27

Computation of de Rham Cohomology

With the tools developed so far, we can compute the cohomology of many manifolds. This chapter is a compendium of some examples.

27.1 Cohomology Vector Space of a Torus

Cover a torus M with two open subsets U and V as shown in Figure 27.1.

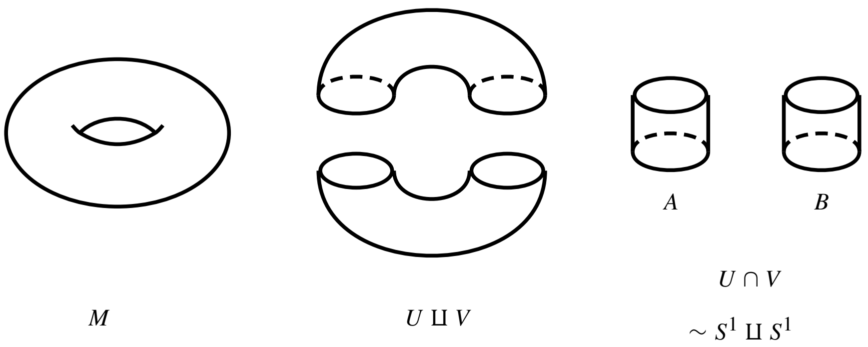

Fig. 27.1. An open cover $\{U, V\}$ of a torus.

Both U and V are diffeomorphic to a cylinder and therefore have the homotopy type of a circle (Problem 26.4). Similarly, the intersection $U \cap V$ is the disjoint union of two cylinders A and B and has the homotopy type of a disjoint union of two circles. Our knowledge of the cohomology of a circle allows us to fill in many terms in the Mayer–Vietoris sequence:

$$
\begin{array}{c|ccccccc}
 & & M & & U \amalg V & & U \cap V \\
\hline
H^2 & \xrightarrow{d_1^*} & H^2(M) & \to & 0 & & \\
H^1 & \xrightarrow{d_0^*} & H^1(M) & \xrightarrow{\gamma} & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\beta} & \mathbb{R} \oplus \mathbb{R} \\
H^0 & 0 \to & \mathbb{R} & \to & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\alpha} & \mathbb{R} \oplus \mathbb{R}
\end{array}
\tag{27.1}
$$

Let $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$ be the inclusion maps. If a is the constant function with value a on U, then $j_U^* a$ is the constant function with the value a on each component of $U \cap V$, that is,

$$j_U^* a = (a, a).$$

Therefore, for $(a, b) \in H^0(U) \oplus H^0(V)$,

$$
\begin{aligned}
\alpha(a, b) &= j_V^* b - j_U^* a \\
&= (b, b) - (a, a) \\
&= (b - a, b - a).
\end{aligned}
$$

Similarly, let us now describe the map

$$\beta\colon H^1(U) \oplus H^1(V) \to H^1(U \cap V) = H^1(A) \oplus H^1(B).$$

Since A is a deformation retract of U, the restriction $H^*(U) \to H^*(A)$ is an isomorphism. If ω_U generates $H^1(U)$, then $j_U^* \omega_U$ is a generator of H^1 on A and on B. Identifying $H^1(U \cap V)$ with $\mathbb{R} \oplus \mathbb{R}$, we write $j_U^* \omega_U = (1, 1)$. Let ω_V be a generator of $H^1(V)$. The pair of real numbers

$$(a, b) \in H^1(U) \oplus H^1(V) \simeq \mathbb{R} \oplus \mathbb{R}$$

stands for $(a\omega_U, b\omega_V)$. Then,

$$
\begin{aligned}
\beta(a, b) &= j_V^*(b\omega_V) - j_U^*(a\omega_U) \\
&= (b, b) - (a, a) \\
&= (b - a, b - a).
\end{aligned}
$$

By the exactness of the Mayer–Vietoris sequence,

$$
\begin{aligned}
H^2(M) &= \operatorname{im} d_1^* && \text{(because } H^2(U) \oplus H^2(V) = 0\text{)} \\
&\simeq H^1(U \cap V)/\ker d_1^* && \text{(by the first isomorphism theorem)} \\
&\simeq (\mathbb{R} \oplus \mathbb{R})/\operatorname{im} \beta \\
&\simeq (\mathbb{R} \oplus \mathbb{R})/\mathbb{R} \simeq \mathbb{R}.
\end{aligned}
$$

Applying Problem 25.2 to the Mayer–Vietoris sequence (27.1), we get

$$1 - 2 + 2 - \dim H^1(M) + 2 - 2 + \dim H^2(M) = 0.$$

Since $\dim H^2(M) = 1$, this gives $\dim H^1(M) = 2$.

