(2) Given an open cover $\{U_\alpha\}_{\alpha \in I}$ of M, there is a partition of unity $\{\rho_\beta\}_{\beta \in J}$ with *compact support*, but possibly with an index set J different from I, such that the support of ρ_β is contained in some U_α.

For a proof see Warner [1, p. 10] or de Rham [1, p. 3].

Note that in (1) the support of ρ_α is not assumed to be compact and the index set of $\{\rho_\alpha\}$ is the same as that of $\{U_\alpha\}$, while in (2) the reverse is true. We usually cannot demand simultaneously compact support and the same index set on a noncompact manifold M. For example, consider the open cover of $\mathbb{R}^1$ consisting of precisely one open set, namely $\mathbb{R}^1$ itself. This open cover clearly does not have a partition of unity with compact support subordinate to it.

The Mayer-Vietoris Sequence

The Mayer-Vietoris sequence allows one to compute the cohomology of the union of two open sets. Suppose $M = U \cup V$ with U, V open. Then there is a sequence of inclusions

$$M \leftarrow U \coprod V \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U \cap V$$

where $U \coprod V$ is the disjoint union of U and V and ∂_0 and ∂_1 are the inclusions of $U \cap V$ in V and in U respectively. Applying the contravariant functor Ω^*, we get a sequence of restrictions of forms

$$\Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \Omega^*(U \cap V),$$

where by the restriction of a form to a submanifold we mean its image under the pullback map induced by the inclusion. By taking the difference of the last two maps, we obtain the *Mayer-Vietoris sequence*

$$(2.2) \qquad 0 \rightarrow \Omega^*(M) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V) \rightarrow 0$$
$$(\omega, \tau) \mapsto \tau - \omega$$

Proposition 2.3. *The Mayer-Vietoris sequence is exact.*

Proof. The exactness is clear except at the last step. We first consider the case of functions on $M = \mathbb{R}^1$. Let f be a C^∞ function on $U \cap V$ as shown in Figure 2.1. We must write f as the difference of a function on U and a function on V. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. Note that $\rho_V f$ is a function on U—to get a function on an open set we must multiply by the partition function of the other open set. Since

$$(\rho_U f) - (-\rho_V f) = f,$$

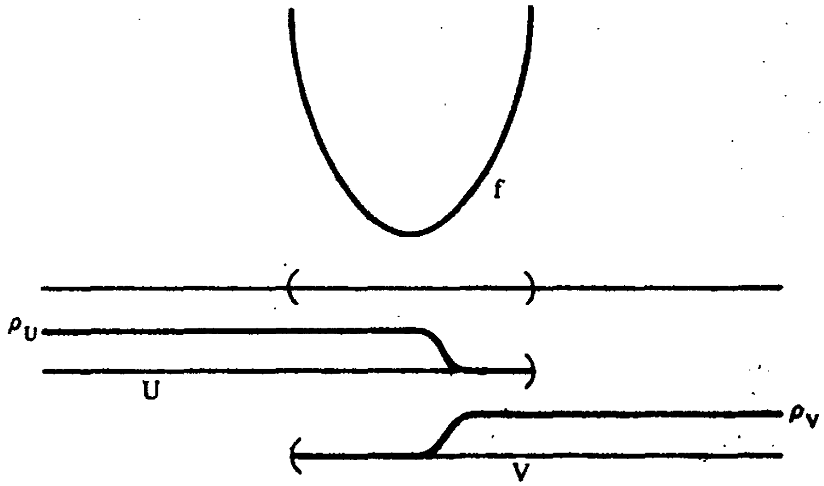

Figure 2.1

we see that $\Omega^0(U) \oplus \Omega^0(V) \to \Omega^0(\mathbb{R}^1) \to 0$ is surjective. For a general manifold M, if $\omega \in \Omega^q(U \cap V)$, then $(-\rho_V\omega, \rho_U\omega)$ in $\Omega^q(U) \oplus \Omega^q(V)$ maps onto ω. □

The Mayer-Vietoris sequence

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$$

induces a long exact sequence in cohomology, also called a Mayer-Vietoris sequence:

$$\text{(2.4)}\qquad \begin{array}{l} \to H^{q+1}(M) \to H^{q+1}(U) \oplus H^{q+1}(V) \to H^{q+1}(U \cap V) \to \cdots \\ \qquad\qquad\qquad\qquad d^* \\ \to H^q(M) \to H^q(U) \oplus H^q(V) \to H^q(U \cap V) \to \end{array}$$

