A partial and biased introduction to topological dynamics

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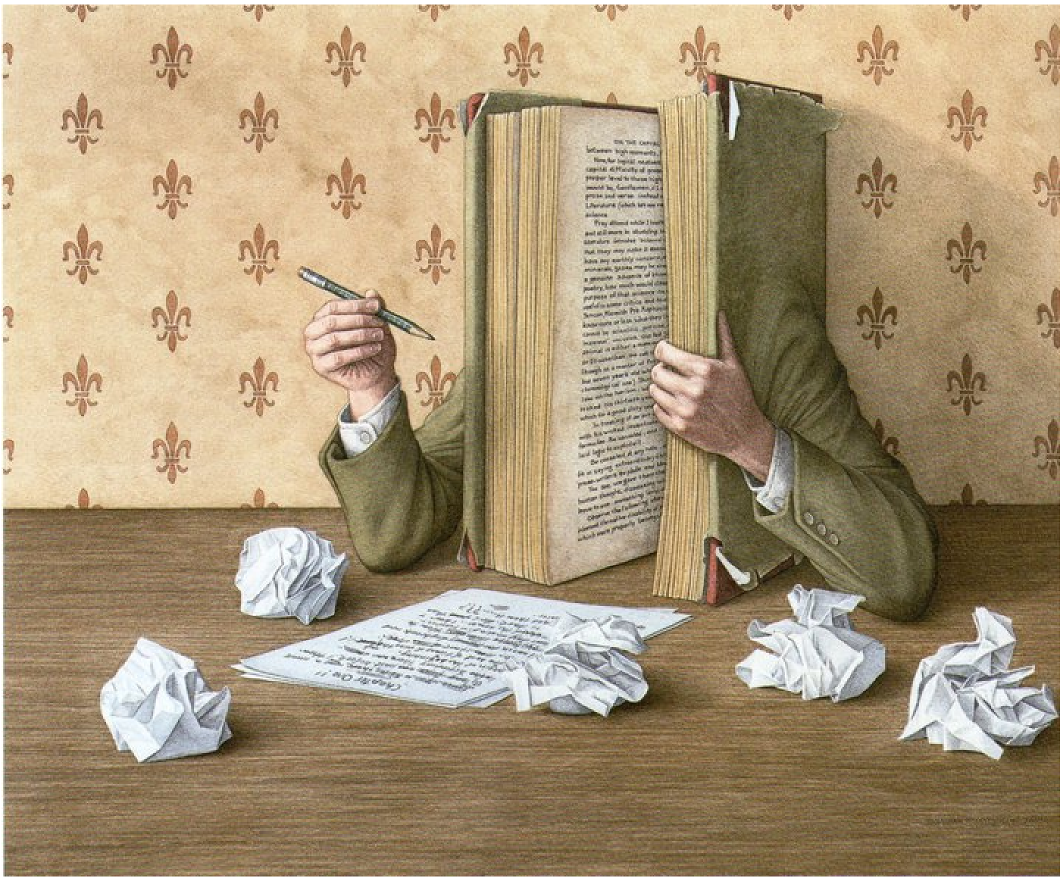

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Before we start

Let us first briefly describe the topic of these notes: we are concerned with actions of (some) Polish groups on (some) topological spaces. Polish groups are a class that generalizes locally compact metrizable groups, where Baire category notions can sometimes be used to help compensate the absence of a well-behaved σ-finite measure. We begin by covering some basic facts and methods involving Polish groups in the first chapter, then we discuss an important class of examples related to first-order logic: automorphism groups of countable (ultrahomogeneous) structures.

Next, we want to develop some general theory for actions of Polish groups on compact Hausdorff spaces, and for that we provide a brief introduction to the theory of uniform spaces (and uniform structures on topological groups) and compactifications in Chapters 3, 4 and 5. With those tools in hand we can establish in Chapter 6 the existence and uniqueness of the *universal minimal flow* $M(G)$ of a topological group G. This object captures some information about the continuous actions of G on compact Hausdorff spaces. In particular, $M(G)$ is trivial iff any continuous action of G on a compact space has a fixed point, a remarkable property known as *extreme amenability*. While this property may seem pathological (it is never satisfied by a nontrivial locally compact Hausdorff topological group), it turns out to hold for many large groups. As an application of our work, we discuss a famous theorem of Kechris–Pestov–Todorčevic which helps understand extreme amenability of automorphism groups of countable structures by relating it to a combinatorial property, the *Ramsey property*.

As the title indicates, these notes by no means provide a complete introduction to topological dynamics (nor do they aim to); I chose to cover the aforementioned subjects in order to brush on several areas of lively contemporary research, and give some tools to tackle the existing literature in these areas. My choices are entirely subjective and should not be taken as meaning anything beyond my hope that the reader will want to know more about the various topics covered.

One word of warning is in order: the reader will need to know some basic descriptive set theory (as usually covered in a first course on the topic) to follow some of the arguments. In particular, Baire category notions are used extensively throughout the text, as are some properties of Borel and analytic sets.

The text is sprinkled with exercises of varying difficulty (all of them are supposed to be feasible, and none of them is actually an open problem, however tempting that was for me). Working on at least some of them is highly recommended in order to get familiarized with the topics covered.

We should perhaps point out that throughout the notes we assume that the axiom of choice holds (in concrete applications one could certainly do with a much weaker axiom like dependent choice, but having the full axiom of choice at our disposal makes it easier to obtain general structural results).

Chapter 1

Polish groups and their actions

Definition 1.1. A *topological group* is a group $(G, \cdot)$ endowed with a topology τ for which the group operations $g \mapsto g^{-1}$ and $(g,h) \mapsto g \cdot h$ are continuous.

This could be stated more concisely by requiring $(g,h) \mapsto g \cdot h^{-1}$ to be continuous; in what follows we will simply write gh instead of $g \cdot h$ (and use $\cdot$ for group actions) since the group law should always be clear from the context. Similarly, when working with actions we will either write $g \cdot x$ or gx if there is no risk of ambiguity. We denote the neutral element of G either by 1 or 1_G.

Definition 1.2. A *Polish group* is a topological group whose underlying topology is Polish.

Example 1.3. Consider the group $\mathfrak{S}_\infty$ of all permutations of ω (the group law being given by composition of maps). See it as a subset of the Baire space ω^ω, and endow it with the induced topology. Explicitly, a basis of neighborhoods of the identity is given by the following clopen subsets (subgroups, actually):

$$U_F = \{\sigma \in \mathfrak{S}_\infty \colon \forall i \in F \ \sigma(i) = i\}$$

where F ranges over all finite subsets of ω.
A compatible distance for this topology is given by

$$d(\sigma, \tau) = \inf\left\{2^{-n} \colon \forall i < n \ \sigma(i) = \tau(i)\right\}$$

Exercise 1.

1. Show that the group operations on $\mathfrak{S}_\infty$ are continuous.
2. Use the sequence $(\sigma_i)_{i<\omega}$ defined by $\sigma_i(n) = n + 1$ for $n \leq i$, $\sigma_i(i+1) = 0$, $\sigma_i(n) = n$ for $n > i + 1$ to show that d is not complete.
3. Show however that the distance ρ defined by $\rho(\sigma, \tau) = d(\sigma, \tau) + d(\sigma^{-1}, \tau^{-1})$ is complete and conclude that $\mathfrak{S}_\infty$ is a Polish group.
4. Prove that $\mathfrak{S}_\infty$ is a G_δ subset of ω^ω to obtain another proof that $\mathfrak{S}_\infty$ is Polish.

To solve the last question of the previous exercise, one needs to know that a subset X of a completely metrizable topological space Y is completely metrizable for the induced topology iff X is a G_δ subset of Y. This is an important result, which we will use several times.

Exercise 2. Prove that a closed subgroup G of $\mathfrak{S}_\infty$ is compact if, and only if, the G-orbit of every element of ω is finite.

Remark 1.4. We see that the topology of $\mathfrak{S}_\infty$ is induced by a distance which is left-invariant but not complete (the distance d above) as well as a distance ρ which is complete but neither left- nor right-invariant. We will see shortly that this is part of a broader phenomenon.

Exercise 3. Prove that a (at most) countable product of Polish groups, endowed with the product topology, is a Polish group.

Clearly a closed subgroup of a Polish group is itself a Polish group; the class of all closed subgroups of $\mathfrak{S}_\infty$ is a rich topic of study, with a strong interaction with model theory.

Exercise 4. Let G be a Polish group. Show that G is isomorphic, as a topological group, to a closed subgroup of $\mathfrak{S}_\infty$ iff 1_G has a basis of neighborhoods consisting of open subgroups. Such Polish groups are called *nonarchimedean*.

Nonarchimedean Polish groups are those that we will be most interested in in these notes; but let us discuss briefly some other examples.

Proposition 1.5. *Let (X, d) be a Polish metric space, and G be its isometry group. Then G, endowed with the pointwise convergence topology, is a Polish group.*

Proof. As in the case of $\mathfrak{S}_\infty$ (which is the isometry group of ω endowed with the discrete metric) one can either argue by defining a complete metric on G, or by proving that it is a G_δ subset of a Polish space. We will use the second approach here (of course in both approaches one also needs to prove that group operations are continuous).
Fix a countable dense subset $(x_i)_{i<\omega}$ of X, then define $\Phi\colon G \to X^\omega$ by $\Phi(g)(i) = g(x_i)$.
By definition of the pointwise convergence topology (which is the topology on G induced by the product topology on X^X) the map Φ is continuous. It is also injective: if $f \neq g \in G$ then there exists $x \in X$ and $\varepsilon > 0$ such that $d(f(x), g(x)) > 2\varepsilon$, and then the triangle inequality implies that $f(x_i) \neq g(x_i)$ for any i satisfying $d(x_i, x) < \varepsilon$, whence $\Phi(f) \neq \Phi(g)$.
Then we note that Φ is a homeomorphism onto its image; let U be open in G, witout loss of generality we may assume that

$$U = \{g \in G \colon \forall a \in A\ d(g(a), f(a)) < \varepsilon_a\}$$

where $f \in G$, $A \subset X$ is finite and each ε_a is > 0.
We have to show that $\Phi(U)$ is open in $\Phi(G)$, to do that we first fix $g \in U$. Then let $\varepsilon > 0$ be such that $d(g(a), f(a)) + 3\varepsilon < \varepsilon_a$ for all $a \in A$. For each $a \in A$ pick i_a such that $d(x_{i_a}, a) < \varepsilon$. Any $h \in G$ such that $d(g(x_{i_a}), h(x_{i_a})) < \varepsilon$ for all $a \in A$ satisfies $d(h(a), f(a)) < \varepsilon_a$ for all $a \in A$, and this gives us an open neighborhood of $\Phi(g)$ contained in $\Phi(U)$, as desired.
What we have proved so far is that the pointwise topology on G is completely understood by looking only at countably many coordinates, hence is is metrizable (more formally, $\Phi(G)$ is metrizable since it is contained in the metrizable topological space X^ω). Recall that convergence of sequences for a product topology is easily described: $(g_n)_{n<\omega}$ converges to g if, and only if, $(g_n(x))_{n<\omega}$ converges to x for all $x \in X$.
Let us now check that the group operations are continuous. Assume that $(g_n)_{n<\omega}$ converges to g in G. Fix $a \in X$ then let $b = g^{-1}(a)$. Fix $\varepsilon > 0$. We have that $a_n = g_n(b)$ converges

to $g(b) = a$, so for n large enough $d(a_n, a) < \varepsilon$. Using that g_n^{-1} is an isometry we obtain that for n large enough $d(g_n^{-1}(a), b) < \varepsilon$, so $(g_n^{-1}(a))_{n<\omega}$ converges to b, and this proves that $g \mapsto g^{-1}$ is continuous. Continuity of $(g,h) \mapsto gh$ is easily proved using the triangle inequality, and we leave those details to the reader.
Finally, we note that $\Phi(G)$ is a G_δ subset of X^ω. Indeed, $f \in X^\omega$ belongs to $\Phi(G)$ if, and only if, it satisfies the following two conditions:

- For all $i, j \in \omega$ $d(f(i), f(j)) = d(x_i, x_j)$ (by completeness, a distance-preserving map defined on a dense subset extends to a distance-preserving map defined on the whole space)
- $\{f(i) \colon i \in \omega\}$ is dense in X (again using completeness, the image of a distance-preserving map is closed, so the map is surjective as soon as it is dense).

This yields the following description of $\Phi(G)$:

$$\Phi(G) = \bigcap_{i,j<\omega} \{f \colon d(f(i), f(j)) = d(x_i, x_j)\} \cap \bigcap_{i<\omega} \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{j<\omega} \{f \colon d(f(j), x_i) < \varepsilon\}$$

Since closed subsets of a metrizable space are G_δ, and a countable intersection of G_δ subsets is again G_δ, we conclude as expected that $\Phi(G)$ is G_δ in X^ω. □

We already mentioned that closed subgroups of Polish groups are themselves Polish groups when endowed with the induced topology; actually, the converse is also true.

Theorem 1.6. *Let G be a Polish group, and H be a Polish subgroup of G, i.e. a subgroup of G which is a Polish group when endowed with the induced topology. Then H is a closed subgroup of G.*

Proof. Since H is Polish, it is a G_δ subset of G. For any $g \in \overline{H}$, H and gH are then dense G_δ subsets of the Polish space $\overline{H}$. The Baire category theorem then implies that $H \cap gH \neq \emptyset$, whence $g \in H$ and it follows that $\overline{H} = H$. □

Definition 1.7. Let X be a Polish space. We say that $A \subseteq X$ is *Baire-measurable* if there exist an open subset O and a meager subset M of X such that $A = O \Delta M$.

We recall that Baire-measurable subsets of X form a σ-algebra; this σ-algebra is the smallest containing open subsets as well as meager sets, and it contains all Borel subsets of X (as well as the analytic and coanalytic subsets).
In contexts where there is no quasi-invariant σ-finite measure (as opposed to the case of locally compact groups which come endowed with the Haar measure) this is a very useful σ-algebra. The meager sets provide us a well-behaved notion of smallness, and the Kuratowski–Ulam theorem (which we discuss in some detail later) is a suitable analogue of the Fubini theorem, though of course the Baire-category theoretic notions are much less quantitative than their measured counterparts.
We recall that a Borel measure on a Polish group G is *left quasi-invariant* if for any g and any Borel A we have $\mu(A) = 0 \Leftrightarrow \mu(gA) = 0$.

Theorem 1.8 (Weil)**.** *Let G be a Polish group, and μ be a Borel σ-finite measure on G which is left quasi-invariant, i.e. such that for any Borel subset A such that $\mu(A) = 0$ one also has $\mu(gA) = 0$ for all $g \in G$. Then G is σ-compact (hence also locally compact since it is Polish).*

Proof. Let A be a Borel subset of G such that $0 < \mu(A) < +\infty$. There exists a compact subset $K \subseteq A$ such that $0 < \mu(K)$[(i)]. Let H be the subgroup of G which is generated by K; since $H = \bigcup_{n<\omega}(K \cup K^{-1})^n$ we have that H is σ-compact.
If H has uncountable index in G, there is an uncountable $A \subset G$ such that $aH \cap bH = \emptyset$ for each $a \neq b$ in A, hence also $aK \cap bK = \emptyset$ for every $a \neq b$. Note that for each $a \in A$ we have $\mu(aK) > 0$.
Since μ is assumed to be σ-finite, there exists a sequence $(B_n)_{n<\omega}$ of subsets of G such that each $\mu(B_n)$ is finite and $G = \bigcup_{n<\omega} B_n$.
For each n, the family $(\mu(B_n \cap aK))_{a\in A}$ is summable, whence for all n there are only at most countably many $a \in A$ such that $\mu(B_n \cap aK) > 0$. But then there are only at most countably many $a \in A$ such that $\mu(aK) > 0$, a contradiction.
Thus H has countable index in G; since H is σ-compact it follows that G is also σ-compact, hence locally compact (applying the Baire category theorem shows that some compact subset of G must have nonempty interior, so every element of G has a compact neighborhood). □

Definition 1.9. Let X be a Polish space, and A be a subset of X. We denote

$$U(A) = \bigcup \{O \text{ open in } X \colon O \setminus A \text{ is meager}\}$$

By definition, $U(A)$ is an open subset of X; using separability (reducing an union of an arbitrary number of open sets to a countable union) it is easy to see that A is always comeager in $U(A)$ (this is actually true in any topological space, though the proof is less direct), by which I mean that $U(A) \setminus A$ is meager. An important fact in descriptive set theory, which we admit here, is that a subset A of a Polish space X is Baire-measurable iff $A \setminus U(A)$ is meager.
Continuity of group operations implies that, for a Polish group G, a subset A of G and $g \in G$ we have $U(A^{-1}) = U(A)^{-1}$ as well as $U(gA) = gU(A)$.

Lemma 1.10 (Pettis). *Let G be a Polish group, and A, B be two subsets of G. Then $U(A)U(B) \subseteq AB$.*

Proof. Let g belong to $U(A)U(B)$. Then $gU(B)^{-1} \cap U(A) \neq \emptyset$, i.e. $U(gB^{-1}) \cap U(A) \neq \emptyset$. Let V denote the nonempty open subset $U(gB^{-1}) \cap U(A)$.
Since A is comeager in $U(A)$, and V is an open subset of $U(A)$, A is also comeager in V; similarly gB^{-1} is comeager in V. Applying the Baire category theorem in V, which is a Polish space for the induced topology, we obtain that $A \cap gB^{-1} \neq \emptyset$, equivalently $g \in AB$. This concludes the proof. □

Corollary. Let G be a Polish group and A a Baire-measurable, non-meager subset of G. Then 1 belongs to the interior of AA^{-1}.

Proof. This is immediate: since $U(A)$ is nonempty, $1 \in U(A)U(A)^{-1} = U(A)U(A^{-1})$. Pettis' lemma then gives $1 \in AA^{-1}$. □

Theorem 1.11 (Banach). *Let G, H be Polish groups and $\varphi \colon G \to H$ be a Baire-measurable group homomorphism. Then φ is continuous.*

(i) This follows from the inner regularity of finite Borel measures on Polish spaces; we omit the details.

Note that this implies that every Borel group homomorphism between Polish groups is measurable, since every Borel map is Baire-measurable.

Proof. It is enough to prove that φ is continuous at 1_G. Let V be an open neighborhood of 1_H, and fix an open neighborhood W of 1_H such that $WW^{-1} \subseteq V$ (continuity of the group operations, as well as the fact that $1_H 1_H = 1_H$ give us the existence of W).
Let $W' = \varphi(G) \cap W$, which is open in $\varphi(G)$. We have $\varphi(G) = \bigcup_{h \in \varphi(G)} hW'$ whence, since the topology of $\varphi(G)$ admits a countable basis, there exists a sequence $(h_n)_{n<\omega}$ of elements of $\varphi(G)$ such that $\varphi(G) = \bigcup_{n<\omega} h_n W'$ (see if necessary the exercise right after this proof).
Fix $g_n \in G$ such that $\varphi(g_n) = h_n$; we then have

$$G = \bigcup_{n<\omega} g_n \varphi^{-1}(W)$$

Since φ is Baire-measurable, $\varphi^{-1}(W)$ is Baire-measurable; and it is not meager since countably many translates of it cover G. By Pettis' lemma, there is an open neighborhood O of 1_G which is contained in $\varphi^{-1}(W)(\varphi^{-1}(W))^{-1}$. But then $\varphi(O)$ is contained in $WW^{-1} \subseteq V$. This proves that φ is continuous at 1_G, hence continuous outright. □

Exercise 5. Let X be a topological space whose topology admits a countable basis, and let $(O_i)_{i \in I}$ be family of open subsets of X. Prove that there is a (at most) countable subset J of I such that $\bigcup_{i \in I} O_i = \bigcup_{i \in J} O_i$.
(One says that X has the *Lindelöff property*)

Exercise 6. Assume that $\varphi \colon \mathbb{R} \to \mathbb{R}$ is Baire-measurable and satisfies $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists $\alpha \in \mathbb{R}$ such that $\varphi(x) = \alpha x$ for all $x \in \mathbb{R}$.

Exercise 7.

1. Let G be a Polish group and H a subgroup of G which is Baire-measurable and non-meager. Prove that H is clopen in G (i.e. both open and closed).

2. Let G, H be two Polish groups and $\varphi \colon G \to H$ a Borel map which is an isomorphism of abstract groups. Prove that φ is a topological group isomorphism (i.e. prove that φ is a homeomorphism).

3. Let G be a group, and τ_1, τ_2 two Polish group topologies on G such that $\tau_1 \subseteq \tau_2$. Prove that $\tau_1 = \tau_2$.

To solve the second question of the above exercise, it is useful to know that the inverse of a Borel bijection between two Polish spaces is Borel (a consequence of the separation theorem for analytic sets).

Exercise 8. Let G, H be two Polish groups, and $\varphi \colon G \to H$ a continuous surjective homomorphism. Prove that φ is an open map.

Let us now come back to the question of existence of metrics inducing the topology of a given Polish group and with additional properties such as left-invariance (or even invariance both on the left and on the right) or completeness.

Theorem 1.12 (Birkhoff–Kakutani)**.** *Let G be a Hausdorff topological group with a countable basis of neighborhoods of* 1*. Then there exists a left-invariant metric d inducing the topology of G.*

Since uniform structures provide a natural framework to prove this result (in a slightly more general version), we postpone the proof of this fact to chapter 4 where we discuss uniform structures on topological groups.
It follows in particular that any Polish group admits a left-invariant metric; however, it is often the case that there is no metric which is both complete and left-invariant. The group $\mathfrak{S}_\infty$ is an example, as follows from our earlier discussion combined with the following fact.

Proposition 1.13. *Let G be a metrizable topological group, and $(g_n)_{n<\omega}$ be a sequence of elements of G. Then $(g_n)_{n<\omega}$ is a Cauchy sequence for* some *compatible left-invariant distance on G iff $(g_n)_{n<\omega}$ is a Cauchy sequence for* any *compatible left-invariant distance on G.*
It follows that G admits a compatible left-invariant complete metric iff any left-invariant compatible metric on G is complete.

Proof. Fix some compatible left-invariant metric d on G; a sequence $(g_n)_{n<\omega}$ is Cauchy for d iff

$$\forall \varepsilon > 0\ \exists N < \omega\ \forall n, m \geq N \quad d(g_n, g_m) < \varepsilon$$

Using left-invariance, this is equivalent to

$$\forall \varepsilon > 0\ \exists N < \omega\ \forall n, m \geq N \quad d(g_m^{-1} g_n, 1) < \varepsilon$$

Let $(U_i)_{i<\omega}$ be a basis of neighborhoods of 1; the above property is equivalent to the following statement:

$$\forall i < \omega\ \exists N < \omega\ \forall n, m \geq N \quad g_m^{-1} g_n \in U_i$$

The previous line only depends on the topology of G, and not on the choice of left-invariant metric inducing it, which proves our claim. □

So any two left-invariant metrics on a Polish group G have the same Cauchy sequences; this suggests considering the completion of (G, d) for some left-invariant metric d on G. The group product extends to a continuous operation (this follows from the argument in the next proof), turning this object into a semigroup (well-worth studying in many cases!), however the inverse operation typically does not extend. It it interesting already to understand what happens in the case of $\mathfrak{S}_\infty$ and give an explicit description of this left-completion of $\mathfrak{S}_\infty$ (see exercise 17 in the next chapter for a more general statement).

Theorem 1.14. *Let G be metrizable topological group, and d a left-invariant metric inducing the topology of G. Define a new compatible distance ρ on G by setting $\rho(g,h) = d(g,h) + d(g^{-1}, h^{-1})$. Denote by $(\hat{G}, \rho)$ the completion of (G, ρ).*
The group operations on G extend to continuous operations on $\hat{G}$, and $(\hat{G}, \rho)$ is a topological group.

Proof. The map $g \mapsto g^{-1}$ is a distance-preserving map from (G, ρ) to itself, thus it extends to a distance-preserving map from $(\hat{G}, \rho)$ to itself. Its image is dense since it contains G, and is closed since $\hat{G}$ is complete, so the inverse map extends to an isometry of $\hat{G}$.
Let us prove that $(g, h) \mapsto gh$ also extends continuously; for this it is enough to prove that if $(g_n)_{n<\omega}$ and $(h_n)_{n<\omega}$ are Cauchy sequences in (G, ρ) then $(g_n h_n)_{n<\omega}$ is also Cauchy in (G, ρ), and actually because of the definition of ρ it is enough to prove that $(g_n h_n)_{n<\omega}$ is Cauchy in (G, d) (since one can then apply this fact to $h_n^{-1} g_n^{-1}$).

We prove the slightly stronger fact that $(g_n h_n)_{n<\omega}$ is d-Cauchy as soon as $(g_n)_{n<\omega}$ and $(h_n)_{n<\omega}$ are both d-Cauchy[(i)]. Note that for all $n, m, p < \omega$ we have

$$\begin{aligned} d(g_n h_n, g_m h_m) &\leq d(g_n h_n, g_n h_p) + d(g_n h_p, g_m h_p) + d(g_m h_p, g_m h_m) \\ &\leq d(h_n, h_p) + d(g_n h_p, g_m h_p) + d(h_p, h_m) \end{aligned}$$

Fix $\varepsilon > 0$, then $p \in \mathbb{N}$ such that for any $n \geq p$ one has $d(h_p, h_n) \leq \varepsilon$.
The map $g \mapsto g h_p$ is continuous at 1, hence there exists $\delta > 0$ such that

$$d(g, 1) \leq \delta \Rightarrow d(g h_p, h_p) \leq \varepsilon$$

Since $(g_n)_{n\in\mathbb{N}}$ is d-Cauchy there exists $N \geq p$ such that for any $n, m \geq N$ we have $d(g_n, g_m) = d(g_m^{-1} g_n, 1) \leq \delta$.
Then $d(g_n h_p, g_m h_p) = d({g_m}^{-1} g_n h_p, h_p) \leq \varepsilon$, and it follows that for any $n, m \geq N$ we have $d(g_n h_n, g_m h_m) \leq 3\varepsilon$.
So the map $(g, h) \mapsto gh$ extends continuously to $\hat{G} \times \hat{G}$; associativity of this binary operation on G, allied with continuity and density, gives us associativity of the operation on $\hat{G}$. The equality $1g = g$ for all $g \in G$ also extends to $\hat{G}$; similarly the equality $gg^{-1} = 1 = g^{-1}g$ extends to $\hat{G}$, proving that every element of $\hat{G}$ has an inverse.
Thus $(\hat{G}, \rho)$ is a topological group. □

Corollary. Let G be a Polish group, and d be a compatible left-invariant metric on G. Then the distance ρ defined by $\rho(g, h) = d(g, h) + d(g^{-1}, h^{-1})$ is a compatible complete metric on G.