As a check, we can also compute $H^1(M)$ from the Mayer–Vietoris sequence using our knowledge of the maps α and β:

$$\begin{aligned}
H^1(M) &\simeq \ker\gamma \oplus \operatorname{im}\gamma && \text{(by the first isomorphism theorem)}\\
&\simeq \operatorname{im} d_0^* \oplus \ker\beta && \text{(exactness of the M–V sequence)}\\
&\simeq (H^0(U\cap V)/\ker d_0^*) \oplus \ker\beta && \text{(first isomorphism theorem for } d_0^*\text{)}\\
&\simeq ((\mathbb{R}\oplus\mathbb{R})/\operatorname{im}\alpha)\oplus\mathbb{R}\\
&\simeq \mathbb{R}\oplus\mathbb{R}.
\end{aligned}$$

27.2 The Cohomology Ring of a Torus

A torus is diffeomorphic to the quotient of $\mathbb{R}^2$ by the integer lattice $\Lambda = \mathbb{Z}^2$. The quotient map

$$\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$$

induces a pullback map on differential forms,

$$\pi^*\colon \Omega^*(\mathbb{R}^2/\Lambda) \to \Omega^*(\mathbb{R}^2).$$

Since $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$ is a local diffeomorphism, it is a submersion at each point. By Problem 18.7, $\pi^*\colon \Omega^*(\mathbb{R}^2/\Lambda) \to \Omega^*(\mathbb{R}^2)$ is an inclusion.

For $\lambda \in \Lambda$, define $\ell_\lambda\colon \mathbb{R}^2 \to \mathbb{R}^2$ to be translation by λ,

$$\ell_\lambda(p) = p + \lambda,\ \ p \in \mathbb{R}^2.$$

A differential form $\bar\omega$ on $\mathbb{R}^2$ is said to be *invariant under translation by* $\lambda \in \Lambda$ if $\ell_\lambda^*\bar\omega = \bar\omega$.

Proposition 27.1. *The image of the inclusion map* $\pi^*\colon \Omega^*(\mathbb{R}^2/\Lambda) \to \Omega^*(\mathbb{R}^2)$ *is the subspace of differential forms on* $\mathbb{R}^2$ *invariant under translations by elements of* Λ.

Proof. For all $p \in \mathbb{R}^2$,

$$(\pi\circ\ell_\lambda)(p) = \pi(p+\lambda) = \pi(p).$$

Hence, $\pi\circ\ell_\lambda = \pi$. By the functoriality of the pullback,

$$\pi^* = \ell_\lambda^*\circ\pi^*.$$

Thus, for any $\omega \in \Omega^k(\mathbb{R}^2/\Lambda)$, $\pi^*\omega = \ell_\lambda^*\pi^*\omega$. This proves that $\pi^*\omega$ is invariant under translations ℓ_λ for all $\lambda \in \Lambda$.

Conversely, suppose $\bar\omega \in \Omega^k(\mathbb{R}^2)$ is invariant under translations ℓ_λ for all $\lambda \in \Lambda$. For $p \in \mathbb{R}^2/\Lambda$ and $v_1,\dots,v_k \in T_p(\mathbb{R}^2/\Lambda)$, define

$$\omega_p(v_1,\dots,v_k) = \bar\omega_{\bar p}(\bar v_1,\dots,\bar v_k) \tag{27.2}$$

for any $\bar{p} \in \pi^{-1}(\{p\})$ and $\bar{v}_1, \ldots, \bar{v}_k \in T_{\bar{p}}\mathbb{R}^2$ such that $\pi_* \bar{v}_i = v_i$. Any other point in $\pi^{-1}(\{p\})$ may be written as $\bar{p} + \lambda$ for some $\lambda \in \Lambda$. By invariance,

$$\bar{\omega}_{\bar{p}} = (\ell_\lambda^* \bar{\omega})_{\bar{p}} = \ell_\lambda^*(\bar{\omega}_{\bar{p}+\lambda}).$$