We recall again the definition of the coboundary operator d^* in this explicit instance. The short exact sequence gives rise to a diagram with exact rows

$$\begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 \to & & \Omega^{q+1}(M) & \to & \Omega^{q+1}(U) \oplus \Omega^{q+1}(V) & \to & \Omega^{q+1}(U \cap V) & \to 0 & \\ & & d\uparrow & & d\uparrow & & d\uparrow & & \\ 0 \to & & \Omega^q(M) & \to & \Omega^q(U) \oplus \Omega^q(V) & \to & \Omega^q(U \cap V) & \to 0 & \\ & & & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \\ & & & & \xi & & \omega & & d\omega = 0 \end{array}$$

Let $\omega \in \Omega^q(U \cap V)$ be a closed form. By the exactness of the rows, there is a $\xi \in \Omega^q(U) \oplus \Omega^q(V)$ which maps to ω, namely, $\xi = (-\rho_V\omega, \rho_U\omega)$. By the commutativity of the diagram and the fact that $d\omega = 0$, $d\xi$ goes to 0 in $\Omega^{q+1}(U \cap V)$, i.e., $-d(\rho_V\,\omega)$ and $d(\rho_U\,\omega)$ agree on the overlap $U \cap V$. Hence $d\xi$ is the image of an element in $\Omega^{q+1}(M)$. This element is easily seen to be closed and represents $d^*[\omega]$. As remarked earlier, it can be shown that $d^*[\omega]$ is independent of the choices in this construction. Explicitly we see that the coboundary operator is given by

$$d^*[\omega] = \begin{cases} [-d(\rho_V\,\omega)] & \text{on} \quad U \\ \;\;[d(\rho_U\,\omega)] & \text{on} \quad V. \end{cases} \tag{2.5}$$

We define the *support* of a form ω on a manifold M to be the smallest closed set Z so that ω restricted to Z is not 0. Note that in the Mayer-Vietoris sequence $d^*\omega \in H^*(M)$ has support in $U \cap V$.

EXAMPLE 2.6 (The cohomology of the circle). Cover the circle with two open sets U and V as shown in Figure 2.2. The Mayer-Vietoris sequence gives

$$\begin{array}{cccccc} & S^1 & & U \amalg V & & U \cap V \\ H^2 & 0 & & 0 & & 0 \\ H^1 & & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & d^* & & \\ H^0 & & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\ \delta\ } & \mathbb{R} \oplus \mathbb{R} \end{array}$$

The difference map δ sends (ω, τ) to $(\tau - \omega, \tau - \omega)$, so im δ is 1-dimensional. It follows that ker δ is also 1-dimensional. Therefore,

$$H^0(S^1) = \ker \delta = \mathbb{R}$$
$$H^1(S^1) = \operatorname{coker} \delta = \mathbb{R}.$$

We now find an explicit representative for the generator of $H^1(S^1)$. If $\alpha \in \Omega^0(U \cap V)$ is a closed 0-form which is not the image under δ of a closed form in $\Omega^0(U) \oplus \Omega^0(V)$, then $d^*\alpha$ will represent a generator of $H^1(S^1)$. As α we may take the function which is 1 on the upper piece of $U \cap V$ and 0 on