Proof. It is clear that ρ is a compatible metric on G since the inverse map is continuous. But then G is a Polish subgroup of $(\hat{G}, \rho)$, hence G is closed in $\hat{G}$, which is only possible if $G = \hat{G}$. □

Note that any locally compact Polish group admits a compatible metric which is both left-invariant and proper, i.e. such that closed balls are compact; such a metric is automatically complete. So the lack of compatible complete left-invariant metric is again a phenomenon which happens only in "large" Polish groups.
Now we want to understand how to form quotients of Polish groups. For that, we recall the following classical result, variously attributed to Sierpinski or Hausdorff.

Theorem 1.15. *Let X be a Polish space, Y a metrizable space and $f : X \to Y$ a continuous, surjective and open map. Then Y is Polish.*

The constructions in the rest of this chapter would be more naturally written down using uniform structures, and should probably be revisited once we have covered that topic later on. We still work trough them now using metrics as a way to prepare ourselves for the use of uniformities later.

Definition 1.16. Let G be a Polish group, and let d be a right-invariant compatible metric ρ on G. Let H be a closed subgroup of G. We endow G/H with the metric

$$d(fH, gH) = \inf_{h_1, h_2 \in H} \rho(fh_1, gh_2) = \inf_{h \in H} \rho(fh, g)$$

and denote by $\pi : G \to G/H$ the natural surjection $g \mapsto gH$.

[(i)]Note however that in general $(g, h) \mapsto gh$ is not d-uniformly continuous!

Exercise 9. Show that d is indeed a metric, and that d induces the quotient topology on G/H. (since π is clearly continous, the last statement amounts to claiming that π is open).

Then $(G/H, d)$ is metrizable and is an open image of a Polish space, thus it is Polish in its own right.

Theorem 1.17. *Let G be a Polish group, and H a closed normal subgroup of G. Then G/H, endowed with the quotient topology, is a Polish group.*

Proof. We have to prove that the group operations are continous on G/H. Let $(f_n)_{n<\omega}$ and $(g_n)_{n<\omega}$ be such that $f_nH \to fH$ and $g_nH \to gH$ in $(G/H, d)$.
By definition of d, we can find sequences $(h_n)_{n<\omega}$ and $(\tilde{h}_n)_{n<\omega}$ in H such that f_nh_n converges to f and $g_n\tilde{h}_n$ converges to g in G.
Then $(f_nh_n)(g_n\tilde{h}_n)^{-1}$ converges to fg^{-1}; but we have

$$(f_nh_n)(g_n\tilde{h}_n)^{-1} = f_nh_n\tilde{h}_n^{-1}g_n^{-1} = f_ng_n^{-1}(g_nh_n\tilde{h}_n^{-1}g_n^{-1})$$

Since $g_nh_n\tilde{h}_n^{-1}g_n^{-1}$ belongs to H for all $n < \omega$, we conclude that $f_ng_n^{-1}H = (f_nH)(g_nH)^{-1}$ converges to $fg^{-1}H$.
This establishes continuity of $(fH, gH) \mapsto (fH)(gH)^{-1}$, and we are done. □

Corollary. Let G, H be two Polish groups and $\varphi \colon G \to H$ a surjective homomorphism. Then φ induces an isomorphism of topological groups from $G/\ker\varphi$ onto H.

Proof. By definition of the quotient topology, φ induces a continuous injective morphism $\tilde{\varphi}$ from $G/\ker\varphi$, which is onto since φ is onto. Thus $\tilde{\varphi}$ is an isomorphism of abstract groups between two Polish groups which is continuous, and we saw in an earlier exercise that this implies that $\tilde{\varphi}$ is an isomorphism of topological groups. □

Of course, now that we have singled out a class of groups we are interested in, we want to make them act on structures; in these notes the main focus will be continuous actions on compact Hausdorff spaces but many other examples are of interest, such as the diagonal conjugation action of G on G^n given by $g \mapsto (g_1, \ldots, g_n) = (gg_1g^{-1}, \ldots, gg_ng^{-1})$, unitary representations, measure-preserving actions, actions by permutations on countable sets...

Exercise 10. Let X be a Polish space, and G be a group of homeomorphisms of X. Prove that the following conditions are equivalent:

(i) There exists $x \in X$ with a dense orbit.

(ii) For any two nonempty open sets U, V of X there exists $g \in G$ such that $gU \cap V \neq \emptyset$.

If these two equivalent conditions are satisfied, one says that G acts *topologically transitively*.

Exercise 11 (The first $0-1$ topological law)**.** Let X be a Polish space, and G be a group of homeomorphisms acting on X topologically transitively. Prove that any subset of X which is both Baire-measurable and G-invariant is either meager or comeager.

In the previous exercise we did not even require the action to be jointly continuous (only that each $x \mapsto gx$ is continuous) but that is what we will require most of the time.

Definition 1.18. Let G be a topological group acting on a topological space X. We say that the action is *continuous* if $(g, x) \mapsto gx$ is continuous.

Note that, if a Polish group G acts on X continuously and topologically transitively, then each G-orbit is Baire-measurable (it is clearly analytic; actually it is even Borel, see below) thus each orbit is either meager or comeager. Since distinct orbits are disjoint, there can only exist at most one comeager orbit.
We now turn to discussing some related questions; even though we will not need it, we note the following fact.

Theorem 1.19 (Miller). *Let X be a Polish space, and G a Polish group acting in a Borel way on X (i.e. $(g, x) \mapsto gx$ is Borel). Then for any x in X the stabilizer G_x is a closed subgroup of G, whence the orbit Gx is Borel.*

Proof. Begin by fixing $x \in X$. Once we have proved that G_x is closed, we are done: indeed G/G_x is a Polish space, and the map $g \mapsto gx$ induces an injective Borel map from G/G_x to Gx, so Gx is Borel as an injective Borel image of a Polish space.
Without loss of generality, we may assume that G_x is dense in G; since G_x is Borel, hence Baire-measurable, Pettis' lemma ensures that $G_x = G$ as soon as G_x is non meager. Let us assume for a contradiction that G_x is meager; since G_x is dense in G an application of the $0-1$ topological law tells us that each Baire-measurable $A \subseteq G$ such that $AG_x = A$ is either meager or comeager (the action of G_x on G by right translation is topologically transitive).
Let us now fix a countable basis $(U_n)_{n<\omega}$ for the topology of X, and consider

$$A_n = \{g \in G \colon gx \in U_n\}$$

Each A_n is Borel, and either meager or comeager by the remark above since $A_n G_x = A_n$. Further, for each $g \in G$ we have

$$gG_x = \{h \colon hx = gx\} = \bigcap_{\{n \colon g \in A_n\}} A_n$$

Since gG_x is meager, it follows that for each g there exists some meager A_n containing g. But then G is contained in a countable union of meager sets, contradicting the Baire category theorem. □

Let us now give a criterion for the existence of a comeager orbit.

Lemma 1.20 (Rosendal). *Assume that G is a Polish group acting continuously and topologically transitively on a Polish space X. Then the following conditions are equivalent:*

(i) There exists a comeager orbit.

(ii) For any nonempty open subset V of G, the set $\{x \in X \colon Vx$ is somewhere dense$\}$ is dense in X.

(iii) For any open $V \ni 1$ in G and any nonempty open subset U of X there exists a nonempty open $U' \subseteq U$ such that for every nonempty open $W_1, W_2 \subseteq U'$ one has $VW_1 \cap W_2 \neq \emptyset$.

Proof. Assume that Gx is comeager, and let $V \subseteq G$ be nonempty open. Then there exist $(g_n)_{n<\omega}$ in G such that $G = \bigcup_n g_n V$, so $Gx = \bigcup_n g_n Vx$, whence Vx is not meager and thus somewhere dense.
Assume that the second condition above holds, and fix $1 \in V \subseteq G$ open and $U \subseteq X$ nonempty open. Using continuity of the action, we may find an open neighborhood $\tilde{V}$ of 1 contained in V and a nonempty $\tilde{U} \subseteq U$ such that $\tilde{V}\tilde{U} \subseteq U$.
Pick some $1 \in V_1 \subseteq \tilde{V}$ symmetric open such that $V_1 V_1 \subseteq \tilde{V}$. There exists $x \in \tilde{U}$ such that $\overline{V_1 x}$ has nonempty interior, and $V_1 x \subseteq U$ so its closure contains some nonempty open $U' \subseteq U$. Pick W_1, W_2 nonempty open and contained in U'. There exists $g, h \in V_1$ such that $gx \in W_1$ and $hx \in W_2$, whence $hg^{-1}W_1 \cap W_2 \neq \emptyset$. Since $hg^{-1} \in V_1 V_1^{-1} = V_1 V_1 \subseteq \tilde{V}$, we have proved the second implication.
Finally, assume that the first item above is false, i.e. there is no comeager orbit. Since the action of G is topologically transitive, all orbits are meager. For every $x \in X$, there exists a countable family $(F_n)_{n<\omega}$ of closed subsets with empty interior such that $Gx \subseteq \bigcup_n F_n$. For some n the set $\{g \colon gx \in F_n\}$ must have nonempty interior in G by Baire's theorem, so there exists some nonempty open subset O of G such that Ox is nowhere dense.
Translating O if necessary, we obtain that for all x there exists an open $V \ni 1$ such that Vx is nowhere dense. We may restrict V to range over some fixed countable basis of neighborhoods $\mathcal{V}$ of 1, and we have obtained

$$X = \bigcup_{V \in \mathcal{V}} \{x \colon Vx \text{ is nowhere dense}\}$$

Applying Baire's theorem again, there exists some V in $\mathcal{V}$ such that $\{x \colon Vx \text{ is nowhere dense}\}$ is not meagre; this set is Borel so it must be comeager in some nonempty open subset U of X.
Assume that the third condition above holds, and apply this to V and U to find a nonempty U' witnessing that. We have that $\{x \in U' \colon Vx \cap W \neq \emptyset\}$ is dense open for any nonempty open W in U', whence for a generic element of U' the closure of Vx contains U'. But for a generic element of U (hence also of U') the closure of Vx is nowhere dense. This yields the desired contradiction. □

This theorem enables us to detect whether there exists a comeager orbit without knowing a priori which x is such that Gx is comeager. If we want to understand whether a given orbit Gx is comeager, or use to our advantage the fact that it is comeager, then the next theorem is very useful.

Theorem 1.21 (Effros). *Let G be a Polish group acting continuously on a Polish space X. For every $x \in X$ the following conditions are equivalent:*

(i) The map $g \mapsto gx$, from G to Gx, is open.

(ii) Gx is a G_δ subset of X.

(iii) Gx is nonmeager in its relative topology.

(iv) Gx is comeager in $\overline{Gx}$.

Proof. If $g \mapsto gx$ is open then Gx is a metrizable, continuous open image of a Polish space hence it is Polish, thus G_δ in X. Similarly, if Gx is G_δ then it is itself Polish, thus nonmeager in its relative topology.
If Gx is nonmeager in its relative topology then it is nonmeager in $\overline{Gx}$, so it is comeager since the left-translation action of G on $\overline{Gx}$ is topologically transitive.
The only remaining thing to prove is that if Gx is comeager in $\overline{Gx} =: Y$ then $g \mapsto gx$ is open from G to Gx.
Denote by G_x the stabilizer of x. The orbit map $\varphi\colon G \to Gx$ is continuous, and induces an injective continuous map $\tilde{\varphi}$ from the Polish space G/G_x to Gx. Thus Gx is Borel in Y (we already knew that) and the map $\psi\colon gx \mapsto gG_x$ is Borel. We want to prove that ψ is actually continuous (this immediately implies the desired result since the quotient map from G to G/G_x is open by definition of the quotient topology).
Since ψ is Borel, there is a dense G_δ subset Ω of Y such that ψ is continuous on $\Omega \cap Gx$ (extend ψ to be constant on $Y \setminus Gx$, then use the general fact that for any Borel map f between Polish spaces there exists a dense G_δ set on which the restriction of f is continuous).
Using as usual the notation $\forall^* x P(x)$ to mean that $\{x\colon P(x) \text{ is true}\}$ is comeager, we have

$$\forall g \in G\, \forall^* y \in Y\; gy \in \Omega$$

The Kuratowski–Ulam theorem (which we discuss after this proof) then gives

$$\forall^* y \in Y\, \forall^* g \in G\; gy \in \Omega$$

In other words, the set $\Sigma = \{y \in Y\colon \forall^* g \in G\; gy \in \Omega\}$ is comeager in Y, and this set is G-invariant by definition. Since Gx is nonmeager it must intersect Σ, thus be contained in Σ by G-invariance of Σ.
Now, let $(y_n)_{n<\omega}$ be a sequence of elements of Y which converges to some $y \in Y$. For each n the set $\{g\colon gy_n \in \Omega\}$ is comeager in G, as is $\{g\colon gy \in \Omega\}$. Hence there exists $g \in G$ such that $gy_n \in \Omega$ for all n and $gy \in \Omega$. Since ψ is continuous on $\Omega \cap Y$, we conclude that $\psi(gy_n)$ converges to $\psi(gy)$ in G/G_x. Since ψ is G-equivariant we conclude as desired that $\psi(y_n)$ converges to $\psi(y)$. □

To end this chapter, we pursue the analogy between Baire category and measure and prove a useful generalization of the Kuratowski–Ulam theorem. First we disuss a notion which provides an analogue of measure-preserving maps.

Definition 1.22. Let $f\colon X \to Y$ be Polish spaces and $f\colon X \to Y$ a continuous map. We say that $x \in X$ is *locally dense* for f if for every neighborhood U of x the set $\overline{f(U)}$ is a neighborhood of $f(x)$.

Exercise 12 (King). Let $f\colon X \to Y$ be Polish spaces and $f\colon X \to Y$ a continuous map.

(i) For each $r > 0$ set $U_r = \left\{x \in X\colon \exists \varepsilon < r\; f(z) \in \mathrm{Int}\overline{f(B(z,\varepsilon))}\right\}$ where $B(z,\varepsilon)$ is the open ball of radius $\varepsilon > 0$ for some (fixed) compatible metric on X.

Show that each U_r is open.

(ii) Prove that $\bigcap_{r>0} U_r$ coincides with the set of points of local density for f. Hence this set is G_δ in X.

Lemma 1.23 ("Dougherty's lemma"). *Assume X, Y are Polish spaces, $f\colon X \to Y$ is continuous and the set of points which are locally dense for f is dense in X. Then $f(X)$ is not meager.*

Proof. Assume for a contradiction that points of local density are dense and $f(X) \subseteq \bigcup_n F_n$, with each F_n a closed subset of Y with empty interior. Since each $f^{-1}(F_n)$ is closed and those sets cover X, the Baire category theorem assures us that for some n there is a nonempty open U contained in $f^{-1}(F_n)$. But then U must contain a point of local density for f, so $\overline{f(U)} \subseteq F_n$ is a neighborhood of $f(x)$, contradicting the fact that F_n has empty interior. □

This notion is often used to prove that maps do not have a meager image; of course one does not need points of local density to be dense for the image to be non meager, but the next lemma proves that those points do have to exist.

Lemma 1.24. *Let $f\colon X \to Y$ be a continuous map between Polish spaces, and let $A \subseteq X$ be the set of points which are not locally dense for f. Then $f(A)$ is meager.*

Proof. Fix a basis $(U_n)_{n<\omega}$ for the topology of Y. For every $x \in A$ there exists n such that $x \in U_n$ and $\overline{f(U_n)}$ is not a neighborhood of $f(x)$. Hence $f(x) \in \overline{f(U_n)} \setminus \mathrm{Int}(\overline{f(U_n)})$. Each $F_n = \overline{f(U_n)} \setminus \mathrm{Int}(\overline{f(U_n)})$ is closed, has empty interior, and we have shown that $f(A)$ is contained in $\bigcup_n F_n$. □

Exercise 13. Let X, Y be two Polish spaces, and $f\colon X \to Y$ be continuous. Show that the following conditions are equivalent:

(i) For every meager $A \subseteq Y$, $f^{-1}(A)$ is meager.

(ii) For every comeager $A \subseteq Y$, $f^{-1}(A)$ is comeager.

(iii) For every dense open $A \subseteq Y$, $f^{-1}(A)$ is dense.

(iv) For every nonempty open $U \subseteq X$, $f(U)$ is not meager.

(v) For every nonempty open $U \subseteq X$, $f(U)$ is somewhere dense.

As an example, every continuous open map satisfies the previous conditions.

Definition 1.25. Let X, Y be Polish spaces and $f\colon X \to Y$ be a continuous map. We say that f is *category-preserving* if f satisfies one of the equivalent conditions in the previous exercise.

Proposition 1.26. *Assume that X, Y are Polish spaces and $f\colon X \to Y$ is continuous. Then f is category preserving iff the set of points which are locally dense for f is dense in X.*

Proof. Assume that points of local density are dense. Then any nonempty open U contains a point of local density, so $f(U)$ is somewhere dense. Hence f is category preserving.
Conversely, Assume that $U \subseteq X$ is nonempty, open, and does not contain any point of local density for f. Then by Lemma 1.24 $f(U)$ is meager, so f is not category-preserving. □

Lemma 1.27. *Let X, Y be Polish spaces and $f\colon X \to Y$ be a continuous map. Then there exists a dense G_δ subset A of Y such that $f\colon f^{-1}(A) \to A$ is open.*
(Of course it could happen that $f^{-1}(A)$ is empty)

Proof. Fix a countable basis $(U_n)_{n<\omega}$ for the topology of X. Each $f(U_n)$ is analytic, so we may find an open O_n and a meager M_n in Y such that $f(U_n) = O_n \Delta M_n$.
Let $B = Y \setminus \bigcup_n M_n$. Since B is comeager, it contains a dense G_δ subset A. By construction, for every n we have $f(U_n \cap f^{-1}(A)) = O_n \cap A$, so each $f(U_n \cap f^{-1}(A))$ is open in A. This implies that for every open $U \subseteq X$ the set $f(U \cap f^{-1}(A))$ is open in A. □

We conclude this chapter by proving a generalization of the Kuratowski–Ulam theorem, which corresponds to the case where f below is the projection map from $X = X_1 \times X_2$ to $Y = X_1$; note that projection maps are continuous and open by definition of the product topology, hence category-preserving.
The Kuratowski–Ulam theorem is the statement that, for a Baire-measurable $\Omega \subseteq X_1 \times X_2$ one has

$$(\forall^*(x_1, x_2)\, \Omega(x_1, x_2)) \Leftrightarrow (\forall^* x_1\, \forall^* x_2\, \Omega(x_1, x_2))$$

This is the analogue, for Baire category, of the Fubini theorem.

Theorem 1.28. *Let X, Y be Polish spaces and $f\colon X \to Y$ be continuous and category-preserving. Assume that $\Omega \subseteq X$ is Baire measurable. The following assertions are equivalent:*

(i) *Ω is comeager in X.*

(ii) $\Big\{y\colon \Omega \cap f^{-1}(\{y\}) \text{ is comeager in } f^{-1}(\{y\})\Big\}$ *is comeager in Y.*

Using category quanfifiers:

$$(\forall^* x \in X\, \Omega(x)) \Leftrightarrow (\forall^* y \in Y\, \forall^* x \in f^{-1}(\{y\})\, \Omega(x))$$

Proof. We give the proof for f open, and leave the general case as an exercise.
We begin by proving (i) ⇒ (ii). By Baire's theorem, it is enough to prove that implication for Ω dense and open in X.
So we assume that Ω is dense open in X. Since each $\Omega \cap f^{-1}(\{y\})$ is then open in $f^{-1}(\{y\})$, it is enough to prove that

$$\forall^* y \in Y\, \Omega \cap f^{-1}(\{y\}) \text{ is dense in } f^{-1}(\{y\})$$

Fix a countable basis $(U_n)_{n<\omega}$ for the topology of X. The previous condition is equivalent to

$$\forall^* y \in Y\, \forall n\, (f^{-1}(\{y\}) \cap U_n \neq \emptyset) \Rightarrow (\Omega \cap f^{-1}(\{y\}) \cap U_n) \neq \emptyset$$

Applying the Baire category theorem, this in turn amounts to

$$\forall n\, \forall^* y \in Y\, (f^{-1}(\{y\}) \cap U_n \neq \emptyset) \Rightarrow (\Omega \cap f^{-1}(\{y\}) \cap U_n) \neq \emptyset$$

Fix $n < \omega$, and denote

$$A_n = \Big\{y\colon (f^{-1}(\{y\}) \cap U_n \neq \emptyset) \Rightarrow (\Omega \cap f^{-1}(\{y\}) \cap U_n) \neq \emptyset\Big\}$$

Since f is open, A_n is the intersection of an open set and a closed set, so it is G_δ. Our aim is to prove that it is comeager, so our job amounts to proving that each A_n is dense.

Pick a nonempty open V in Y. If $V \not\subseteq f(U_n)$ then it is immediate that V meets A_n (any element not in $f(U_n)$ belongs to A_n) so we may assume $V \subseteq f(U_n)$.
Then $f^{-1}(V) \cap U_n$ is non-empty open, so $f^{-1}(V) \cap U_n \cap \Omega$ is nonempty. Pick some element $x \in f^{-1}(V) \cap U_n \cap \Omega$ and let $y = f(x)$. Then $y \in A_n \cap V$, proving that A_n is dense.

To prove (ii) $\Rightarrow$ (i), assume for a contradiction that Ω is Baire measurable, satisfies the condition of (ii) but is not comeager. Then there exists a nonempty open O in X such that $\Omega \cap O$ is meager. Applying (i) $\Rightarrow$ (ii) in the Polish space O, we obtain

$$\forall^* y \in f(O) \quad \Omega \cap O \cap f^{-1}(\{y\}) \text{ is meager in } f^{-1}(\{y\})$$

But (ii) and the openness of $f(O)$ imply that

$$\forall^* y \in f(O) \quad \Omega \cap f^{-1}(\{y\}) \text{ is comeager in } f^{-1}(\{y\})$$

So there exists $y \in f(O)$ such that $\Omega \cap O \cap f^{-1}(\{y\})$ is both meager and comeager in $f^{-1}(\{y\})$, a contradiction. □

Exercise 14. Complete the proof of Theorem 1.28.
(Hint: combine the result for open f with the existence of a dense G_δ subset A of Y such that $f\colon f^{-1}(A) \to A$ is open)

Bibliographical comments. The material discussed in this chapter is fairly standard, except for the discussion at the end about points of local density and category-preserving maps. For information about Polish spaces and groups, standard references include [Kec95], [Gao09], [BK96]. The first two references also include a discussion of the descriptive-set-theoretic theorems that we used without proof; [Kec95] is a definitive reference on classical descriptive set theory. The notion of category-preserving maps comes from [MT13] (though equivalent notions were considered earlier in various articles) and the fact that points of local density are G_δ is in [Kin00].

Chapter 2

Fraïssé limits and their automorphism groups

We briefly review some model-theoretic vocabulary, since countable first-order structures are a rich source of Polish groups to study.

Definition 2.1. A *language* consists of the following data:

- A set of constant symbols $(c_i)_{i \in I}$.
- A set of relational symbols $(R_j)_{j \in J}$ of arity $n_j \in \omega \setminus \{0\}$.
- A set of function symbols $(f_k)_{k \in K}$ of arity $m_k \in \omega \setminus \{0\}$.

The language is said to be *countable* if I, J and K are each (at most) countable.

Definition 2.2. Given some language $\mathcal{L}$ as above, a $\mathcal{L}$-structure $\mathbf{M}$ is a set M with:

- For each $i \in I$ some element $c_i^{\mathbf{M}}$ of M.
- For each $j \in J$ some subset $R_j^{\mathbf{M}}$ of M^{n_j}.
- For each $k \in K$ some function $f_k^{\mathbf{M}}$ from M^{m_k} to M.

Since there should be little risk of confusion we will usually omit the superscript $^{\mathbf{M}}$ in our notations, and for instance both use c_i for the constant symbol of the language and its interpretation in the structure we are working with.
The structure $\mathbf{M}$ is *countable* if the underlying set M is (at most) countable; we will only work with countable languages and structures. We always assume that our languages contain a special binary symbol $=$, which is always interpreted by the equality on M; most of the time we will not bother mentioning this symbol but it is always there (for instance in the examples below).

Example 2.3.

- We may view pure sets as structures in the language containing only $=$.
- Using an additional binary relational symbol, we can consider ordered sets, graphs...
- Using a binary functional symbol $\cdot$ as well as a constant symbol e, we may consider the class of groups (with e being interpreted by the neutral element)

- The language $(0, 1, +, \cdot)$ is well-suited to study rings and fields (and for fields one might be tempted to add a unary functional symbols for multiplicative inverses).
- The language $(0, 1, \wedge, \vee, {}^c)$ is used to study Boolean algebras.

We have a natural notion of *substructure* of a structure $\mathbf{M}$: a subset N containing each c_i and such that for every $k \in K$ and every $\bar{x} = (x_1, \dots, x_{m_k}) \in N^{m_k}$ we have $f_k(x_1, \dots, x_{m_k}) \in N$. Then one turns N into an $\mathcal{L}$-structure $\mathbf{N}$ by restricting the relations and functions of $\mathcal{L}$ to N. Note that the language employed has an influence on the notion of substructure (with our choices of language above to talk about groups a substructure of a group is not necessarily a subgroup; to address this we might want to add a symbol for the inverse map in our language)

Given an $\mathcal{L}$-structure $\mathbf{M}$ and some subset A of M, the structure $\langle A \rangle$ *generated by* A is the smallest substructure of $\mathbf{M}$ containing A.

We are not going to do any model theory with our structures; our concern will be with their automorphism groups (we should note here that sometimes different structures may induce the same automorphism group).

