Abstract

These are the notes for a (half-semester) spring 2024 course in functional analysis at the University of Turin (Università di Torino), emphasizing a “structural” viewpoint informed by real-valued logic.

Notes on Functional Analysis: A Structural Perspective

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Chapter 0

Background

In Section 2 below, we introduce the notion of *real-valued structure*. Roughly speaking, a real-valued structure consists of one or more sets (called "sorts") and real functions (at least one!) on such sorts —possibly in addition to operations and maps between sorts. Ultimately, elements x, y of some sort X are structurally distinguishable *only* when some real function f on X has different values $f(x) \neq f(y)$ on the arguments x, y.

Real-valued structures are compatible *a priori* with the topology on $\mathbb{R}$. More precisely, each sort X is endowed with the (initial) topology making all "structural" real-valued functions f on X continuous; thus, each sort is a completely regular topological space.

0.1 Lattices of real functions

Throughout this section, X is an arbitrary nonempty set. Let $\mathbb{R}^X$ be the set of all functions $f : X \to \mathbb{R}$. Using "pointwise" operations on such functions, $\mathbb{R}^X$ is a real vector space in the usual manner. We expand $\mathbb{R}^X$ to a unital vector lattice by the following additional operations:

- the constant function ("unity") $\mathbb{1} : x \mapsto 1$ (a "nullary" operation);
- the (binary) lattice operations:
  - pointwise maximum ("join"): $(f, g) \mapsto f \vee g : x \mapsto \max\{f(x), g(x)\}$,
  - pointwise minimum ("meet"): $(f, g) \mapsto f \wedge g : x \mapsto \min\{f(x), g(x)\}$.

For the rest of the section, $\mathbb{R}^X$ is regarded as a unital vector lattice. We further endow $\mathbb{R}^X$ with the operation of *evaluation*

$$\begin{aligned} \mathrm{ev} : \mathbb{R}^X \times X &\to \mathbb{R} \\ (f, x) &\mapsto f(x). \end{aligned}$$

Once $\mathbb{R}^X$ is so endowed, one may "forget" that its elements f are "functions", instead capturing the functional meaning of f —in a structural manner— as $\mathrm{ev}(f, \cdot)$; thus, the lattice of all real functions on X is the structure $\langle \mathbb{R}^X, \mathbb{1}, +, \cdot, \mathrm{ev} \rangle$

A *(real-)function lattice on* X is a vector sublattice $\mathbb{1} \in \mathcal{F} \subseteq \mathbb{R}^X$, i.e., a vector subspace closed under lattice operations (and containing $\mathbb{1}$), and further endowed with the evaluation operation $\mathrm{ev} : \mathcal{F} \times X \to \mathbb{R} : (f, x) \mapsto f(x)$. It is therefore a structure of the form $\langle \mathcal{F}, \mathbb{1}, +, \cdot, \mathrm{ev} \rangle$.

Any subset $\mathcal{F} \subseteq \mathbb{R}^X$ generates a function lattice $\widetilde{\mathcal{F}} \supseteq \mathcal{F}$ which, at a minimum, contains all constant functions $r\mathbb{1}$ ($r \in \mathbb{R}$).

The *supremum norm* (or $\|\cdot\|_\infty$-norm) of $f \in \mathbb{R}^X$ is

$$\|f\|_\infty := \sup_{x\in X} |f(x)| \in [0, \infty].$$

Any function $f \in \mathbb{R}^X$ such that $\|f\|_\infty < \infty$ is called *bounded* (these are precisely the functions f for which there exists $C \geq 0$ such that $|f(x)| \leq C$ for all $x \in X$). Let

$$\mathcal{B}(X) := \{f \in \mathbb{R}^X : \|f\|_\infty < \infty\}$$

be the set of bounded functions of $\mathbb{R}^X$. Clearly, $\mathcal{B}(X)$ is a unital sublattice of $\mathbb{R}^X$.

Initial topology by real functions

Given an arbitrary set X and any subset $\mathcal{F} \subseteq \mathbb{R}^X$, one obtains a natural topology $\mathcal{T}_\mathcal{F}$ on X, called the *initial topology by* $\mathcal{F}$. $\mathcal{T}_\mathcal{F}$ is the topology whose subbasic opens have the form $f^{-1}(G)$ where $G \subseteq \mathbb{R}$ is open. (It suffices to take open intervals $G = (r, s)$ —with, say, rational endpoints $r < s$ even.)