So

$$\begin{aligned}\bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) &= \ell_\lambda^*(\bar{\omega}_{\bar{p}+\lambda})(\bar{v}_1, \ldots, \bar{v}_k) \\ &= \bar{\omega}_{\bar{p}+\lambda}(\ell_{\lambda *}\bar{v}_1, \ldots, \ell_{\lambda *}\bar{v}_k),\end{aligned}$$

which shows that ω_p is well defined, independent of the choice of $\bar{p}$. Thus, $\omega \in \Omega^k(\mathbb{R}^2/\Lambda)$. Moreover, by (27.2), for any $\bar{p} \in \mathbb{R}^2$ and $\bar{v}_1, \ldots, \bar{v}_k \in T_{\bar{p}}(\mathbb{R}^2)$,

$$\begin{aligned}\bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) &= \omega_{\pi(\bar{p})}(\pi_*\bar{v}_1, \ldots, \pi_*\bar{v}_k) \\ &= (\pi^*\omega)_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k).\end{aligned}$$

Hence, $\bar{\omega} = \pi^*\omega$. □

Let (x, y) be the coordinates on $\mathbb{R}^2$. Since for any $\lambda \in \Lambda$,

$$\ell_\lambda^*(dx) = d(\ell_\lambda^* x) = d(x + \lambda) = dx,$$

by Proposition 27.1 the 1-form dx on $\mathbb{R}^2$ is π^* of a 1-form on the torus $\mathbb{R}^2/\Lambda$. Similarly, dy is also π^* of a 1-form on the torus. We denote these 1-forms on the torus by the same symbols dx and dy.

Proposition 27.2. *Let M be the torus $\mathbb{R}^2/\mathbb{Z}^2$. A basis for the cohomology vector space $H^*(M)$ is $1, dx, dy, dx \wedge dy$.*

Proof. Since $\int_M dx \wedge dy = 1$, the closed 2-form $dx \wedge dy$ defines a nonzero cohomology class. By the computation of Section 27.1, $H^2(M) = \mathbb{R}$. So $dx \wedge dy$ is a basis for $H^2(M)$.

It remains to show that the set of closed 1-forms dx, dy on M is a basis for $H^1(M)$. Define two closed curves C_1, C_2 in $M = \mathbb{R}^2/\mathbb{Z}^2$ as the images of the maps

$$\begin{gathered} c_i \colon [0,1] \to M, \\ c_1(t) = [(t, 0)], \qquad c_2(t) = [(0, t)], \end{gathered}$$

(see Figure 27.2). Denote by p the point $[(0,0)]$ in M. Since removing a point does not change the value of an integral and c_1 is a diffeomorphism of the open interval $(0,1)$ onto $C_1 - \{p\}$,

$$\int_{C_1} dx = \int_{C_1 - \{p\}} dx = \int_{(0,1)} c_1^* \, dx = \int_0^1 dt = 1.$$

In the same way, because $c_1^* dy = 0$,

$$\int_{C_1} dy = \int_{C_1 - \{p\}} dy = \int_0^1 c_1^* \, dy = 0.$$

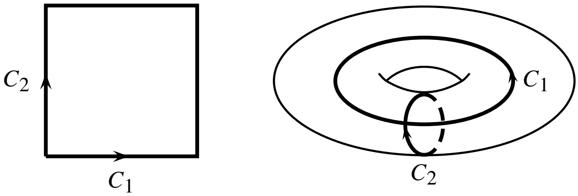

Fig. 27.2. Two closed curves on a torus.