Given $\varphi \colon M \to N$ and $\bar{x} = (x_1, \dots, x_p) \in M^p$ we denote $\varphi(\bar{x}) = (\varphi(x_1), \dots, \varphi(x_p))$.

Definition 2.4. Let $\mathbf{M}$, $\mathbf{N}$ be two $\mathcal{L}$-structures. A map $\varphi \colon M \to N$ is a *morphism* if:

- $\forall i \in I \; f(c_i^{\mathbf{M}}) = c_i^{\mathbf{N}}$.
- $\forall j \in J \; \forall \bar{x} \in M^{n_j} \quad \bar{x} \in R_j^{\mathbf{M}} \Rightarrow \varphi(\bar{x}) \in R_j^{\mathbf{N}}$ (we say that φ is an *embedding* if this implication is an equivalence; embeddings are injective since $x = y \Leftrightarrow \varphi(x) = \varphi(y)$).
- $\forall k \in K \; \forall \bar{x} \in M^{m_k} \quad f^{\mathbf{N}}(\varphi(\bar{x})) = \varphi(f^{\mathbf{M}}(\bar{x}))$.

Definition 2.5. Let $\mathbf{M}$, $\mathbf{N}$ be $\mathcal{L}$-structures. An *isomorphism* is a surjective embedding $f \colon \mathbf{M} \to \mathbf{N}$ (and then f^{-1} is an isomorphism from $\mathbf{N}$ to $\mathbf{M}$).

An *automorphism* of $\mathbf{M}$ is an isomorphism from $\mathbf{M}$ to $\mathbf{M}$.

The automorphisms of $\mathbf{M}$ form a group which we denote $\mathrm{Aut}(\mathbf{M})$.

The universe of $\mathbf{M}$ will almost always be countably infinite, and then we assume it is ω and view $\mathrm{Aut}(\mathbf{M})$ as a subgroup of $\mathfrak{S}_\infty$.

Proposition 2.6. *Let $\mathbf{M}$ be a $\mathcal{L}$-structure with universe ω. Then $\mathrm{Aut}(\mathbf{M})$ is a closed subgroup of $\mathfrak{S}_\infty$. Conversely, every closed subgroup of $\mathfrak{S}_\infty$ is of the form $\mathrm{Aut}(\mathbf{M})$ for some structure with universe ω in some countable relational language.*

Proof. Assume that $g \in \mathfrak{S}_\infty \setminus \mathrm{Aut}(\mathbf{M})$. Then:

- Either $g(c_i) \neq c_i$ for some $i \in I$, and then any h such that $h(c_i) = g(c_i)$ does not belong to $\mathrm{Aut}(\mathbf{M})$;
- Or for some $j \in J$ and some $\bar{x} \in \omega^{n_j}$ we have $\big(\bar{x} \in R_j\big) \not\Leftrightarrow \big(g(\bar{x}) \in R_j\big)$, and then any h such that $h(\bar{x}) = g(\bar{x})$ does not belong to $\mathrm{Aut}(\mathbf{M})$;
- Or for some $k \in K$ and some $\bar{x} \in \omega^{m_k}$ we have $f_k(g(\bar{x})) \neq g(f_k(\bar{x}))$ and then any h such that $h(\bar{x}) = g(\bar{x})$ does not belong to $\mathrm{Aut}(\mathbf{M})$.

In each of the cases above, we found an open subset of $\mathfrak{S}_\infty$ which contains g and has an empty intersection with $\mathrm{Aut}(\mathbf{M})$. This proves that $\mathrm{Aut}(\mathbf{M})$ is closed.
Next, let G be a closed subgroup of $\mathfrak{S}_\infty$. For every k, consider the diagonal action $G \curvearrowright \omega^k$ and let J_k denote the set of orbits for this action. We may then form a countable relational language, with a k-ary relational symbol R_O for each $O \in J_k$, and consider the $\mathcal{L}$-structure$\mathbf{M}$ with universe ω where each R_O is interpreted by O (namely $\bar{x} \in R_O^{\mathbf{M}} \Leftrightarrow \bar{x} \in O$).
By definition, we have $G \leq \mathrm{Aut}(\mathbf{M})$. To see the converse, pick some $h \in \mathrm{Aut}(\mathbf{M})$ and some $\bar{x} \in \omega^k$. Then $h(\bar{x}) \in G\bar{x}$ (since the orbit of x is named in the structure $\mathbf{M}$), i.e. there exists $g \in G$ such that $g(\bar{x}) = h(\bar{x})$. This proves that $h \in \overline{G}$, which is equal to G since G is a Polish (hence closed) subgroup of $\mathfrak{S}_\infty$. □

A particularly interesting case, which will come up later in these notes, is the case where each action $\mathrm{Aut}(\mathbf{M}) \curvearrowright \omega^k$ only has finitely many orbits (so in the language above there are only finitely many relational symbols of any fixed arity). These groups are called *oligomorphic* and correspond to automorphism groups of $\aleph_0$-categorical first-order structures.

Definition 2.7. A $\mathcal{L}$-structure $\mathbf{M}$ is *ultrahomogeneous* if it satisfies the following condition: for any two finitely generated substructures $\mathbf{N}_1$, $\mathbf{N}_2$ of $\mathbf{M}$, any isomorphism $\varphi\colon \mathbf{N}_1 \to \mathbf{N}_2$ extends to an automorphism of $\mathbf{M}$.

This is a very strong condition, often only considered in the case of relational structures where "finitely generated" above is equivalent to "finite".

Exercise 15. Prove that:

1. Any pure set is ultrahomogeneous.
2. The set of all rational numbers, seen as an ordered set with its usual order, is ultrahomogeneous.
3. Any infinite dimensional vector space over a finite field is ultrahomogeneous.
4. The infinite countable, atomless Boolean algebra is ultrahomogeneous.

(In each of the above cases, begin by describing precisely the language you are using)

Exercise 16. Prove that every closed subgroup of $\mathfrak{S}_\infty$ is the automorphism group of some ultrahomogeneous structure on ω in a countable relational language.
(It is enough to prove that the structure built in 2.6 is ultrahomogeneous!)

Exercise 17. Let $\mathbf{M}$ be a countable ultrahomogeneous structure, and $G = \mathrm{Aut}(\mathbf{M})$. Prove that the left completion of G is naturally identified with the set of embeddings of $\mathbf{M}$ into itself.

Definition 2.8. Let $\mathbf{M}$ be a $\mathcal{L}$-structure. The *age* of $\mathbf{M}$, denoted by $\mathrm{age}(\mathbf{M})$, is the class of all finitely generated $\mathcal{L}$-structures which are isomorphic to a substructure of $\mathbf{M}$.

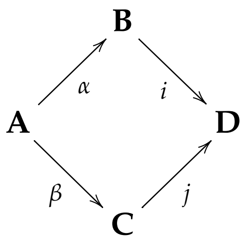
First, some easy observations: fix a $\mathcal{L}$-structure $\mathbf{M}$ and let $\mathcal{K} = \mathrm{age}(\mathbf{M})$. Then:

- For any finitely generated $\mathcal{L}$-structures $\mathbf{A}$, $\mathbf{B}$, if $\mathbf{A}$ embeds into $\mathbf{B}$ and $\mathbf{B} \in \mathcal{K}$ then $\mathbf{A}$ also belongs to $\mathcal{K}$. We say that $\mathcal{K}$ is *hereditary*.

Here, one needs to pay attention that a substructure of a finitely generated structure need not be finitely generated; for instance the free group on two generators admits a subgroup which is free on countably many generators. Of course this phenomenon cannot occur with relational structures. This is why we specified that **A** is finitely generated.

- For any **A**, **B** $\in \mathcal{K}$ there exists **C** $\in \mathcal{K}$ such that both **A** and **B** embed in **C**. We say that $\mathcal{K}$ has the *joint embedding property*.

Definition 2.9. Let $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures. We say that $\mathcal{K}$ has the *amalgamation property* if for any **A**, **B**, **C** $\in \mathcal{K}$ and any embeddings $\alpha\colon \mathbf{A} \to \mathbf{B}$, $\beta\colon \mathbf{A} \to \mathbf{C}$, there exist $\mathbf{D} \in \mathcal{K}$ and embeddings $i\colon \mathbf{B} \to \mathbf{D}$ and $j\colon \mathbf{C} \to \mathbf{D}$ such that the following diagram commutes:



Exercise 18. Prove that the following classes of structures have the amalgamation property.

1. The class of all finite graphs.

2. The class of all finite linear orders.

3. The class of all finite groups.

For the first two, the language has one binary relational symbol (besides equality); for the third one there is a binary symbol for the group operation as well as a constant symbol for the neutral element. At this stage in the notes, the third example requires some group-theoretic knowledge (e.g. that an amalgamated free product of finite groups is residually finite), an alternative argument will come up shortly (see Exercise 23).

Theorem 2.10 (Fraïssé). *Let* **M** *be a ultrahomogeneous structure. Then* $\mathrm{age}(\mathbf{M})$ *has the amalgamation property.*

Proof. Fix $\alpha\colon \mathbf{A} \to \mathbf{B}$ and $\beta\colon \mathbf{A} \to \mathbf{C}$ as in the definition. We may assume that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are substructures of M. Then β is a partial isomorphism of **M** with domain A and image $\beta(A) \subseteq C$.
By definition of ultrahomogeneity, there exists $g \in \mathrm{Aut}(\mathbf{M})$ such that $g(a) = \beta(a)$ for all $a \in A$. Let **D** be the substructure of **M** generated by **B** and $g^{-1}(\mathbf{C})$. Then **D** is finitely generated; letting $i(b) = b$ for all $b \in B$ and $j(c) = g^{-1}(c)$ for all $c \in \mathbf{C}$ gives us the desired maps i, j. □

Definition 2.11. We say that a structure **M** has the *Fraïssé property* if for any finitely generated substructure **A** of **M** and any embedding $\alpha\colon \mathbf{A} \to \mathbf{B}$ with $\mathbf{B} \in \mathrm{age}(\mathbf{M})$ there exists an embedding $\varphi\colon \mathbf{B} \to \mathbf{M}$ such that $\varphi(\alpha(a)) = a$ for all $a \in A$.

In words: any abstract extension of a copy of **A** which is contained in $\mathrm{age}(M)$ can be realized inside **M** by a substructure which contains **A**. The Fraïssé property is sometimes called the extension property, but we reserve that terminology for something else.

Exercise 19. Prove that ultrahomogeneous structures have the Fraïssé property (use the same argument as in the proof of Theorem 2.10).

Theorem 2.12. *Let* $\mathbf{M}$, $\mathbf{N}$ *be two countable ultrahomogeneous* $\mathcal{L}$*-structures.*
Assume that $\mathrm{age}(\mathbf{M}) = \mathrm{age}(\mathbf{N})$*, let* $\mathbf{A}$ *be a finitely generated substructure of* $\mathbf{M}$ *and* $\alpha\colon \mathbf{A} \to \mathbf{N}$ *an embedding. Then there exists an isomorphism* $g\colon \mathbf{M} \to \mathbf{N}$ *such that* $g_{|\mathbf{A}} = \alpha$*.*

In particular, any two ultrahomogeneous structures with the same age are isomorphic.

Proof. Fix enumerations $(m_k)_{k<\omega}$, $(n_k)_{k<\omega}$ of M, N respectively. We claim that, using the Fraïssé property, it is possible to build inductively an increasing sequence of finitely generated substructures $\mathbf{A}_k$, $\mathbf{B}_k$ of $\mathbf{M}$, $\mathbf{N}$ and isomorphisms $\alpha_k\colon \mathbf{A}_k \to \mathbf{B}_k$ with the following properties:

- $\mathbf{A}_0 = \mathbf{A}$, $\mathbf{B}_0 = \alpha(\mathbf{A})$, $\alpha_0 = \alpha$;
- For each k α_{k+1} extends α_k.
- For all k $m_k \in \mathbf{A}_{2k+1}$.
- For all k $n_k \in \mathbf{B}_{2k+2}$.

Assume for now that this is possible. Then $g = \bigcup \alpha_k$ is an isomorphism from $\bigcup_k \mathbf{A}_k = \mathbf{M}$ to $\bigcup \mathbf{B}_k = \mathbf{N}$ which extends α (the third condition above is there to ensure that the domain of g is M, and the fourth one guarantees that the image of g is N).
To see why this is indeed possible, assume we have built $\mathbf{A}_k$, $\mathbf{B}_k$ and α_k up to some rank p. Assume also that $p = 2q$ is even (the odd case is similar). If $m_q \in A_p$ then we have nothing to do and simply set $\mathbf{A}_{p+1} = \mathbf{A}_p$, $\mathbf{B}_{p+1} = \mathbf{B}_p$ and $\alpha_{p+1} = \alpha_p$.
If $m_q \notin A_p$, then we set $\mathbf{A}_{p+1} = \langle A_p, m_q \rangle$ and use the Fraïssé property of $\mathbf{N}$ and the fact that $\mathbf{A}_{p+1} \in \mathrm{age}(\mathbf{M}) = \mathrm{age}(\mathbf{N})$ to find $n \in N$ such that α_p extends to an isomorphism α_{p+1} from $\langle A_p, m_q \rangle$ to $\langle B_p, n \rangle$ such that $\alpha_{p+1}(m_q) = n$. We let $\mathcal{B}_{p+1} = \langle B_p, n \rangle$ and move on to the next step. □

Given that we only used the Fraïssé property of $\mathbf{M}$, $\mathbf{N}$ above, the same argument implies that any countable structure with the Fraïssé property is ultrahomogeneous (so this property is equivalent to ultrahomogeneity).

Definition 2.13. A class $\mathcal{K}$ of finitely generated $\mathcal{L}$-structures is said to be a *Fraïssé class* if it satisfies the following conditions:

- $\mathcal{K}$ contains at most countably many isomorphism types.
- $\mathcal{K}$ has the joint embedding property.
- $\mathcal{K}$ is hereditary.
- $\mathcal{K}$ has the amalgamation property.

Note that here we allow the degenerate case where every structure with age contained in $\mathcal{K}$ is finitely generated (in particular, $\mathcal{K}$ could be a finite set of relational structures and have a largest element); this will not play a part later on but there is no reason to exclude it at this point.

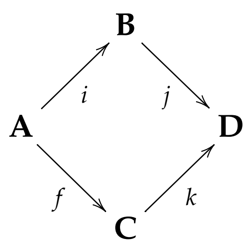
Exercise 20. Let $\mathcal{L} = (R_q)_{q \in \mathbb{Q}^+}$ be the countable language with a binary relational symbol R_q for each positive rational q. Any metric space whose distance takes values in $\mathbb{Q}$ can be seen as an $\mathcal{L}$-structure, by setting $(x, y) \in R_q \Leftrightarrow d(x, y) = q$.
Show that the class of finite $\mathbb{Q}$-valued metric spaces, seen as $\mathcal{L}$-structures as above, is a Fraîssé class.

We already saw that the age of any ultrahomogeneous structure is a Fraïssé class. We turn to establishing the converse of that statement.

Theorem 2.14 (Fraïssé). *Let $\mathcal{K}$ be a Fraïssé class in a countable language $\mathcal{L}$. There there exists a unique (up to isomorphism) $\mathcal{L}$-structure $\mathbf{F}_{\mathcal{K}}$ which is ultrahomogeneous, (at most) countable, and such that* age$(\mathbf{M}) = \mathcal{K}$.
This structure is called the Fraïssé limit *of $\mathcal{K}$.*

Note that uniqueness (up to isomorphism) of a ultrahomogeneous structure with a given age has already been established.
The previous theorem is only really interesting if there are non-finitely generated structures whose age is contained in age$(\mathcal{K})$.

Proof. We claim that one can build an increasing sequence of structures $\mathbf{F}_i \in \mathcal{K}$ in such a way that if $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and $f \colon \mathbf{A} \to \mathbf{F}_i$ is an embedding, then there exists j and an embedding $g \colon \mathbf{B} \to \mathbf{F}_j$ which extends f.
Assume for the moment that such a construction is possible, and let $\mathbf{F} = \bigcup_i \mathbf{F}_i$. Then age$(\mathcal{F}) \subseteq \mathcal{K}$. For any $\mathbf{A} \in \mathcal{K}$ there exists (by the joint embedding property) some $\mathbf{B} \in \mathcal{K}$ such that both $\mathbf{F}_0$ and $\mathbf{A}$ embed in $\mathbf{B}$. Then the identity map from $\mathbf{F}_0$ to itself extends to an embedding of $\mathbf{B}$ in some $\mathbf{F}_j$; hence $\mathbf{F}_j$ contains a copy of $\mathbf{A}$ and age$(\mathbf{F}) = \mathcal{K}$. Then the above condition precisely says that $\mathbf{F}$ satisfies the Fraïssé property, and we are done.
Note that there are only countably many quadruples $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f)$ with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $\mathbf{A} \leq \mathbf{B}$ and $f \colon \mathbf{A} \to \mathbf{C}$ an embedding. So in order to perform the desired construction we simply need to be able, given such a quadruple, to produce $\mathbf{D}$ containing $\mathbf{C}$ such that f extends to an embedding of $\mathbf{B}$ into $\mathbf{D}$; and this is precisely what is provided by the following amalgamation diagram,where i is the inclusion map from A to B:



Indeed, identifying $\mathbf{C}$ with $k(\mathbf{C})$, j is the desired embedding of $\mathbf{B}$ into $\mathbf{D}$. □

Exercise 21.
1. Let $\mathcal{K}$ be the class of all finite graphs (i.e. symmetric, irreflexive binary relations) and let $\mathbf{R}$ be its Fraïssé limit. Show that $\mathbf{R}$ is (up to isomorphism) the unique countable graph such that for any finite $A, B \subset R$ there exists an element $x \in R$ with an edge linking x to each element of A, and no edge linking x to any element of B.

2. We build a graph with vertex set ω as follows: for each $i < j \in \omega$ we put an edge between i and j with probability $1/2$ (all those choices being made independently).

Prove that almost surely the resulting graph is isomorphic to $\mathbf{R}$ (this explains why $\mathbf{R}$ is often called the *random graph*).

Exercise 22. A *tournament* is an oriented graph (asymetric, irreflexive binary relation) with the property that for any pair of distinct points x, y there is an edge from x to y or an edge from y to x (equivalently, a complete graph with an orientation of its edges). Prove that the class of finite tournaments is a Fraïssé class.

Exercise 23. We build an increasing sequence of groups $(G_i)_{i<\omega}$ as follows: let G_0 be the permutation group on 3 elements. Assuming G_i has been built, embed G_i in the permutation group $\mathfrak{S}(G_i)$ via the left-translation action of G_i on itself, and set $G_{i+1} = \mathfrak{S}(G_i)$. Then define $\mathbf{H} = \bigcup_i G_i$.

1. Prove that $\mathbf{H}$ is locally finite (i.e. any finitely generated subgroup is finite), ultrahomogeneous, and that $\text{age}(\mathbf{H})$ is the class of all finite groups. The group $\mathbf{H}$ is called *Hall's universal locally finite group*.

2. Prove that the class of finite groups satisfies the amalgamation property.

3. Prove that any two elements of $\mathbf{H}$ with the same order are conjugate in $\mathbf{H}$.

4. Prove that $\mathbf{H}$ is simple.

Exercise 24. We already saw that the class of finite linear orders is a Fraïssé class. Show that its limit $\mathbf{M} = (M, <)$ is the unique countable linear ordering which is

- *dense*, i.e. for any $x < y \in M$ there exists $z \in M$ such that $x < z < y$.

- *without endpoints*, i.e. M has neither a maximum nor a minimum.

Conclude that $\mathbb{Q}$, with its usual ordering, is the Fraïssé limit of finite linear orders, and that is is up to isomorphism the unique countable linear order which is dense and without endpoints (Cantor's theorem).

It is interesting to analyse the interplay between combinatorial properties of $\mathcal{K}$ and properties of the automorphism group of its Fraïssé limit. Let us give an example to conclude this chapter.

Definition 2.15. Let $\mathcal{K}$ be a Fraïssé class in a relational language $\mathcal{L}$. We say that $\mathcal{K}$ has the *extension property* if for any $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A}$ embeds in $\mathbf{B}$ and every partial automorphism of $\mathbf{A}$ extends to an isomorphism of $\mathbf{B}$.

Hrushovski proved that the class of finite graphs has the extension property; since then many other examples have been obtained, in particular following work of Herwig–Lascar (for instance Solecki proved that this property holds for the class of finite metric spaces with rational distances). This property is however in general very difficult to establish.

Theorem 2.16 (Kechris–Rosendal)**.** *Let $\mathcal{K}$ be a Fraïssé class in a relational language (thus all elements of $\mathcal{K}$ are finite), $\mathbf{F}$ its Fraïssé limit, and $G = \text{Aut}(\mathbf{F})$. Then the following are equivalent:*

- *$\mathcal{K}$ has the extension property.*

- *There exists an increasing sequence of compact subgroups* $G_n \leq \mathrm{Aut}(\mathbf{F})$ *whose union is dense in* $\mathrm{Aut}(\mathbf{F})$.

As usual, this result is only interesting if $\mathcal{K}$ contains structures of arbitrarily large finite cardinality, i.e. if the underlying set of **F** is infinite (otherwise **F** is a finite structure, and both properties above trivially hold).

Proof. We assume that **F** is infinite (and countable, of course). We assume that $\mathcal{K}$ has the extension property and denote $G = \mathrm{Aut}(\mathbf{F})$. Given n, consider the set

$$\Omega_n = \{(g_1, \ldots, g_n) \in G^n : \forall x \in F \ \langle g_1, \ldots, g_n \rangle x \text{ is finite}\}$$

Given some finite $A \subset F$, the set $\{\bar{g} \in G^n : \langle \bar{g} \rangle A = A\}$ is open, since if $\bar{g}$ is in that set and $\bar{h}$ coincides with $\bar{g}$ on A then $\bar{h}$ also belongs to that set. So saying that a given x has a finite orbit under $\langle \bar{g} \rangle$ is an open condition. Thus Ω_n is a countable intersection of open sets, hence is G_δ.
The extension property of $\mathcal{K}$ guarantees that Ω_n is dense in G^n; thus a generic element $\bar{g}$ of G^n belongs to Ω_n, equivalently the closed subgroup generated by $\bar{g}$ is compact (recall that a closed subgroup $G \leq \mathfrak{S}_\infty$ is compact iff every point has a finite G-orbit).
It follows that for a generic sequence $(g_n) \in G^\omega$, we have for all n that $\overline{\langle g_1, \ldots, g_n \rangle}$ is compact.
In any Polish group, the set of all $(g_n) \in G^\omega$ which generate a dense subgroup is dense G_δ; applying the Baire category theorem, we conclude that a generic sequence in G^ω induces an increasing sequence of compact subgroups with a dense union.

Conversely, assume that $(G_i)_{i<\omega}$ is an increasing sequence of compact subgroups with a dense union, and let $\mathbf{A} \in \mathcal{K}$. We may view **A** as a (finite) substructure of **F**. Let $f_1, \ldots, f_k$ enumerate all partial automorphisms of **A**; by ultrahomogeneity of **F** we can extend each f_j to some $\tilde{f}_j \in \mathrm{Aut}(\mathbf{F})$. By density of $\bigcup_i G_i$, for all j there exists some i_j and $g_j \in G_{i_j}$ which coincides with $\tilde{f}_j$ on A, hence coincides with f_j on its domain. Let $n = \max\{i_j : j \in \{1, \ldots k\}\}$. Then $g_j \in G_n$ for each n; and each g_j induces an automorphism of the finite structure $G_n\mathbf{A}$ (since G_n is compact, every element of **F** has a finite G_n-orbit). □

Bibliographical comments. For more on Fraïssé classes the reader can consult courses on model theory such as [TZ12] or [Hod93]. The survey [Mac11] is also a good introductory source for some of the material mentioned here, and much more. The connection between nonarchimedean groups and closed subgroups of $\mathfrak{S}_\infty$ comes from [BK96]. Regarding the extension property, much work has built on [HL00], including very recent papers like [EHN21] and [HKN22]. The fact that the class of finite rational metric spaces has the extension property comes from [Soł05].
Note that whether the extension property holds for finite tournaments is still an open problem.

Chapter 3

Uniform spaces

Definition 3.1. Let X be a set; denote $\Delta_X = \{(x,x) \colon x \in X\}$. A *uniform structure*, also called a *uniformity* on X, is a set $\mathcal{U}$ of subsets of $X \times X$ such that:

- For all $U \in \mathcal{U}$ one has $\Delta_X \subseteq U$.
- For all $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$ (where $U^{-1} = \{(x,y) \colon (y,x) \in U\}$)
- For all $U, V \in \mathcal{U}$ $U \cap V \in \mathcal{U}$.
- For all U, V $(U \in \mathcal{U}$ and $U \subseteq V) \Rightarrow V \in \mathcal{U}$.
- For all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$
  (where $V \circ V = \{(x,y) \colon \exists z \ (x,z) \in V$ and $(z,y) \in V\}$)

Elements of a uniform structure $\mathcal{U}$ are called *entourages*; we think of them as being neighborhoods of Δ_X. A fundamental example is given by metric spaces: given a metric space (X,d) one can consider the uniformity $\mathcal{U}_d$ whose entourages are the subsets of $X \times X$ containing a set of the form $\{(x,y) \colon d(x,y) < r\}$ for some $r > 0$.

Exercise 25. Show that $\mathcal{U}_d$ is indeed a uniformity. Prove that two distances d_1, d_2 on X are uniformly equivalent iff $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$.
(Recall that distances on X are uniformly equivalent iff the identity map is uniformly continuous in both directions)

Definition 3.2. Given a uniformity $\mathcal{U}$ on a set X, we endow X with a topology by stating that $V \subseteq X$ is open iff

$$\forall x \in V \ \exists U \in \mathcal{U} \quad \{y \colon (x,y) \in U\} \subseteq V$$

We will denote $U[x] = \{y \colon (x,y) \in U\}$.

Exercise 26. Let d be a metric on X, and $\mathcal{U}_d$ the metric uniformity. Show that the topology induced by $\mathcal{U}_d$ is the same as the topology induced by d.

Exercise 27. Let $(X,\mathcal{U})$ be a uniform space, and $x \in X$. Show that the neighborhoods of x for the topology induced by $\mathcal{U}$ are exactly the sets of the form $U[x]$ for some $U \in \mathcal{U}$.