0.2 Kolmogorov and Hausdorff spaces

Throughout these notes, a *topological space* is a pair $\underline{X} = \langle X, \mathcal{T}\rangle$ where X is the "underlying set" of $\underline{X}$, and $\mathcal{T} \subseteq \mathcal{P}(X)$ is the "topology" of open sets of X. The topology $\mathcal{T}$ shall always obey the usual definition: $\mathcal{T}$ is closed under arbitrary unions and finite intersections, and contains both the empty and improper subsets $\emptyset, X$.

The notions of *base of* $\mathcal{T}$, *sub-base of* $\mathcal{T}$, and *topology generated by a collection* $\mathcal{S} \subseteq \mathcal{P}(X)$ are the standard ones. The interior, closure and complement of $S \subseteq X$ are denoted S^o, $\overline{S}$ and $S^\complement$, respectively.

Topological indistinguishability Recall that points $x, y \in X$ are *topologically indistinguishable* if the property:

$$\text{for all } A \in \mathcal{T}: \quad x \in A \Leftrightarrow y \in A$$

holds. The property "x is topologically indistinguishable from y" is clearly an equivalence relation that will be denoted "$x \doteqdot y$".

T_0 spaces A space $\underline{X}$ is said to be *Kolmogorov*, or to satisfy the T_0 *separation axiom*, if it satisfies:

$$(\doteqdot\Rightarrow=) \quad \text{If } x \doteqdot y, \text{ then } x = y.$$

In other words, T_0 means that distinct points $x \neq y$ are topologically distinguishable (exactly one of them belongs to some open A).

Every space $\underline{X}$ has a canonical topological quotient $\underline{X}_{\eqcirc} = \langle X_{\eqcirc}, \mathcal{T}_{\eqcirc} \rangle$ that is Kolmogorov, where $X_{\eqcirc} := X/{\eqcirc}$ is the quotient of X by $\eqcirc$, and $\mathcal{T}_{\eqcirc}$ is the naturally induced topology on $X_{\eqcirc}$. ($\mathcal{T}_{\eqcirc}$ consists of the open sets $\mathcal{A} \subseteq X_{\eqcirc}$ such that $\pi^{-1}(\mathcal{A}) \in \mathcal{T}$, where $\pi : X \to X_{\eqcirc}$ is the quotient map.)

Any Kolmogorov quotient $\underline{X}_{\eqcirc}$ is evidently reduced. If $\underline{X}$ is itself Kolmogorov, then it is homeomorphic to its quotient $\underline{X}_{\eqcirc}$; for this reason, we will also call T_0 spaces *(topologically) reduced.*

1 Remarks. The T_0 separation axiom is non-structural from a "purely topological" viewpoint of the space $\underline{X} = \langle X, \mathcal{T} \rangle$. To be more precise, a structural view of the topological space $\underline{X}$ relies on interpreting the membership relation "$x \in A$" between points $x \in X$ and sets $A \in \mathcal{T}$; thus, *sensu stricti,* we regard $\underline{X}$ as a triple $\langle X, \mathcal{T}, \mathtt{E} \rangle$ where $\mathtt{E} = \{(x, A) : x \in A\}$ is the binary relation of membership between points of X and opens in $\mathcal{T}$.

From this perspective, the relation "$x \eqcirc y$" of topological indistinguishability is "intrinsically definable" in $\langle X, \mathcal{T}, \mathtt{E} \rangle$, namely as the property "$(\forall A)(x\mathtt{E}A \leftrightarrow y\mathtt{E}A)$" (with quantification over opens A), whereas the relation "$x = y$" of set-theoretic equality between points is not, because it means "$(\forall \alpha)(\alpha \in x \leftrightarrow \alpha \in y)$" with the quantification over *all sets* α in some model of, say, Zermelo-Fraenkel set theory wherein membership "$\in$" is interpreted. Of course, we could regard topological spaces as quadruples $\underline{X} = \langle X, \mathcal{T}, \mathtt{E}, \mathtt{I} \rangle$ further endowed with the binary relation $\mathtt{I} = \{(x, x) \in X \times X\}$ of equality; however, this approach introduces properties of $\underline{X}$ that are not "purely topological" in the strictest sense.

On the other hand, the relation of indistinguishability of opens $A, B \in \mathcal{T}$ (i.e., $(\forall x)(x \in A \leftrightarrow x \in B)$) does coincide with set theoretic equality, at least if one identifies an element $A \in \mathcal{T}$ with the set $\{x \in X : x\mathtt{E}A\}$.

T_2 (Hausdorff) spaces A space $\underline{X}$ is said to be *Hausdorff*, or to satisfy the T_2 *separation axiom*, if distinct points $x \neq y$ have open neighborhoods $A \ni x$, $B \ni y$ that are disjoint: $A \cap B = \emptyset$.

