\{$rapp. irr. distin. di $G'\}$

$= r \cdot r'$

$= \left|\left\{ \rho_i \otimes \rho'_j \;\middle|\; \begin{array}{l} \forall\, 1 \leq i \leq r \\ \forall\, 1 \leq j \leq r' \end{array} \right\}\right|.$

!!!.

e.g. G gp. finito abeliano.

$\Rightarrow \quad G = G_1 \times G_2 \times \dots \times G_n, \quad G_i = \langle g_i \rangle.$

Calcolare tutte le rapp. irr. di G a coeff. in $\mathbb{C}$!!!