Definition 3.3. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two uniform spaces. A map $f\colon X\to Y$ is *uniformly continuous* if

$$\forall V\in\mathcal{V}\ \exists U\in\mathcal{U}\ \forall x,y\in X\quad (x,y)\in U\Rightarrow (f(x),f(y))\in V$$

We say that a uniform structure is *metrizable* if there is a metric which induces it. Not all uniform structures are metrizable (though they are always induced by a family of pseudo-metrics); even in metrizable settings, it is sometimes the case that uniform structures are more natural objects that metrics (i.e. there are some natural choices of uniform structure, but no canonical choice of metric; we already saw an example of this phenomenon when discussing left-invariant metrics on Polish groups).

Definition 3.4. A uniform structure is Hausdorff if the topology it induces is Hausdorff.

Proposition 3.5. *$(X,\mathcal{U})$ is Hausdorff iff $\bigcap_{U\in\mathcal{U}} U=\Delta_X$.*

Proof. Assume $(X,\mathcal{U})$ is Hausdorff. Let $x\neq y$; there exists $U\in\mathcal{U}$ such that $y\notin U[x]$, equivalently $(x,y)\notin U$ so $\bigcap_{U\in\mathcal{U}} U\subseteq\Delta_X$. The other inclusion is an immediate consequence of the definition of a uniform structure.
Conversely, assume that $\bigcap_{U\in\mathcal{U}} U=\Delta_X$ and let $x\neq y\in X$. There exists $U\in\mathcal{U}$ such that $(x,y)\notin U$, and $V\in\mathcal{U}$ such that $V\circ V^{-1}\subseteq U$. Then $V[x]$ and $V[y]$ are neighborhoods of x,y respectively; if $z\in V[x]\cap V[y]$ then $(x,z)\in V$ and $(z,y)\in V^{-1}$, whence $(x,y)\in U$, a contradiction. So $V[x]$ and $V[y]$ are disjoint. □

The above proof shows that a uniform space is Hausdorff as soon as for any two points $x\neq y$ there exists a neighborhood of x which does not contain y, a property which is in general weaker than being Hausdorff.

Exercise 28. Let $(X,\mathcal{U})$ be a uniform space. Then for any $U\in\mathcal{U}$ the interior of U (for the product topology on $X\times X$ induced by $\mathcal{U}$) belongs to $\mathcal{U}$.

Definition 3.6. Let $(X,\mathcal{U})$ be a uniform space. A *fundamental system* is a family $\mathcal{E}$ of entourages such that any element of $\mathcal{U}$ contains an element of $\mathcal{E}$.

Note that every uniform structure admits a fundamental system consisting of open entourages (start from any fundamental system then consider the interiors of its elements).

Theorem 3.7. *Let $(X,\mathcal{U})$ be a Hausdorff uniform space. Then $\mathcal{U}$ is metrizable iff it admits a countable fundamental system.*

Proof. If d induces $\mathcal{U}$ then the family of entourages $\{(x,y)\colon d(x,y)<\varepsilon\}$, where ε ranges over positive rational numbers, is a countable fundamental system. This proves one implication.
Conversely, assume that $\mathcal{U}$ admits a countable fundamental system. One can then produce another countable fundamental system $\{U_n\}_{n<\omega}$ with the following properties:

- $U_0=X\times X$.
- For all n one has $U_n^{-1}=U_n$.
- For all n one has $U_{n+1}\circ U_{n+1}\circ U_{n+1}\subseteq U_n$.

We may then define, for $x, y \in X$, $\rho(x,y) = \inf\{2^{-n} \colon (x,y) \in U_n\}$.
Then ρ is symmetric, and since $\mathcal{U}$ is Hausdorff we have $\rho(x,y) = 0$ iff $x = y$ (since $(U_n)_n$ is a fundamental system we have $\bigcap_n U_n = \Delta_X$). However ρ need not satisfy the triangle inequality. There is a way to produce a metric from a symmetric weight function: we let

$$d(x,y) = \inf_n \left\{ \sum_{i=0}^{n} \rho(x_i, x_{i+1}) \colon x_0 = x \,, x_{n+1} = y \right\}$$

Clearly d is now a pseudometric and $d \leq \rho$. We claim that $2d \geq \rho$ (which implies that d is actually a metric); accept this for the moment.
Choose ε such that $0 < \varepsilon \leq \frac{1}{2}$, and let $i \geq 1$ be such that $2^{-i-1} \leq \varepsilon \leq 2^{-i}$.
If $(x,y) \in U_{i+1}$ then $d(x,y) \leq \rho(x,y) \leq 2^{-i-1} \leq \varepsilon$. Thus $\{(x,y) \colon d(x,y) \leq \varepsilon\}$ contains U_{i+1}, hence belongs to $\mathcal{U}$.
Conversely, if $d(x,y) \leq \varepsilon$ then $\rho(x,y) \leq 2d(x,y) \leq 2^{-i+1}$, whence $(x,y) \in U_{i-1}$ so U_{i-1} belongs to the uniformity generated by d. It follows that d induces $\mathcal{U}$.
We still have to prove that $2d \geq \rho$. For this, we prove by induction on $n \geq 1$ that for any $x_0, \ldots, x_{n+1} \in X$ one has

$$\sum_{i=0}^{n} \rho(x_i, x_{i+1}) \geq \frac{\rho(x_0, x_{n+1})}{2}$$

Note that the fact that $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ for all n implies that

$$\forall x_1,\, x_2,\, x_3\ \forall \varepsilon > 0 \quad (\rho(x_0, x_1) \leq \varepsilon \wedge \rho(x_1, x_2) \leq \varepsilon \wedge \rho(x_2, x_3) \leq \varepsilon) \Rightarrow \rho(x_0, x_3) \leq 2\varepsilon \qquad (*)$$

This takes care of the cases $n = 0, 1, 2$ in our induction. Assume we have established our property for all $m \leq n-1$ for some integer $n \geq 3$.
Fix $x_0, \ldots, x_{n+1}$ and let $r = \sum_{i=0}^{n} \rho(x_i, x_{i+1})$. If $\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) \leq \frac{r}{2}$ then (by induction) we have $\rho(x_0, x_n) \leq r$, whence we conclude by $(*)$ that $\rho(x_0, x_{n+1}) \leq 2r$. Similarly, we obtain the desired conclusion if $\sum_{i=1}^{n} \rho(x_i, x_{i+1}) \leq \frac{r}{2}$.
Otherwise, there exists some $i \in \{1, \ldots n-1\}$ such that

$$\sum_{j=0}^{i-1} \rho(x_j, x_{j+1}) \leq \frac{r}{2} \text{ and } \sum_{j=i+1}^{n} \rho(x_j, x_{j+1}) \leq \frac{r}{2}$$

(choose the largest i for which $\sum_{j=0}^{i-1} \rho(x_j, x_{j+1}) \leq \frac{r}{2}$).
Again using the inductive assumption, we have $\rho(x_0, x_i) \leq r$, $\rho(x_{i+1}, x_{n+1}) \leq r$ and of course we also have $\rho(x_i, x_{i+1}) \leq r$. Hence $(*)$ gives $\rho(x_0, x_{n+1}) \leq 2r$. □

Proposition 3.8. *Let X be a compact Hausdorff space. Then the set of all neighborhoods of Δ_X (in $X \times X$ endowed with the product topology) is a uniform structure on X.*

Proof. Reviewing the definition of a uniform structure, we see that the only non immediate fact that we have to prove is that if U is a neighborhood of Δ_X then there exists a neighborhood V of Δ_X such that $V \circ V \subseteq U$.

We will make use of the fact that compact Hausdorff spaces are *completely regular*, i.e. for every $x \in F$ and any closed $F \subseteq X$ such that $x \notin F$, there exists a continuous function $f\colon X \to \mathbb{R}$ such that $f(x) = 0$ and f is constant equal to 1 on F. We claim that every neighborhood of Δ_X contains a set of the form $\{(x,y)\colon \forall i \in I\ |f_i(x) - f_i(y)| < \varepsilon_i\}$ with $f_i \in C(X,\mathbb{R})$ and $\varepsilon_i > 0$ for some finite index set I (also, note that each of these sets is an open neighborhood of Δ_X). Once this is proved, the desired result follows easily.
We check the claim. Let A be a neighborhood of Δ_X; by compactness, there is an open covering $(O_i)_{1\leq i\leq n}$ of X such that $\bigcup_{i=1}^n O_i \times O_i \subseteq A$. For each $x \in X$, find i such that $x \in O_i$ and then a continuous map $f_x\colon X \to \mathbb{R}$ such that $f_x(x) = 0$ and $f_x(y) = 1$ for every $y \notin O_i$. Then let $V_x = \left\{y\colon f_x(y) < \frac{1}{2}\right\}$.
By compactness again, there exist $x_1, \ldots, x_p$ such that $V_{x_1}, \ldots, V_{x_p}$ cover X. For $j \in \{1, \ldots, p\}$ denote $g_j = f_{x_j}$. Assume that (x,y) is such that $|g_j(x) - g_j(y)| < \frac{1}{2}$ for all j. Then there exists $j \in \{1,\ldots,p\}$ such that $x \in V_{x_j}$. In turn, there exists $k \in \{1,\ldots,n\}$ such that $x_j \in O_k$; we then have $f_{x_j}(x) < \frac{1}{2}$ and $f_{x_j}(y) < 1$, so $(x,y) \in O_k \times O_k \subseteq A$. □

Theorem 3.9. *Let $(X,\mathcal{U})$ be a compact uniform space, $(Y,\mathcal{V})$ be a uniform space, and $f\colon X \to Y$ a continuous map. Then f is uniformly continuous.*

Proof. Fix $V \in \mathcal{V}$, and choose a symmetric V' such that $V' \circ V' \subseteq V$. Continuity of f implies that for every $x \in X$ there exists $U_x \in \mathcal{U}$ such that $(x,x') \in U_x \Rightarrow (f(x), f(x')) \in V'$. Find some symmetric $U'_x \in \mathcal{U}$ such that $U'_x \circ U'_x \subseteq U_x$.
Recall that $U'_x[x]$ is a neighborhood of x. Hence by compactness there exist $x_1,\ldots,x_n \in X$ such that $X = \bigcup_{i=1}^n U'_{x_i}[x_i]$; let $U = \bigcap_{i=1}^n U'_{x_i}$. By definition of a uniform structure, U belongs to $\mathcal{U}$.
Now, assume that $(x,y) \in U$. Then for some i we have $x \in U'_{x_i}[x_i]$. Also $(x,y) \in U'_{x_i}$ since $U \subset U'_{x_i}$. Thus we have that $(x_i,x) \in U'_{x_i}$ and $(x_i,y) \in U'_{x_i} \circ U'_{x_i} \subseteq U_{x_i}$.
It follows that $(f(x_i), f(x)) \in V'$ and $(f(x_i), f(y)) \in V'$, so $(f(x), f(y)) \in V$. □

Exercise 29. Prove that any compact space has at most one compatible uniform structure. In particular, on any compact Hausdorff space there exists a unique compatible uniformity.

The following fact will be handy when we turn to topological dynamics and need to check uniform continuity of certain maps defined on dense subsets of compact Hausdorff spaces in order to extend them.

Proposition 3.10. *Let $(X,\mathcal{U})$ be a uniform space, and Y a compact Hausdorff space. Then $f\colon (X,\mathcal{U}) \to Y$ is uniformly continuous if, and only if, $g \circ f$ is uniformly continuous for every continuous function $g\colon Y \to \mathbb{R}$.*

Proof. Note that we did not bother mentioning which uniform structure we put on Y, since there is exactly one. One direction in the above equivalence is obvious, since every continous fonction from Y to $\mathbb{R}$ is uniformly continuous and a composition of two uniformly continuous functions is uniformly continuous.
Conversely, assume that f satisfies the above condition, and let V be an entourage for the uniformity on Y. We know that V is a neighborhood of the diagonal, and that reducing V if necessary we may assume that

$$V = \{(y_1,y_2) \in Y\colon \forall i \in I\ |g_i(y_1) - g_i(y_2)| < 1\}$$

for some finite index set I and some continuous functions $g_i \colon Y \to \mathbb{R}$. By assumption, each $g_i \circ f$ is uniformly continuous on $(X, \mathcal{U})$, whence for each $i \in I$ there exists $U_i \in \mathcal{U}$ such that for all $(x_1, x_2) \in U_i$ one has $|g_i \circ f(x_1) - g_i \circ f(x_2)| < 1$.
Then $U = \bigcap_i U_i$ belongs to $\mathcal{U}$ and for all $(x_1, x_2) \in \mathcal{U}$ we have $(f(x_1), f(x_2)) \in V$. □

We now need to present a notion of (Hausdorff) completion of a uniform space; for this we use filters. We first recall the basic definitions.

Definition 3.11. Let X be a set. A subset $\mathcal{F}$ of $\mathcal{P}(X)$ is a *filter* if :

- $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
- If $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2$ belongs to $\mathcal{F}$.
- If $F \in \mathcal{F}$ and $F \subseteq A$ then $A \in \mathcal{F}$.

An *ultrafilter* is a filter which is maximal (among filters) for inclusion.

Intuitively, a filter provides a notion of "large subset": the whole set is large, the emptyset is not large, an intersection of two large sets is still large, and a subset which contains a large subset is large itself.

Exercise 30. Let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is an ultrafilter iff for any $A \subseteq X$ one has either $A \in \mathcal{F}$ of $X \setminus A \in \mathcal{F}$.

A filter on X is *principal* if there exists some $A \subseteq X$ such that $\mathcal{F} = \{F \colon A \subseteq F\}$.
More interesting is the *Fréchet filter*, which is the set of all subsets of X with finite complement (on an infinite set X). Any ultrafilter is either principal (and contains a singleton) or contains the Fréchet filter. Both principal filters and the Fréchet filter may be seen as particular cases of *neighborhood filters*: given a topological space X, that is the set of all neighborhoods of some fixed $x \in X$.

Definition 3.12. Let X be a topological space, and $\mathcal{F}$ be a filter on X. We say that $\mathcal{F}$ *converges to x* if $\mathcal{F}$ contains the neighborhood filter $\mathcal{V}_x$ of x.

Definition 3.13. Let X, Y be two sets, $\mathcal{F}$ a filter on X and $g \colon X \to Y$. The *image filter* $g(\mathcal{F})$ is $\{A \subseteq Y \colon g^{-1}(A) \in \mathcal{F}\}$.

Exercise 31. Assume that X is Hausdorff, and $\mathcal{F}$ is a filter on X which converges to both x and y. Show that $x = y$.

Exercise 32. Let X, Y be two topological spaces, and $f \colon X \to Y$ be a function. Show that f is continuous at$x \in X$ iff the image of any filter that converges to x is a filter that converges to $f(x)$.

Exercise 33. Let X be a topological space, and $(x_n)_{n<\omega} \in X^\omega$. Let $\varphi \colon \omega \to X$ be defined by $\varphi(n) = x_n$. Let $\mathcal{F}$ denote the image under φ of the Fréchet filter on ω.
Show that $(x_n)_{n \in \mathbb{N}}$ converges to x iff $\mathcal{F}$ converges to x.

So convergence of filters generalizes convergence of sequences, and is enough to capture the topology of spaces which are not first countable. One also often sees generalized sequences or nets used for the same purpose; filters have the advantage of being very amenable to set-theoretic manipulations, and probably the disadvantage of being less intuitive.

Proposition 3.14. *Let X be a compact topological space. Then any ultrafilter on X is convergent.*

Proof. Let $\mathcal{F}$ be an ultrafilter on X, and set $\mathcal{A} = \{\overline{F} \colon F \in \mathcal{F}\}$. Then any finite intersection of elements of $\mathcal{A}$ is nonempty, hence by compactness there exists $x_0 \in X$ such that $x_0 \in \overline{F}$ for any $F \in \mathcal{F}$.
Let V be a neighborhood of x_0; if $V \notin \mathcal{F}$ then $X \setminus V \in \mathcal{F}$, and $x_0 \notin \overline{X \setminus V}$, a contradiction. So $V \in \mathcal{F}$, proving that $\mathcal{U}$ converges to x_0. □

Exercise 34. Prove that the previous property characterizes compact spaces, i.e. that a topological space is compact iff any ultrafilter on X is convergent. Use this to prove Tychonoff's theorem: any product of compact spaces is compact (first characterize convergence of a filter on a product space by the convergence of each of its projections).

Definition 3.15. Let $(X,\mathcal{U})$ be a uniform space, and $\mathcal{F}$ be a filter on X. We say that $\mathcal{F}$ is a *Cauchy filter* if :

$$\forall U \in \mathcal{U}\ \exists F \in \mathcal{F} \quad F \times F \subseteq U$$

Exercise 35. Let (X,d) be a metric space, and $(x_n)_{n<\omega} \in X^\omega$. Let $\varphi \colon \omega \to X$ be defined by $\varphi(n) = x_n$. Let $\mathcal{F}$ denote the image under φ of the Fréchet filter on ω.
Show that $(x_n)_{n<\omega}$ is a Cauchy sequence iff $\mathcal{F}$ is a Cauchy filter.

Definition 3.16. A uniform space $(X,\mathcal{U})$ is *complete* iff any Cauchy filter on X is convergent.

Exercise 36. Let $(X,\mathcal{U})$ be a uniform space.

1. Prove that any convergent filter is Cauchy (this amounts to proving that any neighborhood filter is a Cauchy filter).

2. Assume that $(X,\mathcal{U})$ is metrizable by some distance d. Prove that $(X,\mathcal{U})$ is complete iff (X,d) is complete.

Definition 3.17. Let $(X,\mathcal{U})$ be a uniform space, and $\mathcal{F}$ be a filter on X. We say that x is an *adherent point* of $\mathcal{F}$ if for any neighborhood V of x and any $F \in \mathcal{F}$ we have $V \cap F \neq \emptyset$.
Show that a Cauchy filter with an adherent point is convergent (to that point).

Exercise 37. Let $(X,\mathcal{U})$ be a compact uniform space. Show that $(X,\mathcal{U})$ is complete.

Exercise 38. We say that a uniform space $(X,\mathcal{U})$ is *totally bounded* if for any $U \in \mathcal{U}$ there exists a finite $A \in X$ such that $X = U[A]$ $(= \bigcup_{x \in A} U[x])$.
Show that $(X,\mathcal{U})$ is compact iff it is both complete and totally bounded.

We now discuss how to build the Hausdorff completion of a uniform space; I do not plan to discuss this on the board. I chose to include the details in these notes for completeness (no pun intended) but they may be skipped without hindering comprehension of the next chapters (one then just needs to accept as a black box that one can complete Hausdorff uniform spaces in essentially the same way one completes metric spaces).

Lemma 3.18. *Let $(X,\mathcal{U})$ be a uniform space. Assume that Y is a dense subset of X such that for every Cauchy filter $\mathcal{F}$ in Y the filter on X generated by $\mathcal{F}$ converges in X. Then $(X,\mathcal{U})$ is complete.*

The filter on X generated by $\mathcal{F}$ mentioned in the statement above is the set of susbets of X which contain an element of $\mathcal{F}$ (equivalently, the image filter of $\mathcal{F}$ under the inclusion map from Y to X).

Proof. Let $\mathcal{F}$ be a Cauchy filter on X. Let $\mathcal{F}^{\mathcal{U}}$ be the filter generated by all the subsets $U[F]$ $(= \bigcup_{x \in F} U[x])$ for some $U \in \mathcal{U}$ and $F \in \mathcal{F}$.
Then $\mathcal{F}^{\mathcal{U}}$ is contained in $\mathcal{F}$, and it is a Cauchy filter: fix $U \in \mathcal{U}$ and choose a symmetric $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. Since $\mathcal{F}$ is Cauchy, there is $F \in \mathcal{F}$ such that $F \times F \subseteq V$. Then $V[F] \times V[F] \subseteq U$: if $(x,y) \in V[F] \times V[F]$ there exists $f_1, f_2 \in F$ such that $(x, f_1) \in V$ and $(y, f_2) \in V$, and since $(f_1, f_2) \in V$ we get $(x,y) \in V \circ V \circ V \subseteq U$.
For any element A of $\mathcal{F}^{\mathcal{U}}$ we have $A \cap Y \neq \emptyset$ since Y is dense in X, so $\mathcal{F}^{\mathcal{U}}$ induces a Cauchy filter on Y (for the uniformity on Y induced by $\mathcal{U}$). By assumption, we obtain that there exists $x \in X$ such that for every neighborhood N of x, every $F \in \mathcal{F}$ and every $U \in \mathcal{U}$ one has $U[F] \cap N \neq \emptyset$.
Let $N = V[x]$ be a neighborhood of x for some $V \in \mathcal{U}$, and let $U \in \mathcal{U}$ be symmetric and such that $U \circ U \subseteq V$. Given $F \in \mathcal{F}$, we can pick $z \in U[F] \cap U[x]$. Then $(x,z) \in U$ and there exists $f \in F$ such that $(z, f) \in U$, hence $(x, f) \in V$, so $F \cap V[x] \neq \emptyset$.
Finally, x is an adherent point of the Cauchy filter $\mathcal{F}$, hence $\mathcal{F}$ converges to x and $(X, \mathcal{U})$ is complete. □

Given a uniform space $(X, \mathcal{U})$, we let $\widehat{X}$ denote the set of all Cauchy filters which are minimal for inclusion (among Cauchy filters).

Lemma 3.19. *For any $x \in X$, the neighborhood filter $\mathcal{V}_x$ is a minimal Cauchy filter.*

Proof. We already know that $\mathcal{V}_x$ is a Cauchy filter. Assume that $\mathcal{F}$ is a Cauchy filter contained in $\mathcal{V}_x$. Let O be a neighborhood of x, which is of the form $U[x]$ for some $U \in U$.
Since $\mathcal{F}$ is Cauchy, there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$; since $\mathcal{F}$ is contained in $\mathcal{V}_x$ we know that F is a neighborhood of x, in particular $x \in F$. Thus $\{x\} \times F \subseteq U$, whence $F \subseteq U[x] = V$. It follows that $V \in \mathcal{F}$ and we are done. □

Note that a minimal Cauchy filter which converges to some x must coincide with the neighborhood filter $\mathcal{V}_x$; hence any nonconvergent minimal Cauchy filter witnesses that our uniform space is not complete. Furthermore, any Cauchy filter contains a minimal one, so to form a completion we must precisely add a limit point for every nonconvergent minimal Cauchy filter.
It is then natural to consider the map $i \colon X \to \widehat{X}$ which maps $x \in X$ to $\mathcal{V}_x$; this map is injective as soon as $(X, \mathcal{U})$ is Hausdorff.
We now want to endow $\widehat{X}$ with a uniform structure $\widehat{\mathcal{U}}$ such that $i(X)$ is dense in $\widehat{X}$ and $(\widehat{X}, \widehat{\mathcal{U}})$ is complete. For some fixed entourage $U \in \mathcal{U}$, we let

$$C(U) = \{(\mathcal{F}, \mathcal{G}) \colon \exists A \in \mathcal{F} \cap \mathcal{G} \; A \times A \subseteq U\}$$

By definition of a Cauchy filter, $C(U)$ contains Δ_Y for every $U \in \mathcal{U}$. We want to prove that $C(U)$ generates a uniform structure $\widehat{\mathcal{U}}$ on Y; we let $\widehat{\mathcal{U}}$ be the set of all subsets of $\widehat{X} \times \widehat{X}$ which contain some $C(U)$. The only thing that really requires checking is the existence of "square roots" in $\widehat{\mathcal{U}}$, i.e. the last axiom in the definition of a uniformity. The next lemma shows that this property holds for $\widehat{\mathcal{U}}$.

Lemma 3.20. *Assume that $V \circ V \subseteq U$. Then $C(V) \circ C(V) \subseteq C(U)$.*

Proof. Assume that $(\mathcal{F}, \mathcal{G}) \in C(V)$ and $(\mathcal{G}, \mathcal{H}) \in C(V)$. Then we can find $A \in \mathcal{F} \cap \mathcal{G}$ such that $A \times A \subseteq V$, and $B \in \mathcal{G} \cap \mathcal{H}$ such that $B \times B \subseteq V$. Let $D = A \cup B$; it belongs to $\mathcal{F} \cap \mathcal{H}$. Now, let (x, y) belong to $D \times D$. There are four possibilities to consider:

(i) $x \in A$ and $y \in A$. Then $(x, y) \in A \times A \subseteq V \subseteq U$.

(ii) $x \in B$ and $y \in B$. Then $(x, y) \in B \times B \subseteq V \subseteq U$.

(iii) $x \in A$ and $y \in B$. Since both A and B belong to $\mathcal{G}$, $A \cap B$ is nonempty so we can pick $z \in A \cap B$. Then $(x, z) \in A \times A$ and $(z, y) \in B \times B$ so $(x, z) \in V$ and $(z, y) \in V$ whence $(x, y) \in V \circ V \subseteq U$.

(iv) The case $x \in B$ and $y \in A$ is dealt with in the same way.

□

Lemma 3.21. *$i(X)$ is dense in $\widehat{X}$.*

Proof. Let $\mathcal{F} \in \widehat{X}$ be a minimal Cauchy filter, and $U \in \mathcal{U}$. Since $\mathcal{F}$ is Cauchy, there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$. In particular for all $x \in F$ we have $F \subseteq U[x]$ so $U[x] \in \mathcal{F}$. Since $U[x]$ is a neighborhood of x we conclude that $(\mathcal{F}, \mathcal{V}_x) \in C(U)$. Hence $\mathcal{V}_x = i(x)$ belongs to the neighborhood $C(U)[\mathcal{F}]$. □

Lemma 3.22. *The map i is uniformly continuous; for any $U \in \mathcal{U}$ and $x, y \in X$ if $(\mathcal{V}_x, \mathcal{V}_y) \in C(U)$ then $(x, y) \in U$.*

It follows from this that if $(X, \mathcal{U})$ is Hausdorff the map i is a uniform isomorphism from $(X, \mathcal{U})$ to $i(X)$ endowed with the uniformity induced by $\widehat{\mathcal{U}}$.