Similarly,

$$\int_{C_2} dx = 0, \qquad \int_{C_2} dy = 1.$$

As x is not a function on the torus M, dx is not necessarily exact on M. In fact, if $dx = df$ for some C^∞ function f on M, then

$$\int_{C_1} dx = \int_{C_1} df = \int_{\partial C_1} f = 0$$

by Stokes' theorem and the fact that $\partial C_1 = \varnothing$. This contradicts the fact that $\int_{C_1} dx = 1$. Thus, dx is not exact on M. By the same reasoning, dy is also not exact on M. Furthermore, the cohomology classes $[dx]$ and $[dy]$ are linearly independent, since if $[dx]$ were a multiple of $[dy]$, then $\int_{C_1} dx$ would have to be a multiple of $\int_{C_1} dy = 0$. By Section 27.1, $H^1(M)$ is two dimensional. Hence, dx, dy is a basis for $H^1(M)$. □

The ring structure of $H^*(M)$ is transparent from this proposition. Abstractly it is the algebra

$$\bigwedge(a,b) := \mathbb{R}[a,b]/(a^2, b^2, ab+ba), \qquad \deg a = 1, \deg b = 1,$$

called the *exterior algebra* on two generators a and b of degree 1.

27.3 The Cohomology of a Surface of Genus g

Using the Mayer–Vietoris sequence to compute the cohomology of a manifold often leads to ambiguities, because there may be several unknown terms in the sequence. We can resolve these ambiguities if we can describe explicitly the maps occurring in the Mayer–Vietoris sequence. Here is an example of how this might be done.

Lemma 27.3. *Suppose p is a point in a compact oriented surface M without boundary, and $i\colon C \to M - \{p\}$ is the inclusion of a small circle around the puncture (Figure 27.3). Then the restriction map*

$$i^*\colon H^1(M - \{p\}) \to H^1(C)$$

is the zero map.

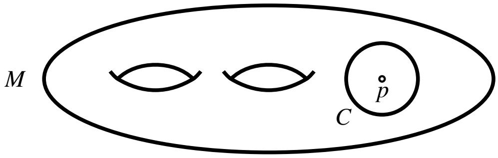

Fig. 27.3. Punctured surface.

Proof. An element $[\omega] \in H^1(M - \{p\})$ is represented by a closed 1-form ω on $M - \{p\}$. Because the linear isomorphism $H^1(C) \simeq H^1(S^1) \simeq \mathbb{R}$ is given by integration over C, to identify $i^*[\omega]$ in $H^1(C)$, it suffices to compute the integral $\int_C i^*\omega$.

If D is the open disk in M bounded by the curve C, then $M - D$ is a compact oriented surface with boundary C. By Stokes' theorem,

$$\int_C i^*\omega = \int_{\partial(M-D)} i^*\omega = \int_{M-D} d\omega = 0$$

because $d\omega = 0$. Hence, $i^* \colon H^1(M - \{p\}) \to H^1(C)$ is the zero map. □

Proposition 27.4. *Let M be a torus, p a point in M, and A the punctured torus $M - \{p\}$. The cohomology of A is*

$$H^k(A) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ \mathbb{R}^2 & \text{for } k = 1, \\ 0 & \text{for } k > 1. \end{cases}$$

Proof. Cover M with two open sets, A and a disk U containing p. Since A, U, and $A \cap U$ are all connected, we may start the Mayer–Vietoris sequence with the $H^1(M)$ term (Proposition 25.3(ii)). With $H^*(M)$ known from Section 27.1, the Mayer–Vietoris sequence becomes

	M	$U \amalg A$	$U \cap A \sim S^1$
H^2	$\xrightarrow{d_1^*}$ $\mathbb{R}$	$\to H^2(A) \to$	0
H^1	$0 \to \mathbb{R} \oplus \mathbb{R}$	$\xrightarrow{\beta} H^1(A) \xrightarrow{\alpha}$	$H^1(S^1)$

Because $H^1(U) = 0$, the map $\alpha \colon H^1(A) \to H^1(S^1)$ is simply the restriction map i^*. By Lemma 27.3, $\alpha = i^* = 0$. Hence,

$$H^1(A) = \ker\alpha = \operatorname{im}\beta \simeq H^1(M) \simeq \mathbb{R} \oplus \mathbb{R}$$

and there is an exact sequence of linear maps

$$0 \to H^1(S^1) \xrightarrow{d_1^*} \mathbb{R} \to H^2(A) \to 0.$$

Since $H^1(S^1) \simeq \mathbb{R}$, it follows that $H^2(A) = 0$. □

Proposition 27.5. *The cohomology of a compact orientable surface Σ_2 of genus 2 is*

$$H^k(\Sigma_2) = \begin{cases} \mathbb{R} & \text{for } k = 0, 2, \\ \mathbb{R}^4 & \text{for } k = 1, \\ 0 & \text{for } k > 2. \end{cases}$$