By the philosophy stated in Remark 1 above, it is structurally more natural to consider topological spaces that are *Hausdorff modulo indistinguishability ("modulo $\eqcirc$").* Explicitly, $\underline{X}$ is Hausdorff modulo $\eqcirc$ if whenever $x \not\eqcirc y$ then x, y have disjoint open neighborhoods.

2 Remark. The distinction between the indistinguishability "$x \eqcirc y$" and set-theoretical equality "$x = y$" relations is both nonstructural and unimportant for our purposes. For all structural purposes, using the indistinguishability relation $\eqcirc$ as a structural substitute for ordinary set-theoretic equality $=$ is appropriate. In practice, it is often convenient to work in reduced topological spaces (also, in reduced pseudometric and, more generally, in reduced real-valued structures that will be introduced in subsequent sections) where structural indistinguishability agrees with equality. We will deal *only* with properties (e.g., Hausdorff) that are meaningful modulo indistinguishability, which will always be the "finest" (most restrictive) structural relation in use and, for all intents and purposes, a suitable substitute (in our context) for set-theoretic equality.

0.3 Compact and compact Hausdorff spaces

Compactness by open coverings

An *open cover* of a topological space $\underline{X}$ is an arbitrary collection $(G_i)_{i \in I}$ of open subsets $G_i \subseteq X$ such that $\cup_{i \in I} G_i = X$. Such cover is *finite* if the index set I is finite (i.e., (G_i) is a cover by finitely many opens of X). A *subcover* of $(G_i)_{i\in I}$ is any subcollection $(g_j)_{j\in J}$ for some index collection $J \subseteq I$.

$\underline{X}$ is *compact* if every open cover $(G_i)_{i\in I}$ has a finite subcover $(G_j)_{j\in J}$.

Compactness by the finite-intersection property

Let X be any set. A *FIP ("finite-intersection property") family* $(F_i)_{i\in I}$ of subsets of X is one whose every finite subfamily $(F_{i_j})_{j<n}$ has nonempty intersection $\bigcap_{j<n} F_{i_j} \neq \emptyset$.

3 Proposition. *A topological space $\underline{X} = \langle X, \mathcal{T}\rangle$ is compact if and only if every FIP family $(F_i)_{i\in I}$ of* closed *subsets of X has nonempty intersection $\bigcap_{i\in I} F_i \neq \emptyset$.* □

Compactness in metric spaces

Let $\underline{X} = \langle X, \mathrm{d}\rangle$ be a metric space (therefore, a Hausdorff topological space). The open (resp., closed) ball of radius $r > 0$ with center $x \in X$ is denoted $B_x(r)$ (resp., $B_x[r]$). A subset $S \subseteq X$ is *ε-dense* if $(B_x(\varepsilon) : x \in S)$ covers X. $\underline{X}$ is *totally bounded* if, for every $\varepsilon > 0$, some finite subset $S \subseteq X$ is ε-dense in X.

4 Proposition. *A metric space $\underline{X}$ is compact if and only if it is totally bounded.*

5 Proposition. *A metric space $\underline{X}$ is compact if and only if every sequence in X has an accumulation point.*

In particular, every compact metric space is complete (every Cauchy sequence in $\underline{X}$ converges).

(Every sequence in a compact topological space has an accumulation point, but a space in which every sequence has an accumulation point —i.e., a "countably compact space"— need not be compact.)

0.4 Filters and ultrafilters

Let $X \neq \emptyset$ be any nonempty set. A *filter* $\mathfrak{F}$ *on* X is a collection of subsets $S \subseteq X$ (called $\mathfrak{F}$*-large sets*) such that:

- $X \in \mathfrak{F}$ (i.e., X is $\mathfrak{F}$-large);
- $\emptyset \notin \mathfrak{F}$ (i.e., $\emptyset$ is not $\mathfrak{F}$-large —hence, is "$\mathfrak{F}$-small");
- If $S, T \in \mathfrak{F}$, then $S \cap T \in \mathfrak{F}$ (the intersection of two $\mathfrak{F}$-large sets is large);
- If $S \in \mathfrak{F}$ and $S \subseteq T \subseteq X$, then $T \in \mathfrak{F}$ (a superset of an $\mathfrak{F}$-large set is itself $\mathfrak{F}$-large).

An *ultrafilter* $\mathfrak{U}$ *on* X is a filter that has no proper extensions to a larger filter on X; equivalently, $\mathfrak{U}$ is a filter with the maximality property:

- for all $S \subseteq X$, either $S \in \mathfrak{U}$, or $X \setminus S =: S^{\complement} \in \mathfrak{U}$.