Proof. Fix $U \in \mathcal{U}$, and let $V \in \mathcal{U}$ be open, symmetric and such that $V \circ V \subseteq U$. Assume that $(x, y) \in V$. Then $V[y]$ is a neighborhood of x and y; and $V[y] \times V[y] \subseteq U$. So $(\mathcal{V}_x, \mathcal{V}_y) \in C[U]$, and this proves that i is uniformly continuous.
The second assertion follows immediately from the definition: if $(\mathcal{V}_x, \mathcal{V}_y) \in C(U)$ there exists a neighborhood A of both x and y such that $A \times A \subseteq U$, in particular $(x, y) \in U$. □

Lemma 3.23. *The uniform space $(\widehat{X}, \widehat{\mathcal{U}})$ is Hausdorff.*

Proof. Assume that $(\mathcal{F}, \mathcal{G}) \in \bigcap_{U \in \mathcal{U}} C(U)$; we have to prove that $\mathcal{F} = \mathcal{G}$.
Let $\mathcal{H}$ be the filter generated by sets of the form $F \cup G$ for $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Clearly $\mathcal{H}$ is a filter which is contained in both $\mathcal{F}$ and $\mathcal{G}$; if we prove that $\mathcal{H}$ is a Cauchy filter then by minimality of $\mathcal{F}$ and $\mathcal{G}$ we will conclude that $\mathcal{F} = \mathcal{H} = \mathcal{G}$.
Choose $U \in \mathcal{U}$. There exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$, as well as $G \in \mathcal{G}$ such that $G \times G \subseteq U$; by assumption on $(\mathcal{F}, \mathcal{G})$ there also exists $A \in \mathcal{F} \cap \mathcal{G}$ such that $A \times A \subseteq U$. Replacing F by $F \cap A$, G by $G \cap A$ we obtain that there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that $F \times F \subseteq U$, $G \times G \subseteq U$, $F \times G \subseteq U$ and $G \times F \subseteq U$.
It follows that $(F \cup G) \times (F \cup G) \subseteq U$, whence $\mathcal{H}$ is a Cauchy filter. □

Lemma 3.24. *The uniform space $(\widehat{X}, \widehat{\mathcal{U}})$ is complete. It is called the* Hausdorff completion *of $(X, \mathcal{U})$.*

Proof. It is enough to prove that for any Cauchy filter $\mathcal{F}$ on $i(X)$ the filter $\widehat{\mathcal{F}}$ that it generates on $\widehat{X}$ is convergent in $\widehat{X}$ (see Lemma 3.18, whose notations we reuse here). Let $\mathcal{G} = i^{-1}(\mathcal{F})$, which is a Cauchy filter on X by Lemma 3.22. Then $\mathcal{G}^{\mathcal{U}}$ is a Cauchy filter on X, contained in $\mathcal{G}$ (see 3.18).
Let us show that $\mathcal{G}^{\mathcal{U}}$ is a minimal Cauchy filter: assume that $\mathcal{H} \subseteq \mathcal{G}$ is also a Cauchy filter, then fix $G \in \mathcal{G}$ and $U \in \mathcal{U}$. There exists $H \in \mathcal{H}$ such that $H \times H \subseteq U$; replacing H by $G \cap H$ if necessary we may assume that $H \subseteq G$. Since $H \times H \subseteq U$ we have $H \subseteq U[H] \subseteq U[G]$, hence $U[G] \in \mathcal{H}$ and we obtain $\mathcal{G}^{\mathcal{U}} \subseteq \mathcal{H}$.
It follows as promised that $\mathcal{G}^{\mathcal{U}}$ is a minimal Cauchy filter (even, the unique minimal Cauchy filter contained in $\mathcal{G}$), and our last step is to prove that $\widehat{\mathcal{F}}$ converges to it in $\widehat{X}$, i.e. that the filter of neighborhoods of $\mathcal{G}^{\mathcal{U}}$ is contained in $\widehat{\mathcal{F}}$.
By definition of $\widehat{\mathcal{F}}$, we need to show that for every $U \in \mathcal{U}$ we have $C(U)[\mathcal{G}^{\mathcal{U}}] \cap i(X) \in \mathcal{F}$, equivalently that $\Sigma := \{x \in X \colon \mathcal{V}_x \in C(U)[\mathcal{G}^{\mathcal{U}}]\} \in \mathcal{G}$.
Explicitly, we have

$$\Sigma = \left\{ x \in X \colon \exists A \in \mathcal{V}_x \cap \mathcal{G}^{\mathcal{U}} \colon A \times A \subseteq U \right\}$$

Pick $U \in \mathcal{U}$ and let $V \in \mathcal{U}$ be symmetric and such that $V \circ V \circ V \subseteq U$. There exists $G \in \mathcal{G}$ such that $G \times G \subseteq V$, hence $V[G] \times V[G] \subseteq V \circ V \circ V \subseteq U$.
For every $x \in G$, $V[G]$ is a neighborhood of x, it belongs to $\mathcal{G}^{\mathcal{U}}$ and $V[G] \times V[G] \subseteq U$; this proves that $G \subseteq \Sigma$, hence $\Sigma \in \mathcal{G}$ and we are done. □

Exercise 39. Let $(X, \mathcal{U})$ be a uniform space, and $(\widehat{X}, \widehat{\mathcal{U}})$ be its Hausdorff completion.
Show that for any complete Hausdorff uniform space $(Y, \mathcal{V})$ and any uniformly continuous $f \colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ there exists a uniformly continuous $\hat{f} \colon \widehat{X} \to Y$ such that $\hat{f} \circ i = f$.
Explain in what sense this property characterizes $(\widehat{X}, \widehat{\mathcal{U}})$.

Exercise 40. Let $(X, \mathcal{U})$ be a uniform space, and Y be a dense subspace of X. Prove that $(\widehat{Y}, \widehat{\mathcal{U}}) = (\widehat{X}, \widehat{\mathcal{U}})$.

Exercise 41. Let $(X, \mathcal{U})$ be a Hausdorff uniform space. Show that $(X, \mathcal{U})$ is totally bounded iff $(\widehat{X}, \widehat{\mathcal{U}})$ is compact.

Bibliographical comments. There are many sources regarding uniform structures, for instance [Eng89]. The books [AT08] and[RD81] are specifically concerned with topological groups.

Chapter 4

Uniform structures on Polish groups

Definition 4.1. Let G be a topological group. We introduce the following four uniformities on G:

(i) The *left uniformity* $\mathcal{U}_l$ is generated by the entourages $\{(g,h)\colon g^{-1}h \in U\}$, where U ranges over all neighborhoods of 1.

(ii) The *right uniformity* $\mathcal{U}_r$ is generated by the entourages $\{(g,h)\colon gh^{-1} \in U\}$, where U ranges over all neighborhoods of 1.

(iii) The *upper uniformity* $\mathcal{U}_+$ is the coarsest uniformity refining both the left and right uniformities; the sets $\{(g,h)\colon gh^{-1} \in U \text{ and } g^{-1}h \in U\}$, where U ranges over all neighborhoods of 1, form a fundamental system of entourages for $\mathcal{U}_+$.

(iv) The *lower uniformity* or *Roelcke uniformity* $\mathcal{U}_{Roelcke}$ is the finest uniformity coarser that both the left and right uniformities; a fundamental system for $\mathcal{U}_{Roelcke}$ is given by sets of the form $\{(g,h)\colon h \in UgU\}$, where U ranges over all neighborhoods of 1.

Note that in general $\mathcal{U}_l$ and $\mathcal{U}_r$ do not coincide and then all four uniformities above are distinct. Clearly left-translations are uniform isomorphisms of $(G,\mathcal{U}_l)$, right-translations are uniform isomorphisms of $(G,\mathcal{U}_r)$ and the inverse map is a uniform isomorphism of $(G,\mathcal{U}_+)$. Given that $\mathcal{U}_r$ and $\mathcal{U}_l$ do not coincide in general, the next result is maybe a little surprising.

Exercise 42. Show that for each g the map $h \mapsto hg$ is a uniform isomorphism of $(G,\mathcal{U}_l)$, and the map $h \mapsto gh$ is a uniform isomorphism of $(G,\mathcal{U}_r)$.
Prove that each left translation, as well as each right translation, is a uniform isomorphism for $\mathcal{U}_+$ and $\mathcal{U}_{Roelcke}$.

Proposition 4.2. *Let (G,τ) be a topological group. Each of the four uniformities above induces the topology of G.*

Proof. Clearly, $\tau_{Roelcke} \subseteq \tau_l$, $\tau_r \subseteq \tau_+$.
Assume that $A \subseteq G$ is τ_+-open and let $g \in A$. Then there exists a τ-neighborhood U of 1 such that A contains $\{h\colon g^{-1}h \in U \text{ and } gh^{-1} \in U\} = gU \cap U^{-1}g$, which is a τ-neighborhood of g. Hence $\tau_+ \subseteq \tau$.
Conversely, let $A \subseteq G$ be τ-open and let $g \in A$. The map $\varphi\colon (f_1,f_2) \mapsto f_1 g f_2$ is τ-continuous, and $\varphi(1,1) = g$. Hence there exists a neighborhood U of 1 such that $\varphi(U \times U) \subseteq A$. So

$UgU \subseteq A$, and UgU is a $\tau_{Roelcke}$-neighborhood of g. This proves that $\tau \subseteq \tau_{Roelcke}$, which concludes the proof. □

Theorem 4.3 (Birkhoff–Kakutani). *Let G be a Hausdorff topological group such that 1 has a countable basis of neighborhoods. Then G admits a compatible left-invariant metric.*

Proof. All four uniformities defined above are Hausdorff and admit a fundamental system of entourages as soon as G is first-countable. So theorem 3.7 implies that all four uniformities are metrizable under our assumptions on G.
Now, let d be a bounded metric inducing $\mathcal{U}_l$. While there is no reason for d to be left-invariant, we can consider a new metric ρ on G defined by $\rho(g,h) = \sup\{d(kg,kh) \colon k \in G\}$. This metric is left-invariant and $id \colon (G,\rho) \to (G,d)$ is 1-Lipschitz hence $id \colon (G,\rho) \to (G,\mathcal{U}_l)$ is uniformly continuous.
To see the converse, fix $\varepsilon > 0$. Since d induces $\mathcal{U}_l$, there exists a neighborhood U of 1 such that for all h one has $g^{-1}h \in U \Rightarrow d(g,h) \leq \varepsilon$. But then whenever $g^{-1}h \in U$ we have $\rho(g,h) \leq \varepsilon$, and this proves that $id \colon (G,\mathcal{U}_l) \to (G,\rho)$ is uniformly continuous. □

Exercise 43. Let G be a metrizable topological group and d be a left-invariant compatible metric. Prove that the uniformity induced by d coincides with $\mathcal{U}_l$.
Use this to recover the result that two left-invariant metrics inducing the same topology have the same Cauchy sequences.

Exercise 44. Let G be a locally compact, first-countable, Hausdorff topological group. Prove that for any left-invariant metric d on G there is $r > 0$ such that closed balls of radius less than r are compact. Use this to prove that any left-invariant metric on G is complete [(i)].

Exercise 45. Let G be a first-countable, Hausdorff topological group. Prove that G admits a compatible bi-invariant metric if, and only if, 1 admits a basis of neighborhoods which are conjugacy invariant (i.e. $gUg^{-1} = U$ for each $g \in G$ and each U in the basis).

Exercise 46. Let G be a first-countable, Hausdorff topological group. Prove that $(g,h) \mapsto gh$ is left-uniformly continuous (i.e. uniformly continuous from $(G,\mathcal{U}_l) \times (G,\mathcal{U}_l)$ to $(G,\mathcal{U}_l)$) iff it is right-uniformly continuous iff G admits a compatible bi-invariant metric.
Contrast this with the result of Exercise 42.

Exercise 47. Let G be a Polish group. Prove that $(G,\mathcal{U}_+)$ is complete (one says that G is *Raikov-complete*).
Prove that if $\mathcal{U}_l$ and $\mathcal{U}_r$ coincide then they are both complete (note, however, that this is only an implication and not an equivalence).

Proposition 4.4 (Solecki). *Let G be a Polish group such that $(G,\mathcal{U}_l)$ is precompact (i.e. its completion is compact). Then G is compact.*

Proof. Here we may simply work with sequences since $\mathcal{U}_l$ is metrizable (and the completion is obtained by taking the metric completion of (G,d_l) for some compatible left-invariant metric d_l on G). Let $(g_n)_{n<\omega}$ be a sequence of elements of G.

[(i)] Actually a second countable, locally compact, Hausdorff group admits a left-invariant metric where every closed ball is compact, but that is harder to prove. The first version of this exercise mistakenly asked to show an even stronger, and false, result...

Since $(G, \mathcal{U}_l)$ is precompact, $(g_n)_{n<\omega}$ admits a subsequence $(g_{\varphi(n)})_{n<\omega}$ which is Cauchy in $\mathcal{U}_l$; applying this to $(g^{-1}_{\varphi(n)})$, we obtain that $(g_n)_{n<\omega}$ admits a subsequence $(g_{\psi(n)})_{n<\omega}$ which is Cauchy both for $\mathcal{U}_l$ and for $\mathcal{U}_r$, hence Cauchy for $\mathcal{U}_+$. Since $\mathcal{U}_+$ is complete, we conclude that $(g_{\psi(n)})_{n<\omega}$ is convergent. Hence G is compact. □

The situation is completely different for the Roelcke uniformity.

Definition 4.5. A topological group is *Roelcke precompact* if the Hausdorff completion of $(G, \mathcal{U}_{Roelcke})$ is compact.

Note that this amounts to saying that $(G, \mathcal{U}_{Roelcke})$ is totally bounded, i.e. that for every $V \in \mathcal{U}_{Roelcke}$ there exists a finite subset $F \subseteq G$ such that $G = V[F]$. Unpacking this a little further, we obtain that G is Roelcke precompact iff for every nonempty open U there exists a finite F such that $UFU = G$.

Proposition 4.6. *Let G be a topological group which is both locally compact and Roelcke precompact. Then G is compact.*

Proof. Since G is locally compact, there exists a compact subset K of G with a nonempty interior. By Roelcke precompactness $G = KFK$ for some finite set $F \subseteq G$, and KFK is compact since $(f, g, h) \mapsto fgh$ is continuous. □

However, many large Polish groups of interest are Roelcke-precompact, and there is a rich connection with model theory.

Theorem 4.7. *Every oligomorphic subgroup of $\mathfrak{S}_\infty$ is Roelcke precompact.*

Recall that $G \leq \mathfrak{S}_\infty$ is oligomorphic iff there are finitely many orbits for the action $G \curvearrowright \omega^k$ for each k; oligomorphic closed subgroups of $\mathfrak{S}_\infty$ are exactly the automorphism groups of $\aleph_0$-categorical countable structures (in particular, automorphism groups of Fraïssé limits in a finite relational language are oligomorphic).

Proof. Let U be a neighborhood of 1; we may assume that $U = \{g \in G \colon \forall i \leq n \ g(i) = i\}$ is a clopen subgroup. We need to find a finite subset F of G such that $G = UFU$, i.e. prove that there are only finitely many disjoint double cosets UgU.
Double cosets UgU are in bijection with orbits for the diagonal action $G \curvearrowright G/U \times G/U$. Indeed, any orbit for this action contains an element of the form (U, gU) (by translating on the first coordinate); and (U, gU), (U, hU) belong to the same orbit iff there exists k such that $kU = U$ and $kgU = hU$, i.e. iff there exists $k \in U$ such that $kgU = hU$, which is equivalent to $g \in UhU$. So assigning to the orbit of (U, gU) the double coset UgU gives the desired bijection.
Since U is the pointwise stabilizer of $\{0, \dots, n\}$, $G \curvearrowright G/U$ can be seen as the action of G on a subset of ω^{n+1} (the orbit of $(0, \dots, n)$); hence $G \curvearrowright G/U \times G/U$ is a subaction of $G \curvearrowright \omega^{n+1} \times \omega^{n+1}$, which has only finitely many orbits since G is oligomorphic. So there are finitely many double cosets UgU and we are done. □

For instance, $\mathfrak{S}_\infty$ itself is Roelcke precompact, as are the automorphism groups of the random graph, of $(\mathbb{Q}, <)$, and of any Fraïssé limit in a finite relational language. This gives many examples of noncompact, Roelcke precompact Polish groups (there are also some interesting connected examples, and a strong connection with continuous logic, though we will not develop that here).

Exercise 48 (Tsankov)**.** Prove that the following conditions are equivalent, for a subgroup $G \leq \mathfrak{S}_\infty$:

(i) G is Roelcke precompact.

(ii) For every continuous action $G \curvearrowright X$ of G on a countable, discrete set X with finitely many orbits, the diagonal action $G \curvearrowright X^n$ has finitely many orbits for all n.

(iii) G is isomorphic (as a topological group) to an inverse limit of oligomorphic subgroups of $\mathfrak{S}_\infty$.

(For the last implication, show first that an inverse limit of Roelcke precompact groups is still Roelcke precompact)

Exercise 49. Prove that a Roelcke precompact subgroup of $\mathfrak{S}_\infty$ has only (at most) countably many open subgroups.

Theorem 4.8 (Tsankov)**.** *Assume that G is a Roelcke precompact topological group which acts isometrically on a metric space (X, d) so that $g \mapsto g \cdot x$ is continuous for each $x \in X$. Then every G-orbit is bounded.*
In particular, a left-invariant continous pseudometric on any Roelcke precompact topological group is always bounded (this applies to every left-invariant continuous metric on a Roelcke precompact Polish group).

Proof. Fix $x_0 \in X$ and let $\varphi(g) = d(x_0, g \cdot x_0)$. We claim that φ is Roelcke-uniformly continuous (that is, both left- and right-uniformly continuous).
Granting that, there exists $\hat{\varphi}\colon \widehat{(G, \mathcal{U}_{Roelcke})} \to \mathbb{R}$ uniformly continuous such that $\hat{\varphi} \circ i = \varphi$ (where i denotes the map $x \mapsto \mathcal{V}_x$ from $(G, \mathcal{U}_{Roelcke})$ to its Hausdorff completion). Since $\widehat{(G, \mathcal{U}_{Roelcke})}$ is compact, the image of φ is then contained in a compact subset of $\mathbb{R}$, hence is bounded.
Now we establish our claim. Let f, g, h belong to G. We have

$$\begin{aligned} d(x_0, fgh \cdot x_0) &= d(f^{-1} \cdot x_0, gh \cdot x_0) \\ &\leq d(f^{-1} \cdot x_0, x_0) + d(x_0, g \cdot x_0) + d(g \cdot x_0, gh \cdot x_0) \\ &\leq d(x_0, f \cdot x_0) + d(x_0, g \cdot x_0) + d(x_0, h \cdot x_0) \end{aligned}$$

We conclude that for all f, g, h one has

$$d(x_0, fgh \cdot x_0) - d(x_0, g \cdot x_0) \leq d(x_0, f \cdot x_0) + d(x_0, h \cdot x_0)$$

Applying this inequality to f^{-1}, fgh, h^{-1} we obtain

$$d(x_0, g \cdot x_0) - d(x_0, fgh \cdot x_0) \leq d(x_0, f^{-1} \cdot x_0) + d(x_0, h^{-1} \cdot x_0) = d(x_0, f \cdot x_0) + d(x_0, h \cdot x_0)$$

Finally we obtain that for every f, g, h

$$|d(x_0, fgh \cdot x_0) - d(x_0, g \cdot x_0)| \leq d(x_0, f \cdot x_0) + d(x_0, h \cdot x_0)$$

Fix $\varepsilon > 0$. Since $g \mapsto gx_0$ is continuous, there exists a neighborhood U of 1 such that $d(u \cdot x_0, x_0) \leq \varepsilon$ for all $u \in U$. This yields that for all (g, h) such that $h \in UgU$ we have $|\varphi(g) - \varphi(h)| \leq 2\varepsilon$, proving that φ is Roelke-uniformly continuous as claimed. $\square$

Bibliographical comments. One can again consult [RD81] or [AT08] for more information about uniformities on topological group. The interest in Roelcke precompact Polish groups stems in large part from work of Uspenskij [Usp02] and then Rosendal [Ros09], Tsankov [Tsa12] and Ben Yaacov–Tsankov [BT16]. The latter paper in particular should appeal to anyone with an interest in the interaction between continuous logic and Polish groups.

Chapter 5

Compactifications

Definition 5.1. We let $\beta\omega$ be the space of all ultrafilters on ω, which we endow with the topology whose basic open sets are of the form $[A] = \{p \in \beta\omega \colon A \in p\}$, with A a nonempty subset of ω.

Equivalently, we endow the space of ultrafilters on ω with the topology induced by the product topology on $2^{\mathcal{P}(\omega)}$, identifying each ultrafilter with its characteristic function. Indeed, a basic open set for this topology is of the form

$$U_{F_1,F_2} = \{p \colon \forall A \in F_1 \; p(A) = 1 \text{ and } \forall B \in F_2 \; p(B) = 0\}$$

where F_1, F_2 are finite subsets of $\mathcal{P}(\omega)$. A filter contains every element of F_1 iff it contains $B = \bigcap_{A \in F_1} A$; and an ultrafilter contains no element of F_2 iff it contains $C = \bigcap_{A \in F_2}(\omega \setminus A)$. Letting $A = B \cap C$, we see that $\beta\omega \cap U_{F_1,F_2} = [A]$.

Proposition 5.2. *(i) The space $\beta\omega$ is compact Hausdorff.*

(ii) The set of principal ultrafilters is dense in $\beta\omega$[(ii)].

(iii) The closure of any open subset of $\beta\omega$ is open (one says that $\beta\omega$ is extremally disconnected*). Clopen subsets of $\beta\omega$ are of the form $[A]$ for some $A \subseteq \omega$.*

(iv) For any two disjoint open subsets U, V of $\beta\omega$ one has $\overline{U} \cap \overline{V} = \emptyset$.

(v) For any convergent sequence $(p_n)_{n<\omega}$ in $\beta\omega$, there exists some N such that $p_n = p_N$ for each $n > N$.

Proof. Each of the following sets is a clopen subset of $2^{\mathcal{P}(\omega)}$ (below A, B are subsets of ω):

- $\Sigma_1 = \{p \colon p(\omega) = 1 \text{ and } p(\emptyset) = 0\}$.
- $\Sigma_2(A, B) = \{p \colon (p(A) = 1 \text{ and } p(B) = 1) \Rightarrow p(A \cap B) = 1\}$.
- $\Sigma_3(A, B) = \{p \colon (p(A) = 1 \text{ and } A \subseteq B) \Rightarrow p(B) = 1\}$
- $\Sigma_4(A) = \{p \colon p(A) = 1 \text{ or } p(\omega \setminus A) = 1\}$

(ii) So below we identify each $n \in \omega$ with the corresponding element in $\beta\omega$, thus viewing ω as a dense subset of $\beta\omega$.

And we have
$$\beta\omega = \Sigma_1 \cap \bigcap_{A,B} \Sigma_2(A,B) \cap \bigcap_{A,B} \Sigma_3(A,B) \cap \bigcap_{A} \Sigma_4(A)$$

This proves that $\beta\omega$ is closed in $2^{\mathcal{P}(\omega)}$, hence compact Hausdorff.
Next, let A be a nonempty subset of ω. For any $n \in A$ the principal ultrafilter associated to n belongs to $[A]$, proving that ω is dense in $\beta\omega$.
Let U be open in $\beta\omega$, and set $A = U \cap \omega$. Since U is open we have $\overline{U} = \overline{A}$. We claim that $\overline{A} = [A]$, which is clopen (it is open and its complement is $[X \setminus A]$, which is open). To see this, note first that $A \subset [A]$ by definition. Conversely, let $p \in \beta\omega$ contain A, and let $[B]$ be a basic open neighborhood of p. Then both A and B belong to p, so $A \cap B \in p$ and is in particular nonempty. For any $n \in A \cap B$ we have $n \in A \cap [B]$, whence $A \cap [B] \neq \emptyset$. It follows that $p \in \overline{A}$. So $\overline{A} = [A]$ as promised.
Assume U, V are disjoint open. Then $U \cap \omega = A$ and $V \cap \omega = B$ are disjoint nonempty subsets of ω, hence $[A] \cap [B] = \emptyset$. Given what we just proved above, $\overline{U} \cap \overline{V} = [A] \cap [B] = \emptyset$.
Finally, let $(p_n)_{n<\omega}$ be a convergent sequence. If it is not eventually constant, we may up to some extraction assume that it is an injective sequence which converges to some $p \notin \{p_n \colon n < \omega\}$. We may then inductively build a sequence of open subsets $(O_n)_{n<\omega}$ of $\beta\omega$ with $p_n \in O_n$ for all n and $O_n \cap O_m = \emptyset$ for all $n \neq m$. Let $U = \bigcup_n O_{2n}$ and $V = \bigcup_n O_{2n+1}$. Then U and V are disjoint open but $p \in \overline{U} \cap \overline{V}$, a contradiction. □

Proposition 5.3. *Let X be a compact Hausdorff topological space, and $f \colon \omega \to X$ a function. Then there exists a continuous $\hat{f} \colon \beta\omega \to K$ which extends f.*

Proof. Given $p \in \beta\omega$, $f(p)$ is an ultrafilter on X, hence is convergent since X is compact. We may then set $\hat{f}(p) = \lim f(p)$. Clearly $\hat{f}$ extends f, and it remains to prove that $\hat{f}$ is continuous.
Fix $p \in \beta\omega$, let $x = \hat{f}(p)$, and let V be a neighborhood of x. Since X is compact Hausdorff it is regular, i.e. there exists a neighborhood W of x such that $\overline{W} \subseteq V$.
By definition of convergence of filters, W belongs to $f(p)$, equivalently $\{n \colon f(n) \in W\}$ belongs to p. Let $A = \{n \colon f(n) \in W\}$; $A \in p$ so $p \in [A]$. For every $q \in [A]$ we have $W \in f(q)$, so $\lim f(q) \in \overline{W}$ hence $\hat{f}(q) \in V$ for all $q \in [A]$, proving that $\hat{f}$ is continuous at p. □

Exercise 50. Explain in what sense the above proposition characterizes $\beta\omega$ among all compact spaces in which ω densely embeds.

The space $\beta\omega$ is called the *Stone-Čech* compactification of ω. As we saw, it is a compact space to which ω maps densely (and injectively) so that all bounded functions from ω to $\mathbb{R}$ (for instance) extend continuously. This leads us to the general notion of a *compactification*.