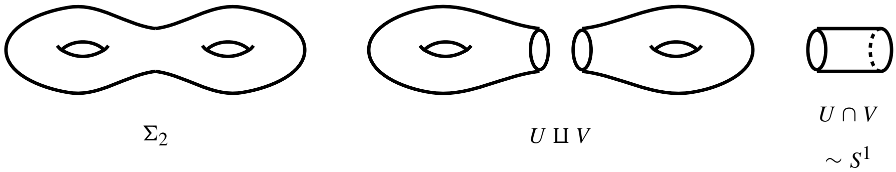

Fig. 27.4. An open cover $\{U, V\}$ of a surface of genus 2.

Proof. Cover Σ_2 with two open sets U and V as in Figure 27.4. The Mayer–Vietoris sequence gives

	M	$U \amalg V$	$U \cap V \sim S^1$
H^2	$\to H^2(\Sigma_2) \to$	0	
H^1	$0 \to H^1(\Sigma_2) \to$	$\mathbb{R}^2 \oplus \mathbb{R}^2 \xrightarrow{\alpha}$	$\mathbb{R}$

The map $\alpha\colon H^1(U) \oplus H^1(V) \to H^1(S^1)$ is the difference map

$$\alpha(\omega_U, \omega_V) = j_V^* \omega_V - j_U^* \omega_U,$$

where j_U and j_V are inclusions of an S^1 in $U \cap V$ into U and V, respectively. By Lemma 27.3, $j_U^* = j_V^* = 0$, so $\alpha = 0$. It then follows from the exactness of the Mayer–Vietoris sequence that

$$H^1(\Sigma_2) \simeq H^1(U) \oplus H^1(V) \simeq \mathbb{R}^4$$

and

$$H^2(\Sigma_2) \simeq H^1(S^1) \simeq \mathbb{R}. \qquad \square$$

A genus 2 surface Σ_2 can be obtained as the quotient space of an octagon with its edges identified following the scheme of Figure 27.5.

To see this, first cut Σ_2 along the circle e as in Figure 27.6. Then the two halves A and B are each a torus minus an open disk (Figure 27.7), so that each half can be represented as a pentagon (Figure 27.8).

When A and B are glued together along e, we obtain the octagon in Figure 27.5.

By Lemma 27.3, if $p \in \Sigma_2$ and $i\colon C \to \Sigma_2 - \{p\}$ is a small circle around p in Σ_2, then the restriction map

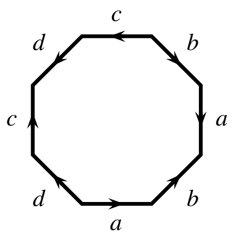

Fig. 27.5. A surface of genus 2 as a quotient space of an octagon.

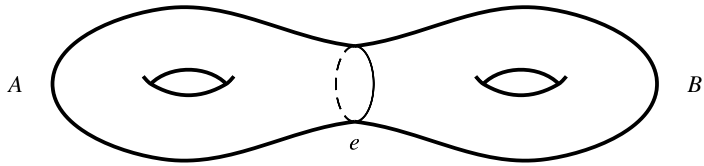

Fig. 27.6. A surface of genus 2 cut along a curve e.

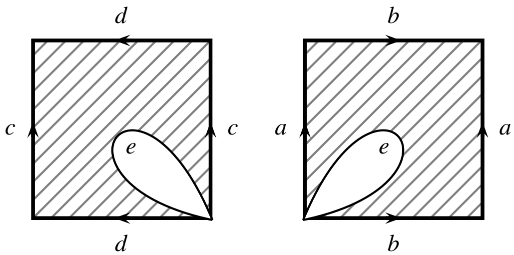

Fig. 27.7. Two halves of a surface of genus 2.

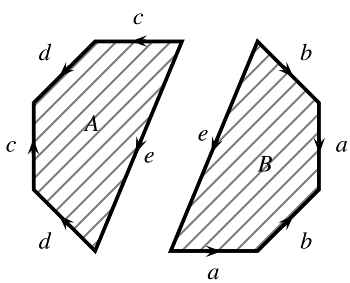

Fig. 27.8. Two halves of a surface of genus 2.