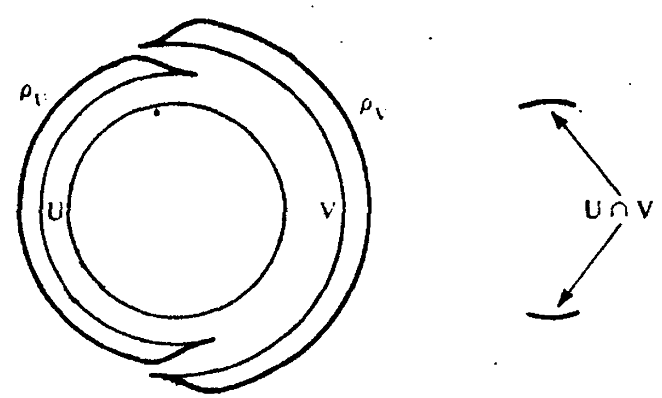

Figure 2.2

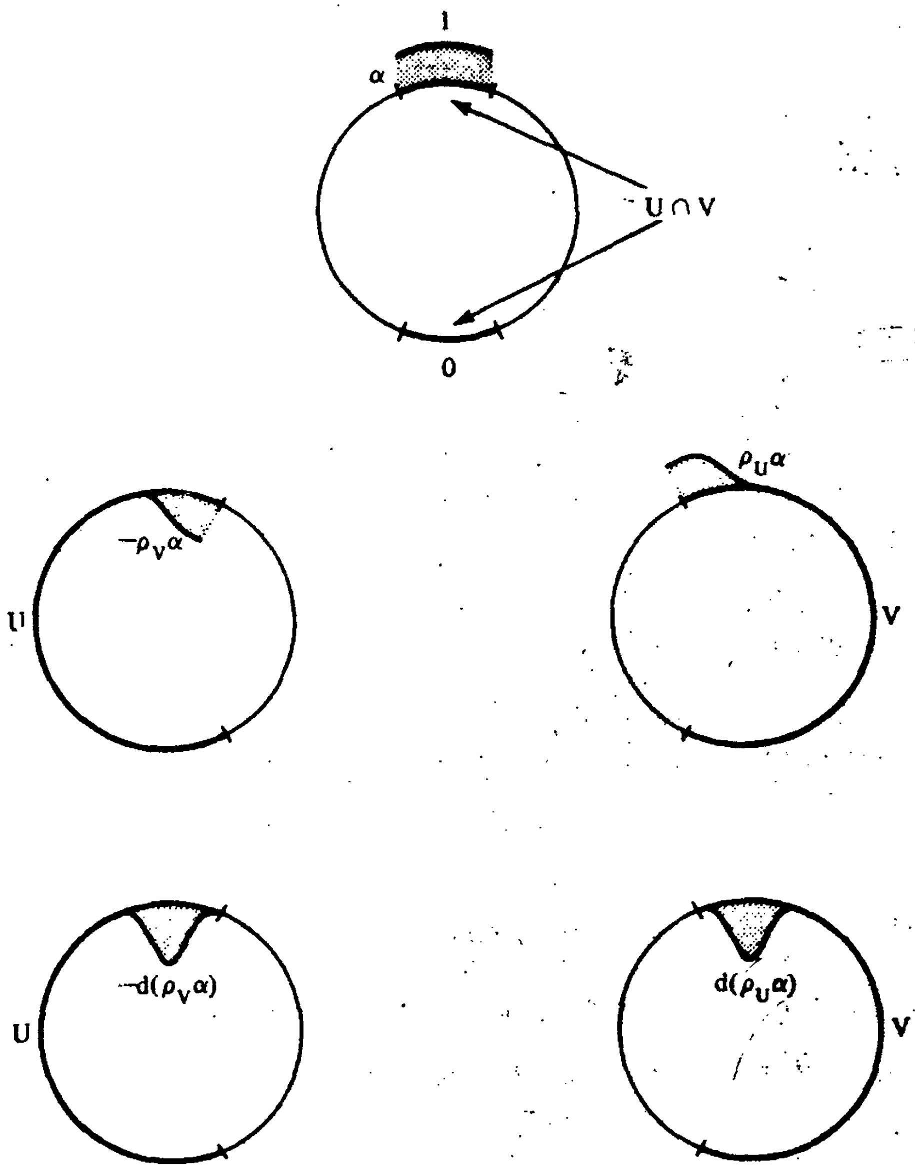

Figure 2.3

the lower piece (see Figure 2.3). Now α is the image of $(-\rho_V\alpha, \rho_U\alpha)$. Since $-d(\rho_V\alpha)$ and $d\rho_U\alpha$ agree on $U \cap V$, they represent a global form on S^1; this form is $d^*\alpha$. It is a bump 1-form with support in $U \cap V$.

The Functor Ω_c^* and the Mayer-Vietoris Sequence for Compact Supports

Again, before taking up the Mayer-Vietoris sequence for compactly supported cohomology, we need to discuss the functorial properties of $\Omega_c^*(M)$, the algebra of forms with compact support on the manifold M. In general the pullback by a smooth map of a form with compact support need not