Definition 5.4. Let X be a topological space. A *compactification* is a pair (Y, φ), where Y is compact and φ is continuous and has a dense image.
We say that (Y, φ) is *proper* if φ is a homeomorphism onto $\varphi(X)$.

Note that we do not even ask that φ is injective, for instance the constant map from X to $\{0\}$ is a compactification in the above sense (beware that in the literature terminologies regarding compactifications vary!).
Given a topological space X, we denote by $C_b(X)$ the $*$-algebra of all bounded continuous functions from X to $\mathbb{C}$ (and simply denote it by $C(X)$ when X is compact). Endowed with the

sup nom $\|\cdot\|_\infty$ this is a unital complex *-algebra, i.e. a complex Banach space equipped with an internal product satisfying the algebra axioms, with a unit element (the constant function 1) such that $\|fg\| \leq \|f\|\|g\|$ and an involution preseving the norm (given by $f^* = \bar{f}$).

Proposition 5.5. *Let* (Y, φ) *be a compactification of a topological space* X*. Then* $\{f \circ \varphi \colon f \in C(Y)\}$ *is a closed unital* $*$*-subalgebra of* $C_b(X)$*.*

Proof. It is immediate that $\{f \circ \varphi \colon f \in C(Y)\}$ is a $*$-algebra which contains 1; since $\varphi(X)$ is dense the map $\Phi\colon f \mapsto f \circ \varphi$ is norm preserving from $(C(Y), \|\cdot\|_\infty)$ to $(C_b(X), \|\cdot\|_\infty)$. Since $C(Y)$ is complete its image under Φ also is, hence it is closed in $C_b(X)$. □

Since we are going to manipulate continuous maps to $\mathbb{C}$, there is no real point in bothering with non-Hausdorff spaces (our maps will factor through a Hausdorff space) so for simplicity we will insist that in this chapter all our spaces are Hausdorff from now on (we could have done that from the start, non-Hausdorff spaces do not play any role in these notes).

Theorem 5.6 (Gelfand–Naimark)**.** *Let* X *be a Hausdorff topological space, and* $A \subseteq C_b(X)$ *be a unital closed* $*$*-subalgebra.*
There exists a Hausdorff compactification (Y, φ) *such that* $A = \{f \circ \varphi \colon f \in C(Y)\}$*.*
This compactification is proper iff A *separates points and closed sets, i.e. iff for every* $x \in X$ *and every closed* $F \not\ni x$ *there exists* $a \in A$ *such that* $a(x) \notin \overline{a(F)}$*.*

Note that in the theorem above A must be isomorphic (as a Banach algebra) with $C(Y)$.

Proof. Fix A as in the statement of the theorem. For every $a \in A$, denote $I_a = \overline{a(X)}$, which is a compact subset of $\mathbb{C}$. Define $Z = \prod_{a \in A} I_a$ and set $\varphi\colon X \to Z$ so that $\varphi(x) = (a(x))_{a \in A}$. Let $Y = \overline{\varphi(X)}$, which is compact. Clearly (Y, φ) is a Hausdorff compactification of X.
Given $a \in A$, let $f_a\colon Y \to \mathbb{C}$ be defined by $f_a(y) = y(a)$. It is continuous and by definition for every $x \in X$ we have $f_a(\varphi(x)) = a(x)$, in other words $a = f_a \circ \varphi$.
Note that $a \mapsto f_a$ is a $*$-algebra morphism from A to $C(Y)$. Indeed, for every $a_1, a_2, a_3 \in A$, every $\lambda \in \mathbb{C}$ and every $x \in X$ we have

$$f_{(\lambda a_1 + a_2)a_3}(\varphi(x)) = ((\lambda a_1 + a_2)a_3)(x) = ((\lambda f_{a_1} + f_{a_2})f_{a_3})(\varphi(x))$$

Since $\varphi(X)$ is dense in Y we obtain $f_{(\lambda a_1 + a_2)a_3} = (\lambda f_{a_1} + f_{a_2})f_{a_3}$. Similarly we check that $f_{a^*} = (f_a)^*$ for every $a \in A$.
So $\widehat{A} = \{f_a \colon a \in A\}$ is a $*$-subalgebra of $C(Y)$, and for every $a \in A$ we have (again by density of $\varphi(X)$ in Y)

$$\|f_a\|_\infty = \sup\{|f_a(\varphi(x))| \colon x \in X\} = \sup\{|a(x)| \colon x \in X\} = \|a\|_\infty$$

It follows that $\widehat{A}$ is closed in $C(Y)$ (it is isometric to A, which is complete). By definition, $\widehat{A}$ separates points: if $y_1 \neq y_2 \in Y$ then for some a we have $y_1(a) \neq y_2(a)$, i.e. $f_a(y_1) \neq f_a(y_2)$. By the Stone–Weierstrass theorem, it follows that $\widehat{A} = C(Y)$. Finally,

$$\{f \circ \varphi \colon f \in C(Y)\} = \{f_a \circ \varphi \colon a \in A\} = A$$

This takes care of the first part of the statement. Now, assume that the compactification (Y, φ) is proper and let $x \in X$, F closed such that $x \notin F$. Then $\overline{\varphi(F)}$ is a compact subset of Y

which does not contain $\varphi(x)$, so there exists $f \in C(Y)$ such that $f(\varphi(x)) \notin \overline{f(\varphi(F))}$. Since $f \circ \varphi$ belongs to A, this proves that A separates points and closed sets.
To conclude the proof, assume that A separates points and closed sets (whence φ is injective), and let U be an open subset of X and $x \in U$. There exists $a \in A$ such that $a(x) \notin \overline{a(X \setminus U)}$. For some $f \in C(Y)$ we have $a = f \circ \varphi$. Then $V = \left\{ y \colon f(y) \notin \overline{a(X \setminus U)} \right\}$ is open in Y, contains $\varphi(x)$, and $\varphi(X) \cap V \subseteq \varphi(U)$. Hence $\varphi \colon X \to \varphi(X)$ is open. □

We have established that Hausdorff compactifications of a Hausdorff space X correspond to closed unital $*$-subalgebras of $C_b(X)$. On the algebra side, we have a natural partial ordering given by inclusion (with $C_b(X)$ as its maximum). We now discuss what this partial order corresponds to for Hausdorff compactifications.

Theorem 5.7. *Let X be a Hausdorff topological space, and (Y_1, φ_1), (Y_2, φ_2) be two Hausdorff compactifications of X. Let $A_1 = \{f \circ \varphi_1 \colon f \in C(Y_1)\}$ and $A_2 = \{f \circ \varphi_2 \colon f \in C(Y_2)\}$.*
Then the following conditions are equivalent:

(i) $A_2 \subseteq A_1$.

(ii) There exists a continuous $\psi \colon Y_1 \to Y_2$ such that $\varphi_2 = \psi \circ \varphi_1$.

Proof. One implication is immediate: indeed, if there exists a continuous $\psi \colon Y_1 \to Y_2$ such that $\varphi_2 = \psi \circ \varphi_1$ then we have

$$A_2 = \{f \circ \varphi_2 \colon f \in C(Y_2)\} = \{(f \circ \psi) \circ \varphi_1 \colon f \in C(Y_2)\} \subseteq \{f \circ \varphi_1 \colon f \in C(Y_1)\} = A_1$$

Assume that $A_2 \subseteq A_1$. Note that if x, x' are such that $\varphi_2(x) \neq \varphi_2(x')$ then there exists $a \in A_2$ so that $a(x) \neq a(x')$ (because $C_b(Y_2)$ separates points). Since $A_2 \subseteq A_1$ there exists $g \in C_b(Y_1)$ so that $g \circ \varphi_1 = a$, hence $\varphi_1(x) \neq \varphi_1(x')$.
This means that we can define $\psi \colon \varphi_1(X) \to \varphi_2(X)$ by setting $\psi(\varphi_1(x)) = \varphi_2(x)$. Certainly we guaranteed that $\psi \circ \varphi_1 = \varphi_2$, but we do not know yet how to extend ψ to Y_1, or whether it is continuous.
Both issues are taken care of at once if we prove that ψ is uniformly continuous from $\varphi_1(X)$ to Y_2 (for the uniformities coming from the compact topologies of Y_1, Y_2) since Y_2 is complete for its unique compatible uniformity. By proposition 3.10 this is equivalent to proving that for every confinuous $f \colon Y_2 \to \mathbb{R}$, $f \circ \psi$ is uniformly continuous on $\varphi_1(X)$.
By definition, we have for all x that $f \circ \psi(\varphi_1(x)) = f \circ \varphi_2(x)$. Thus $f \circ \psi \circ \varphi_1 \in A_2$, so the exists $g \in C(Y_1)$ such that $f \circ \psi \circ \varphi_1 = g \circ \varphi_1$. Since g is uniformly continuous on Y_1, $f \circ \psi$ is uniformly continuous on $\varphi_1(X)$. Hence ψ is uniformly continuous, so it extends to a continuous map $\psi \colon \varphi_1(X) \to \varphi_2(X)$. □

In particular, two Hausdorff compactifications (Y_1, φ_1) and (Y_2, φ_2) which correspond to the same subalgebra of $C_b(X)$ are such that there exist a continuous $\psi \colon Y_1 \to Y_2$ such that $\psi(\varphi_1(x)) = \varphi_2(x)$ for all $x \in X$, and a continuous $\tilde{\psi} \colon Y_2 \to Y_1$ such that $\tilde{\psi}(\varphi_2(x)) = \varphi_1(x)$ for all x. By density it follows that $\tilde{\psi} = \psi^{-1}$, so ψ is a homeomorphism from $Y_1 \to Y_2$ such that $\varphi_2 = \psi \circ \varphi_2$. When that situation occurs we say that the two compactifications are *equivalent*.
Up to equivalence of compactifications, there is a unique compactification associated to any closed unital $*$-subalgebra A of $C_b(X)$; for $A = C_b(X)$ we obtain the *Stone–Čech compactification* $(\beta X, \beta)$ of X, which is characterized by the following universal property: for every

continuous map $\varphi\colon X \to Y$ with Y a compact Hausdorff space, there exists a continuous application $\psi\colon \beta_X \to Y$ such that $\varphi = \psi \circ \beta$.
In the particular case where $X = \omega$ with the discrete topology, we recover the space $\beta\omega$ which we already discussed, with the natural embedding of ω inside $\beta\omega$.

Proposition 5.8. *Let $(X,\mathcal{U})$ be a uniform space. The set A of all uniformly continuous functions from $(X,\mathcal{U}) \to \mathbb{C}$ is a closed unital $*$-subalgebra of $C_b(X)$.*

Proof. Clearly A contains the constant function 1 and if $f \in A$ then also $\bar{f} \in A$. To see that it is closed, one only needs to check that a uniform limit of uniformly continuous bounded functions is still uniformly continuous, and we leave that to the reader. It is also clear that A is closed under linear combinations. Closure under product follows from a general result about uniform spaces, see exercise 51 below. □

Exercise 51. Let $(X,\mathcal{U})$ be a uniform space. Assume that $f,g\colon X \to \mathbb{C}$ are uniformly continuous and bounded. Prove that fg is uniformly continuous.

Definition 5.9. Let $(X,\mathcal{U})$ be a uniform space. The *Samuel compactification* of $(X,\mathcal{U})$ is the compactification induced by the algebra of all uniformly continuous bounded functions from $(X,\mathcal{U})$ to $\mathbb{C}$.

It is tempting to consider the compactification βG associated to $C_b(G)$, however there is in general no way to extend the group operations to this compactification. We now need to know which algebras are associated to continuous group actions. Whenever G acts on X by homeomorphisms, it also acts on $C(X)$ via $(g\varphi)(x) = \varphi(g^{-1}x)$.

Lemma 5.10. *Assume that X is compact Hausdorff. Then the action $G \curvearrowright X$ is continuous iff $G \curvearrowright C(X)$ is continuous.*

Proof. Assume that G acts on X continuously. Certainly each $\varphi \mapsto g\varphi$ is continuous since it is isometric. So we need to prove continuity of the action at each $(1_G, \varphi)$, i.e. that for any $\varphi \in C(X)$ and $\varepsilon > 0$ there exists a neighborhood U of 1_G and $\delta > 0$ such that

$$g \in U \text{ and } \|\psi - \varphi\| \leq \delta \Rightarrow \|g\psi - \varphi\| \leq \varepsilon$$

Since $\|g\psi - g\varphi\| = \|\psi - \varphi\|$, this reduces to proving that for any φ and ε there exists a neighborhood U of 1 such that $\|g\varphi - \varphi\| \leq \varepsilon$ for all $g \in U$. Since φ is uniformly continuous, there exists a neighborhood V of Δ_X such that $|\varphi(x) - \varphi(y)| \leq \varepsilon$ for all $(x,y) \in V$. By continuity of the action, for every $x \in X$ there exists an open $O_x \ni x$ and a symmetric neighborhood U_x of 1_G such that $(y, gy) \in V$ for all $g \in U_x$ and all $y \in O_x$. Applying compactness we cover X by $O_{x_1}, \ldots, O_{x_n}$ and then $U = \bigcap_i U_{x_i}$ is a neighborhood of 1_G such that $(y, gy) \in V$ for all $y \in X$. It follows that for all $x \in X$ and all $g \in U$ we have $|\varphi(x) - \varphi(gx)| \leq \varepsilon$.
Conversely, assume that $G \curvearrowright C(X)$ is continuous, and let $x \in X$ and $O \ni x$ be open in X. Since X is compact Hausdorff, there exists a continuous $\varphi\colon X \to \mathbb{R}$ such that $\varphi(x) = 0$ and $\{y\colon \varphi(y) < 1\} \subseteq O$. Let $V = \{y\colon \varphi(y) < \frac{1}{2}\}$, and U a symmetric open neighborhood of 1_G such that $\|g\varphi - \varphi\| \leq \frac{1}{2}$ for all $g \in U$. Then for every $g \in U$ and every $y \in V$ we have

$$\varphi(gy) \leq \|g^{-1}\varphi - \varphi\| + \varphi(y) < 1$$

hence $gy \in O$. This proves that $(g, x) \mapsto gx$ is continuous at $(1, x)$ and we are done. □

If G acts continuously on a compact Hausdorff X, and $x_0 \in X$, the pair $(\overline{Gx_0}, i_{x_0})$ defined by $i_{x_0}(g) = gx_0$ is a compactification of G. Then $\{g \mapsto f(gx_0) \colon f \in C(X)\}$ is a closed, unital $*$-subalgebra A of $C_b(G)$, and by the previous proposition the natural action of G on A is continuous. In particular, for every $a \in A$ we have $\lim_{g \to 1} ga = a$ for every a in A, which means that for every $\varepsilon > 0$ there exists an open neighborhood U of 1_G such that

$$\forall g \in U \, \forall h \in G \, |a(g^{-1}h) - a(h)| \leq \varepsilon$$

This implies that a is right-uniformly continuous (i.e. continuous for the right uniformity $\mathcal{U}_r$ on G). Indeed, if U is as above and $hg^{-1} \in U$ then $|a(g) - a(h)| = |a((hg^{-1})^{-1}h) - a(h)| \leq \varepsilon$. Let us sum up what we just observed.

Proposition 5.11. *Let G be a topological group acting continuously on a compact Hausdorff space X, and fix $x_0 \in X$.*
Then for every $\varphi \in C(X)$ the map $g \mapsto \varphi(gx_0)$ is right-uniformly continuous.

Note that this amounts to saying that each map $g \mapsto gx$ is uniformly continuous from $(G, \mathcal{U}_r)$ to X endowed with its unique uniform structure.

Definition 5.12. Let G be a topological group. Denote by $\mathrm{RUC}_b(G)$ the $*$-algebra of all right-uniformly continuous functions from G to $\mathbb{C}$.

Proposition 5.13. *The following facts hold.*

(i) $\mathrm{RUC}_b(G)$ *is G-invariant for the action $G \curvearrowright C_b(G)$ (defined by $(g\varphi)(x) = \varphi(g^{-1}x)$).*

(ii) The action of G on $\mathrm{RUC}_b(G)$ *is continuous.*

(iii) If $A \subseteq C_b(G)$ is such that $\lim_{g \to 1} g \cdot f = f$ *for every $f \in A$, then* $A \subseteq \mathrm{RUC}_b(G)$.

Proof. (i) To check G-invariance, fix $g \in G$ and $\varphi \in \mathrm{RUC}_b(G)$. Given $\varepsilon > 0$, there exists an open $U \ni 1$ such that $|\varphi(uh) - \varphi(h)| \leq \varepsilon$ for all $u \in U$ and all $h \in G$.
This amounts to saying that $|(g\varphi)(guh) - (g\varphi)(gh)| \leq \varepsilon$ for all $u \in U$ and $h \in G$, i.e. that $|(g\varphi)(gug^{-1}k) - (g\varphi)(k)| \leq \varepsilon$ for all $u \in U$ and all $k \in G$. Since gUg^{-1} is an open neighborhood of 1 we conclude that $g\varphi \in \mathrm{RUC}_b(G)$.
(ii) Fix $\varepsilon > 0$ and $\varphi \in \mathrm{RUC}_b(G)$. We need to find $\delta > 0$ and $U \ni 1_G$ open such that

$$\forall \psi \, \forall g \quad (g \in U \text{ and } \|\varphi - \psi\| \leq \delta) \Rightarrow \|g\psi - \varphi\| \leq \varepsilon$$

This is straightforward: we have for all $h \in G$

$$|g\psi(h) - \varphi(h)| \leq \|\psi - \varphi\| + |\varphi(g^{-1}h) - \varphi(h)|$$

Since $\varphi \in \mathrm{RUC}_b(G)$, there exists a neighborhood U of 1_G such that for every $g \in U$ and every $h \in G$ we have $\|\varphi(g^{-1}h) - \varphi(h)| \leq \frac{\varepsilon}{2}$. This U along with $\delta = \frac{\varepsilon}{2}$ are what we were looking for.
(iii) is immediate from the definition of $\mathrm{RUC}_b(G)$ (and has already been pointed out above). □

Definition 5.14. Let G be a topological group. We denote by $(S(G), i)$ the Samuel compactification of $(G, \mathcal{U}_r)$.

Exercise 52. Let G be a Polish group. Prove that the compactification $(S(G), i)$ is proper. (This fact actually holds for any Hausdorff topological group, which makes for a slightly more challenging exercise)

Recall that we realized $S(G)$ "concretely" by considering

$$Y = \prod_{\varphi \in \mathrm{RUC}_b(G)} \overline{\varphi(G)}\,, \quad i(g)(\varphi) = \varphi(g)\,, \quad S(G) = \overline{i(G)}$$

Since $\mathrm{RUC}_b(G)$ is G-invariant, it follows that $G \curvearrowright Y$ via $(g \cdot y)(\varphi) = y(g^{-1}\varphi)$. Note that each $y \mapsto g \cdot y$ is a homeomorphism.

Exercise 53. Prove that the above formula indeed defines an action of G on $S(G)$, and that $i(gh) = g \cdot i(h)$ for all $g, h \in G$.

Proposition 5.15. *The action $G \curvearrowright S(G)$ is continuous.*

Proof. Let $i \colon G \to S(G)$. The map $\varphi \mapsto \varphi \circ i$ induces a $*$-algebra isomorphism from $C(S(G))$ to $\mathrm{RUC}_b(G)$, and the previous exercise shows that i is G-equivariant.
Since $G \curvearrowright \mathrm{RUC}_b(G)$ is continuous, we conclude that $G \curvearrowright C(S(G))$ is continous (i carries all the structure of one, including the action, onto the other) so $G \curvearrowright S(G)$ is continuous by Lemma 5.10. □

Exercise 54. Let $A \subseteq \mathrm{RUC}_b(G)$ be a G-invariant, closed, unital $*$-subalgebra. Let X_A be the compactification associated to X_A. Prove that the left-translation of G on itself extends to a continuous action of G on X_A.

Exercise 55. Assume G is discrete (countable if you wish) and let βG be its Stone–Čech compactification.
Prove that $G \curvearrowright S(G)$ is simply the translation action $G \curvearrowright \beta G$, where $A \in g \cdot p \Leftrightarrow g^{-1}A \in p$.

Bibliographical comments. The book [dVri93] is an encyclopedical reference about topological dynamics, in particular Appendix D contains information about compactifications, Many books cover this material, for instance the very first chapter of Folland's book [Fol16] (though with a somewhat different formulation). Uspenskij's paper [Usp02] gives good reasons to care about compactifications of topological groups.

Chapter 6

The greatest ambit and universal minimal flow

Definition 6.1. Let G be a topological group. A *G-flow* is a continuous action $G \curvearrowright X$ where X is compact Hausdorff and nonempty.
A *G-ambit* is a pair (X, x_0) where $x_0 \in X$, $X = \overline{Gx_0}$ is compact Hausdorff and $G \curvearrowright X$ is continuous (i.e. a G-ambit is a G-flow where we named a point with a dense orbit).

We identify G with its image $i(G)$ in $S(G)$ (which is a slight abuse of notation if G is not Hausdorff, since then $S(G)$ is not a proper compactification of G; but for Hausdorff topological groups this is not an issue, and as usual we only care about the Hausdorff case: since our actions are on compact Hausdorff spaces, every G action factors through the greatest Hausdorff quotient of G so that is the only group we really see acting).

Theorem 6.2. *Let (X, x_0) be a G-ambit. Then there exists a unique continuous map $\pi\colon S(G) \to X$ which is G-equivariant and such that $\pi(1_G) = x_0$.*

Note that π above is automatically surjective, since $S(G)$ is compact ang Gx_0 is dense in X. We say that $(S(G), 1_G)$ is the *greatest ambit* of G.

Proof. Consider the map $\varphi\colon g \mapsto g \cdot x_0$. Then (X, φ) is a compactification of G, and since the action $G \curvearrowright X$ is continuous we obtain that $\{f \circ \varphi\colon f \in C(X)\} \subseteq \mathrm{RUC}_b(G)$. This gives us the existence of a continuous $\pi\colon S(G) \to X$ such that $\pi \circ i = \varphi$.
We then have $\pi(1_G) = x_0$. Also, for every $g, h \in G$ we have $\pi(gh) = gh \cdot x_0 = g \cdot \pi(h)$. By continuity of $G \curvearrowright S(G)$, density of G on $S(G)$ and continuity of π we conclude that for every $g \in G$ and every $p \in S(G)$ we have $\pi(g \cdot p) = g \cdot \pi(p)$. This proves that π is G-equivariant.
Uniqueness of π is immediate, since G is dense on $S(G)$ and on G we have $\pi(g) = g \cdot x_0$. □

Definition 6.3. Let G be a topological group. A G-flow X is *minimal* if every G-orbit is dense in X.

Note that if X is minimal then (X, x_0) is an ambit for every $x_0 \in X$.

Exercise 56. (i) Prove that $G \curvearrowright X$ is minimal iff the only closed G-invariant sets are $\emptyset$ and X iff the only open G-invariant sets are $\emptyset$ and X iff for every nonempty open $U \subseteq X$ there exists a finite $F \subseteq G$ such that $X = F \cdot U$.

(ii) Let X be a G-flow. Prove that the set of all nonempty closed G-invariant subsets of X, ordered by $\supseteq$, is inductive. Apply Zorn's lemma to prove that there exists a nonempty closed G-invariant $Y \subseteq X$ such that $G \curvearrowright Y$ is minimal.

Definition 6.4. Let G be a topological group. A minimal G-flow X is *universal* if for any minimal G-flow Y there exists a continuous equivariant map $\pi \colon X \to Y$.

Proposition 6.5. *Any minimal subflow of $S(G)$ is universal.*

Proof. Let Y be a minimal G-flow, and $y \in Y$. Let M be a minimal subflow of $S(G)$. There exists a G-equivariant map $\pi \colon S(G) \to Y$ such that $\pi(1_G) = y$, and it is onto since $G \curvearrowright Y$ is minimal.
We also have $\pi(M) = Y$ since $\pi(M)$ is a subflow of Y and Y is mimnimal. Hence the restriction of π to M witnesses that M is universal. □

To obtain existence of a universal minimal flow, we could also have taken the product of a set of representatives of all possible G-flows, and taken a minimal component in that product. So there is nothing unexpected in the existence of a universal minimal flow; but it turns out to be unique up to isomorphism, which makes it an interesting object to study. To prove this we are going to introduce some additional structure on $S(G)$ (though again there are other possible arguments).

Definition 6.6. Let p, q belong to $S(G)$.
There exists a unique continuous G-equivariant $\pi_q \colon S(G) \to S(G)$ such that $\pi_q(1) = q$. We denote $p \cdot q = \pi_q(p)$ (often we simply write it as pq).

The existence and uniqueness of π_q come from considering $(\overline{Gq}, q)$ as an ambit and applying the universal property of $S(G)$.

Proposition 6.7. *The map $(p, q) \mapsto p \cdot q$ is associative and extends the group action of G on $S(G)$. Furthermore, $p \mapsto p \cdot q$ is continuous for all $q \in S(G)$.*

One says that $S(G)$ is a right topological semigroup (right translations are continuous). Beware however that in general left translations are not continuous!

Proof. Since $p \cdot q = \pi_q(p)$ and π_q is continuous, the continuity of $p \mapsto p \cdot q$ for all $q \in S(G)$ is by definition.
Given $g \in G$ and $q \in S(G)$ we have $g \cdot q = \pi_q(g) = \pi_q(g 1_G) = g\pi_q(1_G) = gq$.
To check associativity, let $p, q, r \in S(G)$. We have $\pi_r \circ \pi_q(1) = \pi_r(q) = q \cdot r$, and $\pi_r \circ \pi_q$ is G-equivariant. Hence $\pi_{q \cdot r} = \pi_r \circ \pi_q$, and it follows that

$$p \cdot (q \cdot r) = \pi_{q \cdot r}(p) = \pi_r(\pi_q(p)) = \pi_q(p) \cdot r = (p \cdot q) \cdot r$$

□

Definition 6.8. We say that $I \subseteq S(G)$ is a *left-ideal* if I is nonempty, closed and $S(G)I \subseteq I$.
We say that a left-ideal is minimal if it contains no proper left-ideal.

