22.5 Stokes' Theorem

Let M be an oriented manifold of dimension n with boundary ∂M. We give ∂M the boundary orientation.

Theorem 22.8 (Stokes' theorem). *For any $(n-1)$-form ω with compact support on the oriented n-dimensional manifold M,*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. Choose an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M in which each U_α is diffeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$ via an orientation-preserving diffeomorphism. This is possible since any open disk is diffeomorphic to $\mathbb{R}^n$ (see Problem 1.4). Let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. As we showed in the preceding section, the $(n-1)$-form $\rho_\alpha\omega$ has compact support in U_α.

Suppose Stokes' theorem holds for $\mathbb{R}^n$ and for $\mathbb{H}^n$. Then it holds for all the charts U_α in our atlas, which are diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$. Also, note that

$$(\partial M) \cap U_\alpha = \partial U_\alpha.$$

Therefore,

$$\begin{aligned}
\int_{\partial M} \omega &= \int_{\partial M} \sum_\alpha \rho_\alpha \omega && \Big(\sum_\alpha \rho_\alpha = 1\Big) \\
&= \sum_\alpha \int_{\partial M} \rho_\alpha \omega && \Big(\sum_\alpha \rho_\alpha\omega \text{ is a finite sum by Problem 18.5}\Big) \\
&= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega && (\operatorname{supp} \rho_\alpha\omega \text{ is contained in } U_\alpha) \\
&= \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) && (\text{Stokes' theorem for } U_\alpha) \\
&= \sum_\alpha \int_{M} d(\rho_\alpha \omega) && (\operatorname{supp} d(\rho_\alpha\omega) \subset U_\alpha) \\
&= \int_M d\Big(\sum \rho_\alpha \omega\Big) && (\rho_\alpha\omega \equiv 0 \text{ for all but finitely many } \alpha) \\
&= \int_M d\omega.
\end{aligned}$$

Thus, it suffices to prove Stokes' theorem for $\mathbb{R}^n$ and for $\mathbb{H}^n$. We will give a proof only for $\mathbb{H}^2$, as the general case is similar.

Proof of Stokes' theorem for the upper half-plane $\mathbb{H}^2$. Let x, y be the coordinates on $\mathbb{H}^2$. Then the standard orientation on $\mathbb{H}^2$ is given by $dx \wedge dy$, and the boundary orientation on $\partial\mathbb{H}^2$ is given by dx.

The form ω is a linear combination

$$\omega = f(x, y)\, dx + g(x, y)\, dy \tag{22.7}$$

for C^∞ functions f, g with compact support in $\mathbb{H}^2$. Since the supports of f and g are compact, we may choose a real number $a > 0$ large enough so that the supports of f and g are contained in the interior of the square $[-a, a] \times [-a, a]$. We will use the notation f_x, f_y to denote the partial derivatives of f with respect to x and y, respectively. Then

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = (g_x - f_y)\, dx \wedge dy,$$

and

$$\begin{aligned}
\int_{\mathbb{H}^2} d\omega &= \int_{\mathbb{H}^2} g_x\, |dx\, dy| - \int_{\mathbb{H}^2} f_y\, |dx\, dy| \\
&= \int_0^\infty \int_{-\infty}^\infty g_x\, |dx\, dy| - \int_{-\infty}^\infty \int_0^\infty f_y\, |dy\, dx| \\
&= \int_0^a \int_{-a}^a g_x\, |dx\, dy| - \int_{-a}^a \int_0^a f_y\, |dy\, dx|.
\end{aligned} \tag{22.8}$$

In this expression,

$$\int_{-a}^{a} g_x(x, y)\, dx = g(x, y)\Big]_{x=-a}^{a} = 0$$

because supp g lies in the interior of $[-a, a] \times [-a, a]$. Similarly,

$$\int_{0}^{a} f_y(x, y)\, dy = f(x, y)\Big]_{y=0}^{a} = -f(x, 0)$$

because $f(x, a) = 0$. Thus, (22.8) becomes

$$\int_{\mathbb{H}^2} d\omega = \int_{-a}^{a} f(x, 0)\, dx.$$

On the other hand, $\partial\mathbb{H}^2$ is the x-axis and $dy = 0$ on $\partial\mathbb{H}^2$. It follows from (22.7) that $\omega = f(x, 0)\, dx$ when restricted to $\partial\mathbb{H}^2$ and

$$\int_{\partial\mathbb{H}^2} \omega = \int_{-a}^{a} f(x, 0)\, dx.$$

This proves Stokes' theorem for the upper half-plane. □