Note that since I is closed, G is dense in $S(G)$ and the semigroup operation extends the action $G \curvearrowright S(G)$, I is a left-ideal iff $GI = I$ iff I is a subflow of $G \curvearrowright S(G)$. Similarly, minimal left-ideals correspond to minimal subflows of $G \curvearrowright S(G)$. So Zorn's lemma implies that minimal left-ideals exist in $S(G)$.

Lemma 6.9 (Ellis). *Let I be a left-ideal in a compact right-topological semigroup. Then I contains an* idempotent*, i.e. there exists $e \in I$ such that $e \cdot e = e$.*

The previous lemma of course applies to $S(G)$.

Proof. Using Zorn's lemma, we find S minimal among nonempty closed subsets of I such that $S \cdot S \subseteq S$ (note that $I \cdot I \subseteq I$).
For any $x \in S$, we have $S \cdot x \subseteq S$; and $(S \cdot x) \cdot (S \cdot x) \subseteq S^3 \cdot x \subseteq S \cdot x$. By minimality of S, it follows that $S \cdot x = S$ for every $x \in S$. Let $W_x = \{y \in S \colon y \cdot x = x\}$, which is nonempty since $S \cdot x = S$. Then W_x is closed by continuity of $y \mapsto y \cdot x$. Further, by associativity we have $W_x \cdot W_x \subseteq W_x$; hence $W_x = S$ by minimality of S again. Thus $y \cdot x = x$ for all $x \in S$, so $S = \{e\}$ with $e \cdot e = e$. □

Lemma 6.10. *Let I be a minimal left-ideal in $S(G)$, and $e \in I$ be an idempotent. Then for every $p \in I$ we have that $p \cdot e = p$; for every $q \in M$ $p \mapsto p \cdot q$ is a G-equivariant surjection of I onto itself.*

Proof. Note that $I \cdot e$ is a left-ideal contained in I since I is a left ideal and $\cdot$ is associative. Hence $I \cdot e = I$. For every $p \in I$ there exists some $q \in I$ such that $p = q \cdot e$, from which we obtain that $p \cdot e = q \cdot e^2 = q \cdot e = p$.
Similarly for every $q \in I$ we have $I \cdot q = I$, and G-equivariance of $p \mapsto p \cdot q$ is part of its definition. □

The uniqueness of a universal minimal flow is a consequence of the next result.

Theorem 6.11 (Ellis). *Let G be a topological group, and $M \subseteq S(G)$ be a minimal left-ideal. Then M is* coalescent*, i.e. every G-equivariant continuous $\pi \colon M \to M$ is bijective.*

Of course surjectivity of π is immediate by minimality of M, so the interesting point here is that π is injective.

Proof. Let $\pi \colon M \to M$ be G-equivariant and continuous. For all $q \in M$ and all $g \in G$ we have $\pi(g \cdot q) = g\pi(q)$, so by continuity of right translations we obtain $\pi(p \cdot q) = p \cdot \pi(q)$.
Let $e \in M$ be an idempotent, and $p = \pi(e)$. For all $q \in M$ we have

$$\pi(q) = \pi(q \cdot e) = q \cdot \pi(e) = q \cdot p$$

So π is the right -translation by p.
Let q be such that $q \cdot p = e$ (such a q exists because $M \cdot p = M$), and set $\rho(x) = x \cdot q$. Then $\pi(\rho(x)) = \rho(x) \cdot p = x \cdot (q \cdot p) = x$. Hence ρ is injective, and we already knew that it was surjective (by minimality of M, see the previous lemma). Hence ρ is a bijection of M, and from $\pi \circ \rho = id$ we obtain that $\pi = \rho^{-1}$ is bijective. □

Theorem 6.12. *Let G be a topological group. Up to isomorphism, there exists a unique universal minimal G-flow, which we denote by $M(G)$.*

Proof. We already established the existence of a universal minimal G-flow. So, let M be a minimal subflow of $S(G)$, and N another universal minimal G-flow. Applying the definition of universality, we obtain two G-equivariant continuous maps $\varphi \colon M \to N$ and $\psi \colon N \to M$. Then $\psi \circ \varphi \colon M \to M$ is continuous and G-equivariant, hence bijective since M is coalescent. Hence φ is injective, so it is an isomorphism of G-flows. □

We seize the opportunity to mention the following useful fact.

Proposition 6.13. *Let G be a topological group, and M be a minimal subflow of $S(G)$. Then there exists a G-equivariant retraction $r\colon S(G) \to M$.*

Proof. Let $\pi\colon S(G) \to M$ be G-equivariant. Then π maps M into itself, so by coalescence it induces an automorphism of M which we denote by φ. Let $r = \varphi^{-1} \circ \pi$.
Then r is G-equivariant, continuous, maps $S(G)$ onto M, and for every $x \in M$ we have $r(x) = \varphi^{-1}(\pi(x)) = x$. So r is the desired retraction. □

Since we are going to look at universal minimal flows of subgroups of $\mathfrak{S}_\infty$ in the next chapter, we mention a result which implies one only needs to understand what happens for closed subgroups.

Proposition 6.14. *Let G be a Hausdorff topological group, and assume that H is a dense subgroup of G acting on a compact Hausdorff space X.*
Then the action of H extends to a continuous action of G on X.

Proof. We saw in Proposition 5.11 that each map $h \mapsto hx$ is right-uniformly continuous; by compactness of X this map extends to the Hausdorff completion of $(H,\mathcal{U}_r)$, which coincides with the Hausdorff completion of $(G,\mathcal{U}_r)$ since H is dense in G. In particular, for all x we can continuously extend $h \mapsto hx$ to G.
This gives us an action of G on X; we want to prove that this action is by homeomorphisms, i.e. that each mapping $x \mapsto gx$ is continuous. Fix $g \in G$, $x \in X$ and V a neighborhood of gx. By continuity of $H \curvearrowright X$ and compactness of X , there is a neighborhood U of 1 in H and an open W containing gx such that $\overline{UW} \subseteq V$.
Since H is dense in G, $\overline{U}$ is a neighborhood of 1 in G and $k \mapsto kx$ is continuous there is $h \in H$ such that $g \in \overline{Uh}$ and $hx \in W$. Let $O \ni x$ be open and such that $hO \subseteq W$. Then for every $y \in O$ we have (using continuity of $k \mapsto ky$) that $gy \in \overline{Uhy} \subseteq \overline{UW} \subseteq V$.
Now that we know our action is by homeomorphisms, fix $\varphi \in C(X)$ and $\varepsilon > 0$. There is an open neighborhood U of 1 in H such that $\|h\varphi - \varphi\| \leq \varepsilon$ for each $h \in U$. Since each $g \mapsto gx$ is continuous, it follows that $\|g\varphi - \varphi\| \leq \varepsilon$ for each $g \in \overline{U}$. This proves that $G \curvearrowright C(X)$ is continuous, and we are done. □

Exercise 57. Let G be a topological group, and H be a dense subgroup of G. Show that $M(G) = M(H)$, in the sense that both G and H act on the same space and the two H-actions coincide (hence the G-action on $M(G) = M(H)$ is the unique continuous extension of the H-action).

Definition 6.15. A topological group is *extremely amenable* if every G-flow admits a fixed point.

Since every G-flow contains a minimal subflow, and a minimal subflow with a fixed point is a singleton, G is extremely amenable iff $M(G)$ is a singleton. In the next chapter, we discuss a combinatorial characterization of extremely amenable subgroups of $\mathfrak{S}_\infty$.

Bibliographical comments. The book [dVri93] contains a wealth of related material. V. Pestov was the origin of much progress in our understanding of extreme amenability, and his survey [Pes99] still makes for interesting reading. A wealth of material and references concerning extreme amenability is contained in his book [Pes06] which is compulsory reading for anyone interested in extreme amenability.

Chapter 7

A theorem of Kechris–Pestov–Todorčevic

Definition 7.1. A topological space X is 0-*dimensional* if every point of X has a neighborhood basis consisting of clopen subsets.

This is the case for instance for ω, 2^ω, ω^ω and $\beta\omega$.
Consider a subgroup $G \leq \mathfrak{S}_\infty$. For every open subgroup V of G, note that $\ell^\infty(V\backslash G)$, the space of all bounded functions which are invariant when multiplied on the left by an element of V, is a subalgebra of $\mathrm{RUC}_b(G)$ (because V is a neighborhood of 1). And each of these functions takes only countably many values, hinting that $S(G)$ has many clopen sets (an intuition which will be confirmed shortly).

Proposition 7.2. *Let G be a subgroup of $\mathfrak{S}_\infty$, $\varepsilon > 0$, and $\varphi \in \mathrm{RUC}_b(G)$. Then there exists an open subgroup V of G as well as $\psi \in \ell^\infty(V\backslash G)$ which takes only finitely many values and is such that $\|\varphi - \psi\| \leq \varepsilon$.*

Proof. Since $\varphi \in \mathrm{RUC}_b(G)$, there exists a neighborhood V of 1_G such that $|\varphi(vg) - \varphi(g)| \leq \varepsilon$ for every $g \in G$ and every $v \in V$, and we may as well assume that V is an open subgroup of G since those form a basis of neighborhoods of 1_G.
Since $\varphi(G)$ is bounded in $\mathbb{C}$, there exist $z_1, \dots, z_p \in \mathbb{C}$ such that $\varphi(G) \subseteq \bigcup_i D(z_i, \varepsilon)$.
Find a (at most) countable family $(g_i)_{i\in I}$ in G such that $G = \bigsqcup_i Vg_i$. For each $i \in I$ find some $k_i \in \{1, \dots p\}$ such that $|\varphi(g_i) - z_{k_i}| \leq \varepsilon$, then set $\psi(vg_i) = z_{k_i}$ for every $v \in V$ and every $i \in I$. Then $\psi \in \ell^\infty(V\backslash G)$. Given $g \in G$, write it as vg_i for some $v \in V$ and $i \in I$ and observe that

$$|\varphi(g) - \psi(g)| = |\varphi(vg_i) - \psi(vg_i)| \leq |\varphi(vg_i) - \varphi(g_i)| + |\varphi(g_i) - z_{k_i}| \leq 2\varepsilon$$

Hence $\|\varphi - \psi\| \leq 2\varepsilon$ and we are done. □

Proposition 7.3 (Pestov)**.** *Let G be a subgroup of $\mathfrak{S}_\infty$. Then $S(G)$ is 0-dimensional (hence $M(G)$ is also 0-dimensional).*

Proof. Let $p \neq q \in S(G)$. There exists $\varphi \in C(S(G))$ such that $\varphi(p) \neq \varphi(q)$.
Let $|\varphi(p) - \varphi(q)| = 2\varepsilon$. The restriction of φ to G belongs to $\mathrm{RUC}_b(G)$; by the previous lemma there exists $\psi \in \mathrm{RUC}_b(G)$ which takes only finitely many values on G and such that $\|\varphi - \psi| \leq \varepsilon$.
Viewing ψ as a continuous function on $S(G)$, it still takes only finitely many values, and $\psi(p) \neq \psi(q)$. Hence $\psi^{-1}(\{\psi(p)\})$ and $\psi^{-1}(\{\psi(q)\})$ are disjoint clopen sets containing x, y respectively.

Now, let U be an open set containing p. For every $q \in S(G) \setminus U$ we have some disjoint clopen $U_q \ni p$, $V_q \ni q$. By compactness of $S(G) \setminus U$ there exist $q_1, \dots, q_n$ such that $\bigcup_i V_{q_i}$ contains $S(G) \setminus U$, so $\bigcap_i U_{q_i}$ is a clopen neighborhood of p contained in U. □

Next, we identify a family of flows which suffice to understand whether G is extremely amenable.

Definition 7.4. Let G be a subgroup of $\mathfrak{S}_\infty$. For every open subgroup V in G, we denote by X_V the flow obtained by having G act on $2^{V \backslash G}$ by setting

$$g \cdot \varphi(Vh) = \varphi(Vhg)$$

Theorem 7.5 (Kechris–Pestov–Todorčevic). *Let G be a subgroup of $\mathfrak{S}_\infty$. Then G is extremely amenable iff any minimal subflow of any X_V is trivial.*

Proof. One implication is immediate (if G is extremely amenable any minimal flow is trivial). Conversely, assume that $M(G)$ is nontrivial, and let D be a nontrivial clopen subset of $M(G)$. Then $V = \{g \in G \colon gD = D\}$ is a subgroup of G. For every $p \in D$ there exists a neighborhood U_p of 1_G and an open $W_p \ni p$ such that $U_p W_p \subseteq D$. By compactness we have $D \subseteq \bigcup_{i=1}^n W_{p_i}$. Since $\bigcap_i U_{p_i}$ is a neighborhood of 1_G it contains an open subgroup U, and for every $u \in U$ we have $uD \subseteq D$, hence $uD = D$ for all $u \in U$. So $U \leq V$, proving that V is open.

Now we consider $\pi \colon M(G) \to 2^{V \backslash G}$ defined by

$$\pi(p)(Vg) = 1 \Leftrightarrow gp \in D$$

This map is well-defined since $VD = D$, and continuous since D is clopen and $G \curvearrowright M(G)$ is continuous so each $g^{-1}D$ is clopen.

For every $g, h \in G$ and $p \in M(G)$ we have

$$\pi(gp)(Vh) = 1 \Leftrightarrow hgp \in D \Leftrightarrow \pi(p)(Vhg) = 1 \Leftrightarrow (g\pi(p))(Vh) = 1$$

Hence π is G-equivariant, so $\pi(M(G))$ is a subflow of X_V. For $p \in D$ we have $\pi(p)(V) = 1$ while $\pi(p)(V) = 0$ for $p \notin D$, hence $\pi(M(G))$ is nontrivial. □

The only fixed points for $G \curvearrowright 2^{V \backslash G}$ are the constant functions 0 and 1, so the above criterion amounts to saying that for every V and every $c \in 2^{V \backslash G}$ the closure of Gc contains a fixed point. Hence we obtain the following:

Theorem 7.6 (Kechris–Pestov–Todorčevic). *Let G be a subgroup of $\mathfrak{S}_\infty$. Then G is extremely amenable iff for every open subgroup of G and every $c \in 2^{V \backslash G}$ the following holds:*

$$\forall F \subseteq G \text{ finite } \exists g \in G \, \forall f_1, f_2 \in F \; c(Vf_1 g) = c(Vf_2 g)$$

Exercise 58. Provide the details of the proof of the above statement.

Exercise 59. Prove that in the theorem above one may equivalently consider $k^{V \backslash G}$ for any $k \geq 2$, and that one also obtains an equivalent statement by letting V run over some fixed family of open subgroups forming a basis of neighborhoods of 1_G.

Now we turn to a combinatorial interpretation of this property, when G is the automorphism group of some relational ultrahomogeneous structure (we recall that every closed subgroup of $\mathfrak{S}_\infty$ is of this form). So, fix a relational Fraïssé class $\mathcal{K}$ with elements of arbitrarily high (finite) cardinality, denote by $\mathbf{F}$ its limit and let $G = \mathrm{Aut}(\mathbf{F})$.
Given a finite $A \subseteq F$ we denote by G_A the pointwise stabilizer of A. Using ultrahomogeneity we may identify G/G_A with the set of all embeddings of $\mathbf{A}$ into $\mathbf{F}$.

Definition 7.7. Given $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ we denote $\binom{\mathbf{B}}{\mathbf{A}}$ the (finite) set of embeddings of $\mathbf{A}$ in $\mathbf{B}$. For $\mathbf{A} \in \mathcal{K}$ we similarly define $\binom{\mathbf{F}}{\mathbf{A}}$ as the (infinite) set of all embeddings from $\mathbf{A}$ to $\mathbf{F}$.

Here we should warn about differing choices of notations in the litterature: it is probably more common to denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all *substructures* of $\mathbf{B}$ isomorphic to $\mathbf{A}$. As soon as elements of $\mathcal{K}$ have nontrivial automorphism groups, the two definitions differ.
The group G acts on $\binom{\mathbf{F}}{\mathbf{A}}$ by composition, via $g \cdot \alpha(a) = g(\alpha(a))$. The ultrahomogeneity of $\mathbf{F}$ amounts to the statement that $\begin{cases} G/G_A \to \binom{\mathbf{F}}{\mathbf{A}} \\ gG_A \mapsto g_{|A} \end{cases}$ is an isomorphism of G-spaces.
Note that the action $G \curvearrowright 2^{G/G_A}$ defined by $g \cdot x(hG_A) = x(g^{-1}hG_A)$ is isomorphic to $G \curvearrowright 2^{G_A \backslash G}$ (by taking inverses), and we are going to work with the first action rather than the second.

Definition 7.8. Let $\mathbf{A} \in \mathcal{K}$. A *coloring* of $\binom{\mathbf{F}}{\mathbf{A}}$ is a map $\gamma \colon \binom{\mathbf{F}}{\mathbf{A}} \to k$, where $k < \omega$. We similarly define colorings of $\binom{\mathbf{B}}{\mathbf{A}}$ for $\mathbf{A}, \mathbf{B} \in \mathcal{K}$.

So 2-colorings of $\binom{\mathbf{F}}{\mathbf{A}}$ are simply elements of $2^{\binom{\mathbf{F}}{\mathbf{A}}}$, which we already met under the guise of elements of the flow X_{G_A}.

Definition 7.9. A class $\mathcal{K}$ of $\mathcal{L}$-structures has the *Ramsey property (for embeddings)* if for any k and any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that for any k-coloring γ of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\beta \in \binom{\mathbf{C}}{\mathbf{B}}$ such that γ is constant on $\beta \circ \binom{\mathbf{B}}{\mathbf{A}}$.

Exercise 60. Prove that the Ramsey property is equivalent to the property above where one only considers 2-colorings.

We could have similarly defined the Ramsey property for substructures, by asking that for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that any k-coloring of the copies of $\mathbf{A}$ contained in $\mathbf{B}$ is constant on some copy of $\mathbf{B}$ (i.e. takes the same value on all copies of $\mathbf{A}$ contained in some fixed copy of $\mathbf{B}$).

Proposition 7.10. *A class $\mathcal{K}$ has the Ramsey property for embeddings iff it has the Ramsey property for substructures and every $\mathbf{A} \in \mathcal{K}$ has a trivial automorphism group (we say that elements of $\mathcal{K}$ are* rigid*).*

Proof. Note that if every element of $\mathcal{K}$ has a trivial automorphism group then coloring embeddings is the same thing as coloring substructures, since every embedding of some $\mathbf{A}$ into $\mathbf{C}$ is uniquely characterized by its image.
So the above proposition really amounts to the statement that, if some $\mathbf{A} \in \mathcal{K}$ has a nontrivial automorphism group, then $\mathcal{K}$ cannot have the Ramsey property for embeddings.
Assume that some $\mathbf{A} \in \mathcal{K}$ has a nontrivial automorphism group then let $\mathbf{A} = \mathbf{B}$ in the definition of the Ramsey property (for embeddings). We want to prove that no $\mathbf{C}$ in $\mathcal{K}$ can witness the Ramsey property for the pair $(\mathbf{A}, \mathbf{A})$. If $\mathbf{C}$ does not contain a copy of $\mathbf{A}$ this is immediate, so we may assume that $\mathbf{C}$ contains $\mathbf{A}$.
For each $\mathbf{A}' \leq \mathbf{C}$ isomorphic to $\mathbf{A}$, we choose an isomorphism $g_{A'} \colon \mathbf{A}' \to \mathbf{A}$ and for $\alpha \in \binom{\mathbf{C}}{\mathbf{A}}$ we set $\gamma(\alpha) = g_{\alpha(\mathbf{A})} \circ \alpha$ (so our set of colors is $\mathrm{Aut}(\mathbf{A})$, which is finite and does not depend on $\mathbf{C}$).
For every $\beta \in \binom{\mathbf{C}}{\mathbf{A}}$ we have $\beta \circ \binom{\mathbf{A}}{\mathbf{A}} = \beta \circ \mathrm{Aut}(\mathbf{A})$, and γ takes all possible values on this set, so $\mathbf{C}$ cannot witness the Ramsey property for embeddings, hence this property fails. □

The Ramsey property is related to the amalgamation property, as witnessed by the result of the next exercise.

Exercise 61 (Nešetřil)**.** Assume that $\mathcal{K}$ satisfies the joint embedding property as well as the Ramsey property for embeddings (so elements of $\mathcal{K}$ have trivial automorphism group).
Fix $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathcal{K}$ and embeddings $\alpha \colon \mathbf{A} \to \mathbf{B}$, $\beta \colon \mathbf{A} \to \mathbf{C}$. Find $\mathbf{E} \in \mathcal{K}$ containing a copy of both $\mathbf{B}$ and $\mathbf{C}$ and view α, β as maps from A to E.
Given $\mathbf{D} \in \mathcal{K}$, we define a coloring c of $\binom{\mathbf{D}}{\mathbf{A}}$ with colors in $2^{\{B,C\}}$ by declaring that

$$B \in c(f) \Leftrightarrow \text{ There exists an embedding } i \colon \mathbf{B} \to \mathbf{D} \text{ with } i \circ \alpha(A) = f(A)$$

and define similarly when $C \in c(f)$.
Using this coloring, and the Ramsey property for the pair $(\mathbf{A}, \mathbf{E})$, prove that $\mathcal{K}$ satisfies the amalgamation property.

Next, we give an equivalent "infinite" formulation of the Ramsey property.

Proposition 7.11. *Let $\mathcal{K}$ be a Fraïssé class of rigid structures with limit $\mathbf{F}$. Then $\mathcal{K}$ has the Ramsey property iff for any $\mathbf{A}$, $\mathbf{B} \in \mathcal{K}$, any integer k and any k-coloring γ of $\binom{\mathbf{F}}{\mathbf{A}}$ there exists $\beta \in \binom{\mathbf{F}}{\mathbf{B}}$ such that γ is constant on $\beta \circ \binom{\mathbf{B}}{\mathbf{A}}$.*

Proof. Assume first that $\mathcal{K}$ has the Ramsey property. Fix $\mathbf{A}$, $\mathbf{B} \in \mathcal{K}$. If $\binom{\mathbf{B}}{\mathbf{A}}$ is empty we have nothing to prove, so we assume that $\mathbf{A} \leq \mathbf{B}$. We find $\mathbf{C}$ witnessing that the Ramsey property holds for $(\mathbf{A}, \mathbf{B})$ and we realize $\mathbf{C}$ as a substructure of $\mathbf{F}$.

Let γ be a k-coloring of $\binom{\mathbf{F}}{\mathbf{A}}$. Since $C \subset F$, any element of $\binom{\mathbf{C}}{\mathbf{A}}$ can be seen as an element of $\binom{\mathbf{F}}{\mathbf{A}}$. So γ induces a coloring of $\binom{\mathbf{C}}{\mathbf{A}}$, thus there exists $\beta \in \binom{\mathbf{C}}{\mathbf{B}}$ such that γ is constant on $\beta \circ \binom{\mathbf{B}}{\mathbf{A}}$. Viewing β as an elemnt of $\binom{\mathbf{F}}{\mathbf{B}}$, we are done.

Conversely, assume that $\mathcal{K}$ does not have the Ramsey property and fix $\mathbf{A} \leq \mathbf{B}$ witnessing that failure. We may assume that $\mathbf{A}$, $\mathbf{B}$ are substructures of $\mathbf{F}$.

For every substructure $\mathbf{C}$ of $\mathbf{F}$ in which $\mathbf{B}$ embeds we fix a bad 2-coloring $\gamma_{\mathbf{C}}$ of $\binom{\mathbf{C}}{\mathbf{A}}$, i.e. a coloring which is not constant on $\beta \circ \binom{\mathbf{B}}{\mathbf{A}}$ for any $\beta \in \binom{\mathbf{C}}{\mathbf{B}}$. Next, fix an ultrafilter $\mathcal{U}$ on the set of finite subsets of F which is such that for any finite X the set $\{Y : X \subseteq Y\}$ belongs to $\mathcal{U}$.

We define a 2-coloring γ of $\binom{\mathbf{F}}{\mathbf{A}}$ by setting $\gamma(\alpha) = \lim_{\mathcal{U}} \gamma_{\mathbf{C}}(\alpha)$.

Equivalently, $\gamma(\alpha) = \varepsilon \in \{0,1\}$ iff $\{C : \gamma_{\mathbf{C}}(\alpha) = \varepsilon\} \in \mathcal{U}$; of course $\gamma_{\mathbf{C}}(\alpha)$ is not defined for all C, but it is defined as soon as $B \cup \alpha(A) \subseteq C$, and $\{C : B \cup \alpha(A) \subseteq C\} \in \mathcal{U}$.

We now check that γ is a bad coloring of $\binom{\mathbf{F}}{\mathbf{A}}$: let $\beta \in \binom{\mathbf{F}}{\mathbf{B}}$. Then $\{C : \beta(B) \subseteq C\} \in \mathcal{U}$, and for any C in this set we have $\alpha_1, \alpha_2 \in \binom{\mathbf{B}}{\mathbf{A}}$ such that $\gamma_{\mathbf{C}}(\beta \circ \alpha_1) = 0$ and $\gamma_{\mathbf{C}}(\beta \circ \alpha_2) = 1$.

Since $\binom{\mathbf{B}}{\mathbf{A}}$ is finite, there exist $\alpha_1, \alpha_2 \in \binom{\mathbf{B}}{\mathbf{A}}$ such that $\{C : \gamma_{\mathbf{C}}(\beta \circ \alpha_1) = 0\} \in \mathcal{U}$ and $\{C : \gamma_{\mathbf{C}}(\beta \circ \alpha_2) = 1\} \in \mathcal{U}$. Thus $\gamma(\beta \circ \alpha_1) = 0 \neq \gamma(\beta \circ \alpha_2)$, and we are done. □

Theorem 7.12 (Kechris–Pestov–Todorčevic)**.** *Let $\mathcal{K}$ be a Fraïssé class of relational structures with infinite Fraïssé limit $\mathbf{F}$. Then $G = \mathrm{Aut}(\mathbf{F})$ is extremely amenable iff $\mathcal{K}$ has the Ramsey property (for embeddings).*

Proof. Translating theorem 7.5 in terms of $\binom{\mathbf{F}}{\bar{\mathbf{a}}}$ instead of $G_{\bar{a}} \backslash G$, we obtain that G is extremely amenable iff

$$\forall H \subseteq G \text{ finite } \forall A \subseteq F \text{ finite } \forall \gamma \colon \binom{\mathbf{F}}{\mathbf{A}} \to 2 \ \exists g \in G \text{ such that } \gamma \text{ is constant on } g \circ H_{|\mathbf{A}}$$

Assume $\mathcal{K}$ has the Ramsey property, then fix a finite $H \subseteq G$, a finite $A \subseteq F$ and $\gamma \in \binom{\mathbf{F}}{\mathbf{A}}$. Let $B = \bigcup_{h \in H} hA$. Using the Ramsey property we obtain $\beta \in \binom{\mathbf{F}}{\mathbf{B}}$ such that γ is constant on $\beta \circ \binom{\mathbf{B}}{\mathbf{A}}$. Extending β to some $g \in \mathrm{Aut}(\mathbf{F})$ we are done, since $\binom{\mathbf{B}}{\mathbf{A}}$ contains $H_{|A}$.

Conversely, assume that G satisfies the property above, and fix $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$. We may assume that $A \subseteq B \subseteq F$. Then every element α of $\binom{\mathbf{B}}{\mathbf{A}}$ extends to some $g_\alpha \in G$.

Let $H = \left\{ g_\alpha \colon \alpha \in \binom{\mathbf{B}}{\mathbf{A}} \right\}$. Let γ be a coloring of $\binom{\mathbf{F}}{\mathbf{A}}$; by assumption we obtain some $g \in G$ such that γ is constant on $g \circ H_{|A}$. Letting $\beta = g_{|B}$ we obtain that γ is constant on $\beta \circ \binom{\mathbf{B}}{\mathbf{A}}$, witnessing that $\mathcal{K}$ satisfies the Ramsey property. □

It is certainly time to discuss an example...

Theorem 7.13 (Ramsey)**.** *The class of all finite linear orders has the Ramsey property.*

Proof. We claim that it is enough to prove the following classical (infinite version of) Ramsey theorem: for any integer n and any 2-coloring γ of the set $\omega^{[n]}$ of n-element subsets of ω, there exists an infinite subset I of ω such that γ is constant on $I^{[n]}$.
Indeed, given a finite subset $A \subseteq \mathbb{Q}$ of cardinality n, a 2-coloring of $\binom{\mathbb{Q}}{\mathbf{A}}$ is the same thing as a coloring of $\mathbb{Q}^{[n]} \cong \omega^{[n]}$. Since an infinite I obviously contains finite sets of any cardinality, the property above gives us an I witnessing the Ramsey property.
We now prove the classical Ramsey theorem by induction on n. For $n = 1$ the statement above follows from the pigeonhole principle. Assume that we have proved the result up to some n; let $m = n + 1$ and γ be a 2-coloring of $\omega^{[m]}$.
For each $a < \omega$ we let $\gamma_a \colon (\omega \setminus \{a\}))^{[n]} \to 2$ be defined by $\gamma_a(X) = \gamma(X \cup \{a\})$.
By the induction hypothesis, there exists an infinite $I_0 \subseteq \omega \setminus \{0\}$ such that γ_0 is constant on $I_0^{[n]}$. Let $i_1 = \min I_0$, and apply the same argument to find an infinite $I_1 \subseteq I_0 \setminus \{i_1\}$ such that γ_{i_1} is constant on $I_1^{[n]}$. Set $i_2 = \min I_1$ and keep going.
For $i \in I$, denote by ε_i the constant color taken by γ_i on $I_i^{[n]}$. By the pigeonhole principle, there is $\varepsilon \in \{0, 1\}$ and $J \subseteq I$ infinite such that $\varepsilon_j = \varepsilon$ for all $j \in J$.
Let $j_0 < \ldots < j_n$ enumerate a subset of J of cardinality m. Then $\{j_1, \ldots, j_n\}$ is a n-element subset of I_{j_0}, whence $\gamma(\{j_0, \ldots, j_m\}) = \gamma_{j_0}(\{j_1, \ldots, j_m\}) = \varepsilon$. Thus γ is monochromatic on $J^{[m]}$ and we are done. □

The next result (which was a precursor of the Kechris–Pestov–Todorčevic correspondence) is now an immediate consequence.

Corollary (Pestov)**.** $\mathrm{Aut}(\mathbb{Q}, <)$ is extremely amenable.

Exercise 62 (Pestov)**.** Let H be the group of all homeomorphisms of $[0, 1]$, endowed with the topology of uniform convergence.

(i) Let $H_+ = \{g \in H \colon g(0) = 0\}$. Prove that H_+ is extremely amenable.
(Hint: map $\mathrm{Aut}(\mathbb{Q}, <)$ densely and continuously in H_+)

(ii) Use this to compute the universal minimal flow of H.

We conclude this chapter by explaining a general strategy to compute universal minimal flows of Polish groups, in the particular case when they happen to be metrizable (which never happens in the locally compact noncompact case, see [KPT05]).

Definition 7.14. Let G be a topological group and H be a subgroup of G. We endow G/H with the uniformity $\mathcal{U}$ coming from the right uniformity, i.e. the uniformity generated by entourages of the form

$$\{(fH, ufH) \colon u \in U\}$$

for U an open neighborhood of 1_G.
We say that H is *co-precompact* in G if the Hausdorff completion $\widehat{(G/H, \mathcal{U})}$ is compact.

Exercise 63. Prove that H is co-precompact in G iff for every nonempty open V there exists a finite F such that $VFH = G$.

Proposition 7.15. *Let G be a Polish group and H a subgroup of G. The left-translation action of G on G/H extends to a continuous action $G \curvearrowright \widehat{G/H}$.*

Proof. It follows from the proof of Theorem 1.14 that if $(g_n)_{n<\omega}$ and $(k_n)_{n<\omega}$ are $\mathcal{U}_r$-Cauchy then $(g_n k_n)_{n<\omega}$ is $\mathcal{U}_r$-Cauchy. Assume $(g_n)_{n<\omega}$ is Cauchy in $(G, \mathcal{U}_r)$ and $(k_n H)_{n<\omega}$ is Cauchy in $(G/H, \mathcal{U})$.
Then there exists a subsequence $(k_{\varphi(n)})_{n<\omega}$ and $(h_n)_{n<\omega} \in H^\omega$ such that $(k_{\varphi(n)} h_n)_{n<\omega}$ is Cauchy in $(G, \mathcal{U}_r)$, whence $(g_{\varphi(n)} k_{\varphi(n)} h_n)_{n<\omega}$ is Cauchy in $(G, \mathcal{U}_r)$. This implies that $(g_{\varphi(n)} k_{\varphi(n)} H)_{n<\omega}$ is Cauchy in $(G/H, \mathcal{U})$.
It follows that every subsequence of $(g_n k_n H)_{n<\omega}$ admits a Cauchy subsequence, which implies that $(g_n k_n H)_{n<\omega}$ is Cauchy. This proves that $(g, kH) \mapsto gkH$ extends continuously to $\widehat{(G, \mathcal{U}_r)} \times \widehat{(G/H, \mathcal{U})}$, and that is more than we needed. □

Let us discuss an instructive example. Endow the space of linear orderings $\mathrm{LO}(\omega)$ with the topology coming from viewing it as a subset of $2^{\omega\times\omega}$, and have $\mathfrak{S}_\infty$ act on $\mathrm{LO}(\omega)$ via

$$i\,(\sigma \cdot \prec)\,j \Leftrightarrow \sigma^{-1}(i) \prec \sigma^{-1}(j)$$

Note that $\mathrm{LO}(\omega)$ is compact since it is closed in $2^{\omega\times\omega}$.

Lemma 7.16. *The flow $\mathfrak{S}_\infty \curvearrowright \mathrm{LO}(\omega)$ is minimal.*

Proof. Let U be a basic open subset of $\mathrm{LO}(\omega)$, which we can assume to be made up of all orders $\prec$ such that $i_0 \prec i_1 \prec \ldots \prec i_n$ for some $(i_0, \ldots, i_n) \in \omega^{n+1}$.
Let $I = \{i_0, \ldots, i_n\}$ and let F be the finite set of all elements of $\mathfrak{S}_\infty$ whose support is contained in I.
For any ordering $\prec$ of ω, there exists a bijection σ of I such that $i_0 (\sigma \cdot \prec) i_1 \ldots (\sigma \cdot \prec) i_n$ (map the smallest element of $(I, \prec)$ to i_0, the second smallest to i_1, and so on). We may extend σ to an element of F, and we obtain that $\prec \in FU$.
This proves that $\mathrm{LO}(\omega) = FU$, so $\mathfrak{S}_\infty \curvearrowright \mathrm{LO}(\omega)$ is minimal as promised. □

Lemma 7.17. *The flow $\mathfrak{S}_\infty \curvearrowright \mathrm{LO}(\omega)$ is isomorphic to $\mathfrak{S}_\infty \curvearrowright \widehat{\mathfrak{S}_\infty / \mathrm{Aut}(\mathbb{Q})}$.*

In particular, $\mathrm{Aut}(\mathbb{Q})$ is co-precompact in $\mathfrak{S}_\infty$ (this could also be seen directly).

Proof. Fix an ordering $\prec$ of ω such that $(\omega, \prec)$ is dense and without endpoints. Denote by H the stabilizer of $\prec$, which is isomorphic to $\mathrm{Aut}(\mathbb{Q})$.
Let V be the pointwise stabilizer of a finite $F \subset \omega$. Let $<_1, <_2$ be two elements of $\mathfrak{S}_\infty \cdot \prec$.

There exists $\sigma \in \mathfrak{S}_\infty$ and $\tau \in V$ such that $(<_1, <_2) = (\sigma\cdot \prec, \tau\sigma\cdot \prec)$ iff there exists $\tau \in V$ such that $<_1 = \tau\cdot <_2$, which implies that for any $i, j \in F$ we have $(i <_1 j) \Leftrightarrow (i <_2 j)$.
Conversely, if $<_{1|F} = <_{2|F}$ then, since $<_1$ and $<_2$ are dense and without endpoints there exists $\tau \in \mathfrak{S}_\infty$ which is the identity on F and such that $<_1 = \tau\cdot <_2$.
If we denote by E_F the basic entourage $\{(\sigma H, \tau\sigma H) \colon \tau \in V\}$, and by Δ_F the neighborhood of the diagonal $\left\{(<_1, <_2) \colon <_{1|F} = <_{2|F}\right\}$ we just proved that

$$\{(\sigma\cdot \prec, \tau\cdot \prec) \colon (\sigma H, \tau H) \in E_F\} = \Delta_F$$

Since the sets E_F and Δ_F are fundamental systems for the two uniformities under consideration, this proves that $\sigma H \mapsto \sigma\cdot \prec$ is a uniform isomorphism from $\mathfrak{S}_\infty/H$ to $\mathfrak{S}_\infty\cdot \prec$.
This uniform isomorphism extends to a uniform isomorphism from $\mathfrak{S}_\infty \curvearrowright \widehat{\mathfrak{S}_\infty/\mathrm{Aut}(\mathbb{Q})}$ to $\mathfrak{S}_\infty \curvearrowright \mathrm{LO}(\omega)$, which is $\mathfrak{S}_\infty$-equivariant by continuity of the actions and $\mathfrak{S}_\infty$-equivariance on a dense subset. □

The next (and last!) exercise can be seen as an abstract extension of what we just did.

Exercise 64. Let G be a Polish group and X be a G-flow. Assume that $x_0 \in X$ has a comeager orbit and let H be the stabilizer of x_0. Show that $G \curvearrowright \widehat{G/H}$ is uniformly isomorphic to $G \curvearrowright \overline{G\cdot x_0}$ (in particular, H is co-precompact in G).

Theorem 7.18. *Assume that G is a Polish group and H is a closed extremely amenable subgroup. Then $G \curvearrowright \widehat{G/H}$ factors onto every minimal G-flow.*
In particular, if H is co-precompact, extremely amenable and $G \curvearrowright \widehat{G/H}$ is minimal, then it is the universal minimal flow $M(G)$.

Proof. Let $G \curvearrowright X$ be a minimal G-flow. Then $H \curvearrowright X$ is an H-flow, hence has a fixed point x_0. Since $g \mapsto gx_0$ is right uniformly continuous, $gH \mapsto gx_0$ is $\mathcal{U}$-uniformly continuous, so it extends to a continuous map $\pi \colon \widehat{G/H} \to X$.
For every $kH \in G/H$ and every $g \in G$ we have $\pi(gkH) = gkx_0 = g\pi(kH)$. By continuity we obtain $\pi(gy) = g\pi(y)$ for every $y \in \widehat{G/H}$. Hence π is G-equivariant, and it is surjective since $G \curvearrowright X$ is minimal. □

Note that it also follows from the previous result that if H is co-precompact and extremely amenable then every minimal subflow of $\widehat{G/H}$ is the universal minimal flow of G; and it turns out that if one can find such an H then one can also find one such that $G \curvearrowright \widehat{G/H}$ is minimal, enabling one to compute the universal minimal flow of G (see theorem 7.20 below).
The previous result, combined with Lemma 7.17, enables us to compute a new universal minimal flow.

Theorem 7.19 (Glasner–Weiss)**.** *The universal minimal flow of $\mathfrak{S}_\infty$ is $\mathfrak{S}_\infty \curvearrowright \mathrm{LO}(\omega)$.*

Proof. We know that $\mathrm{Aut}(\mathbb{Q})$ is extremely amenable and co-precompact in $\mathfrak{S}_\infty$. Since $\mathfrak{S}_\infty \curvearrowright \widehat{\mathfrak{S}_\infty/\mathrm{Aut}(\mathbb{Q})}$ is isomorphic to $\mathfrak{S}_\infty \curvearrowright \mathrm{LO}(\omega)$, which is minimal, we have identified the universal minimal flow of $\mathfrak{S}_\infty$. □

In particular, the universal minimal flow of $\mathfrak{S}_\infty$ is metrizable. Note that it has a comeager orbit, consisting of all dense linear orders without endpoints. This result actually fits into a broader picture, as the following theorem shows.

Theorem 7.20 (Melleray–Nguyen van Thé–Tsankov; Ben Yaacov–Melleray–Tsankov[(i)]). *Let G be a Polish group. Then the following conditions are equivalent:*

(i) The universal minimal flow $M(G)$ is metrizable.

(ii) There exists a coprecompact, extremely amenable subgroup $G^* \leq G$ such that $M(G) = \widehat{G/G^*}$.

We already discussed the implication (ii) $\Rightarrow$ (i). The new and interesting thing here is that the only way for a Polish group G to have a metrizable universal minimal flow is for G to contain a large extremely amenable subgroup.
Replacing G^* by its closure we may assume that G^* above is closed; then G/G^* is a dense G_δ orbit in $M(G)$ (since G/G^* is a Polish subspace of $\widehat{G/G^*}$). Thus a metrizable minimal flow of a Polish group always has a comeager orbit, and the stabilizer of a point in the comeager orbit must be extremely amenable.

Bibliographical comments. The main theorem of this chapter comes from the paper [KPT05] of Kechris–Pestov–Todorčevic. The last result is a combination of theorems obtained in two papers, [MVT16] and [BMT17]. Another approach to this was proposed in [Zuc17] by A. Zucker (who had obtained an earlier and different proof for subgroups of $\mathfrak{S}_\infty$). Recent work of Zucker and co-authors has brought significant progress, for instance the reader (and the author, if we're being honest) could do worse than spending some time with [Zuc19].

[(i)] It might be tempting to believe that all theorems in this area must be proved in three-person papers. However this is belied by a recent paper of Balko–Chodounský–Dobrinen–Hubička–Konečný–Nešetřil–Zucker...

Chapter 8

Metrizability of the universal minimal flow: existence of a comeager orbit

We conclude these notes by proving a part of theorem 7.20, namely, we show that if the universal minimal flow of a Polish group G is metrizable then there must exist a comeager orbit in $M(G)$. The co-precompact, extremely amenable subgroup G^* whose existence is asserted in Theorem 7.20 then appears as the stabilizer of a point in the generic orbit, as shown in [MVT16]. The proof we give is taken from [BMT17].

Definition 8.1. A *compact topometric space* is a triple (Z, τ, ∂), where Z is a set, τ is a compact Hausdorff topology on Z, and ∂ is a distance on Z such that the following conditions are satisfied:

- the ∂-topology refines τ;
- ∂ is τ-lower semicontinuous, i.e., the set $\{(a,b) \in Z^2 \colon \partial(a,b) \leq r\}$ is τ-closed for every $r \geq 0$.

Lemma 8.2. *Let (Z, τ, ∂) be a compact topometric space. Then ∂ is complete.*

Proof. Let (z_n) be a Cauchy sequence; for all n, define $r_n = \sup\{\partial(z_n, z_m) : m \geq n\}$. Then r_n converges to 0. Let F_n denote the closed ball of radius r_n centred at z_n. Each F_n is τ-closed, hence compact, and since F_n contains z_m for all $m \geq n$, this family has the finite intersection property. By compactness, $\bigcap_{n \in \mathbf{N}} F_n$ is non-empty; it must be a singleton, which is the ∂-limit of the sequence (z_n). □

We now define a topometric structure on the Samuel compactification $S(X)$ of a bounded metric space (X, d) (i.e. the compactification associated to the algebra $\mathrm{UC_b}(X)$): we endow $S(X)$ with its usual compact topology and define ∂ by:

$$\partial(a,b) = \sup\{|f(a) - f(b)| : f \in \mathrm{C_L}(X)\},$$

where $\mathrm{C_L}(X)$ denotes the set of all bounded 1-Lipschitz functions on (X, d). To make sense of this, one must recall that functions in $\mathrm{C_L}(X)$, being bounded and uniformly continuous, uniquely extend to continuous functions on $S(X)$.

Clearly, each $\{(a,b)\colon \partial(a,b) \leq r\}$ is τ-closed. To see why the ∂-topology refines τ, recall that any function in $\mathrm{UC_b}(X)$ is a uniform limit of Lipschitz maps, so τ has a basis of open sets of the form

$$\{a \in S(X) : f_1(a) \in I_1, \dots, f_n(a) \in I_n\},$$

where each f_j belongs to $\mathrm{C_L}(X)$ and each I_j is an open interval. Each set in this basis is ∂-open.

Note that ∂ and d coincide on X, and that elements of $\mathrm{C_L}(X)$ extend to maps on $S(X)$ which are both τ-continuous and ∂-1-Lipschitz (the Lipschitz part is immediate from the definition of ∂). Also, if d is the discrete 0–1 distance on X, then ∂ is the discrete 0–1 distance on $S(X)$.

The following result is the topometric analogue of the fact that for discrete X the only convergent sequences in βX are stationnary.

Theorem 8.3. *Let (X,d) be a bounded metric space. Then every τ-convergent sequence in $S(X)$ is ∂-convergent.*

In order to prove this theorem, we first establish two lemmas. The first is the topometric analogue of the fact that in the discrete case, $S(X)$ is extremally disconnected (see proposition 5.2). If A, B are two subsets of a metric space (Z,d), we denote

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}.$$

Lemma 8.4. *Let U, V be τ-open subsets of $S(X)$. Then we have*

$$\partial(\overline{U}^{\tau}, \overline{V}^{\tau}) = \partial(U,V) = d(U \cap X, V \cap X).$$

Proof. First note that, since X is dense in $S(X)$, we have $\overline{U}^{\tau} = \overline{U \cap X}^{\tau}$ and $\overline{V}^{\tau} = \overline{V \cap X}^{\tau}$. Consider the function $f \in \mathrm{C_L}(X)$ defined by $f(x) = d(x, U \cap X)$. It extends to a τ-continuous, ∂-1-Lipschitz map on $S(X)$, which we still denote by f.

Since $f = 0$ on $U \cap X$, we must also have $f = 0$ on $\overline{U}^{\tau}$ by continuity; similarly, $f \geq d(V \cap X, U \cap X)$ on $V \cap X$, so $f \geq d(V \cap X, U \cap X)$ on $\overline{V}^{\tau}$. Hence f witnesses the fact that $\partial(\overline{U}^{\tau}, \overline{V}^{\tau}) \geq d(V \cap X, U \cap X)$; since $d(U \cap X, V \cap X)$ is equal to $\partial(U \cap X, V \cap X)$ by the definition of ∂, this inequality must in fact be an equality, and we are done. □

Lemma 8.5. *Assume that (a_n) is a sequence in $S(X)$ and $\delta > 0$ is such that $\partial(a_n, a_m) > \delta$ for all $n \neq m$. Then, for every $\varepsilon < \delta/2$, there exist a subsequence (b_n) of (a_n) and τ-open sets (U_n) such that $b_n \in U_n$ and $\partial(U_n, U_m) \geq \varepsilon$ for all $n \neq m$.*

Proof. Let $f \in \mathrm{C_L}(X)$ be such that $|f(a_0) - f(a_1)| > \delta$. The triangle inequality implies that, for all $n > 1$, we have $|f(a_0) - f(a_n)| > \frac{\delta}{2}$ or $|f(a_1) - f(a_n)| > \frac{\delta}{2}$. One of those cases happens infinitely many times. Thus we see that for any such sequence (a_n), there exists $i_0 \in \{0,1\}$, an infinite subset $\{i_n\}_{n \geq 1} \subseteq \mathbf{N} \setminus \{0,1\}$ and $f_0 \in \mathrm{C_L}(X)$ such that $f_0(a_{i_0}) = 0$ and $f_0(a_{i_n}) > \frac{\delta}{2}$ for all $n \geq 1$. Repeating this infinitely many times, we build a subsequence (b_n) of (a_n) and a sequence of maps $f_n \in \mathrm{C_L}(X)$ such that $f_n(b_n) = 0$ for all n and $f_n(b_m) > \frac{\delta}{2}$ for all $n < m$.

Set $U_n = \{a \in S(X)\colon f_n(a) < \frac{\delta}{2} - \varepsilon \text{ and } f_k(a) > \frac{\delta}{2} \text{ for all } k < n\}$. We have $b_n \in U_n$ for all n, and the function f_n witnesses the fact that $\partial(U_n, U_m) \geq \varepsilon$ for all $n < m$. □

Proof of Theorem 8.3. Let (a_n) be a τ-convergent sequence in $S(X)$ with limit a and suppose that it does not admit a ∂-Cauchy subsequence.
Up to an extraction, there exists $\delta > 0$ such that $\partial(a_n, a_m) > \delta$ for all $n \neq m$ and we can apply Lemma 8.5 to obtain a subsequence (b_n) of (a_n) and τ-open subsets U_n of $S(X)$ such that $b_n \in U_n$ and $\partial(U_n, U_m) \geq \delta/2$ for all $n \neq m$. Let

$$U = \bigcup_n U_{2n} \quad \text{and} \quad V = \bigcup_n U_{2n+1}.$$

Then we have both that $\partial(U, V) \geq \delta/2$ and $a \in \overline{U}^\tau \cap \overline{V}^\tau$, which contradicts Lemma 8.4.
Thus every τ-convergent sequence admits a ∂-Cauchy subsequence, which combined with the facts that ∂ is complete and that the ∂-topology refines τ implies the statement of the theorem. □

Corollary. Let $K \subseteq S(X)$ be a subset such that K equipped with the relative τ-topology is a metrizable topological space. Then the ∂-topology and τ coincide on K and in particular, if K is τ-closed, (K, ∂) is compact.

Proof. We already know that the ∂-topology is finer than τ. To see the converse, note that if K is metrizable, its topology is determined by convergence of sequences and then apply Theorem 8.3. □

Now we let G be a Polish group. We fix a bounded, right-invariant distance d_{R} on G. Then (G, d_{R}) is a metric space and we construct its topometric Stone–Čech compactification $(S(G), \tau, \partial)$ as above.
We are now ready to prove the main result of this chapter, namely, that if G is a Polish group such that $M(G)$ is metrizable then there exists a comeager orbit in $M(G)$.

Proof that if $M(G)$ is metrizable then it has a comeager orbit.
We view $M(G)$ as a subflow of $S(G)$ and fix a G-equivariant retraction $r \colon S(G) \to M(G)$ (see Proposition 6.13).
To show that there is a comeagre orbit, we apply Rosendal's criterion (a.k.a. Lemma 1.20). Let $V \ni 1_G$ and $U \subseteq M(G)$ be given. We may assume that $V = \{g : d_{\mathrm{R}}(g, 1_G) < \varepsilon\}$ for some $\varepsilon > 0$. Since $M(G)$ is metrizable, we have that τ and the ∂-topology coincide on $M(G)$ by Corollary 8 and we can find a non-empty τ-open $U' \subseteq U$ of ∂-diameter $< \varepsilon$.
Let $W_1, W_2 \subseteq U'$ be non-empty, open. By choice of U' we have $\partial(W_1, W_2) < \varepsilon$; since $W_1 \subseteq r^{-1}(W_1)$ and $W_2 \subseteq r^{-1}(W_2)$, we also have that $\partial(r^{-1}(W_1), r^{-1}(W_2)) < \varepsilon$. Then Lemma 8.4 tells us that $d_{\mathrm{R}}(r^{-1}(W_1) \cap G, r^{-1}(W_2) \cap G) < \varepsilon$. So we can find $f_1 \in r^{-1}(W_1) \cap G$ and $f_2 \in r^{-1}(W_2) \cap G$ such that $d_{\mathrm{R}}(f_1, f_2) < \varepsilon$, that is, $f_2 f_1^{-1} \in V$. Since $r(f_2) = f_2 f_1^{-1} r(f_1) \in W_2 \cap f_2 f_1^{-1} W_1$, the criterion is verified and we are done. □

Bibliographical comments. The proof given in this chapter is lifted essentially verbatim from [BMT17]; some of the ideas originate in work of Zucker.
Topometric structures occur naturally in several contexts related to continuous logic (such as type spaces); on any Polish group (G, τ) it is interesting to consider the topometric structure (G, τ, ∂), where $\partial(g, h) = \sup\{d(gk, hk) \colon k \in G\}$ for some left-invariant metric d on G. Another, related example is given by considering weak*-topologies on the dual of a normed vector space (or its unit ball), and the metric given by the operator norm.

It seems that it would be useful to have a good notion of “topo-uniform” (topiform?) space since what we really use appears to be the uniform structure associated to the metric rather than the metric itself, but at the moment such a notion has not been worked out.

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