(For no $S \subseteq X$ can any filter $\mathfrak{F}$ contain both S and $S^{\complement}$ since $\mathfrak{F}$ is closed under intersections, but $S \cap S^{\complement} = \emptyset \notin \mathfrak{F}$.)

For any $x \in X$, the *principal filter* $\mathfrak{P}_x$ consists of all sets $x \in S \subseteq X$.

Exercise 1

1. Every principal filter is an ultrafilter.
2. If X is a finite set, then every ultrafilter on X is principal.

The collection of ultrafilters on X is denoted βX; when endowed with a suitable topology, βX is the Stone-Čech compactification of the (discrete) topological space X. (Stone-Čech compactifications are discussed in Section 0.9 below.) If X is infinite, the ultrafilter space βX is very large (it has cardinality $2^{2^{|X|}}$) wherein principal ultrafilters are scant. However, nonprincipal ultrafilters cannot be "constructed", their existence only shown as a consequence of, e.g., Zorn's Lemma.

Topological convergence of filters

Let $\underline{X} = \langle X, \mathcal{T} \rangle$ be a topological space. A filter (or ultrafilter) on $\underline{X}$ is, by definition, a filter or ultrafilter on the underlying pointset X.

We say that $x \in X$ *is a limit point of a filter* $\mathfrak{F}$ if every open neighborhood $G \ni x$ is $\mathfrak{F}$-large, and denoted "$\mathcal{F} \to x$". We may also say that "$\mathcal{F}$ tends to x"; however, for fixed $\mathcal{F}$, the relation $\mathcal{F} \to x$ may hold for many, a single, or no elements $x \in X$. In particular, the set $\{x \in X : \mathcal{F} \to x\} \subseteq X$ is completely determined by the collection $\mathcal{F} \cap \mathcal{T}$ of opens of $\mathcal{F}$, i.e., by the "sub-filter of opens of $\mathcal{F}$" so to speak.

6 Proposition. *A topological space $\underline{X}$ is compact if and only if every ultrafilter $\mathcal{U}$ on X has a limit.*

Exercise 2

Let $\underline{X}$ be Hausdorff. An ultrafilter $\mathfrak{U}$ on X has at most one limit $x \in X$.

We call any ultrafilter with a unique limit *convergent* (to its unique limit).

Tychonoff's Theorem

Topological products

If $X_\cdot := (X_i)_{i \in I}$ is an arbitrary collection of topological spaces $\langle X_i, \mathcal{T}_i \rangle$, the product space $\Pi X_\cdot := \Pi_i X_i$ is the set of all tuples $x_\cdot := (x_i)_{i \in I}$ such that $x_i \in X_i$ for all $i \in I$. The *product topology* on $\Pi X_\cdot$ has subbasic open sets of the form

$$G@i := G \times \prod_{j \neq i} X_j,$$

one such for every open $G \in \mathcal{T}_i$. (In other words, $G@i = \{x_\cdot \in \Pi X_\cdot : x_j \in G\}$.)

For $i \in I$, the natural projection $\pi_i : \Pi X_\cdot \to X_i : x_\cdot \mapsto x_i$ is continuous.

Exercise 3

Show that an ultrafilter $\mathfrak{U}$ on a product space $\Pi X_\cdot$ has a limit $x_\cdot$ if and only if, for each $i \in I$, x_i is a limit of the filter $\mathfrak{U}_i = \pi_i(\mathfrak{U}) := \{S \subseteq X_i : \mathfrak{F} \ni \pi_i^{-1}(S)\}$ (which is necessarily an ultrafilter on X_i).

7 Theorem (Tychonoff)**.** *If $X_\cdot := (X_i)_{i\in I}$ is an arbitrary indexed family of compact spaces, then the topological product space $\Pi X_\cdot$ is compact.*

Proof. Let $\mathcal{U}$ be an ultrafilter on $\Pi X_\cdot$. Since X_i is compact, the ultrafilter $\mathfrak{U}_i = \pi_i(\mathfrak{U})$ of Exercise 3 has a limit $x_i \in X_i$, hence $\mathfrak{F}$ has a limit $x_\cdot = (x_i : i \in I)$. □

0.5 Tychonoff spaces

Let $\underline{X} = \langle X, \mathcal{T}\rangle$ be a topological space. The set of continuous (resp., continuous and bounded) functions $f : X \to \mathbb{R}$ will be denoted $C(X)$ (resp., $C_b(X)$). Clearly, $C_b(X) \subseteq C(X) \subseteq \mathbb{R}^X$ are unital sublattices.

Completely regular topological spaces

The space $\underline{X}$ is *completely regular* if given $x \in X$ and a closed set $F \subseteq X \setminus \{x\}$ there exists $f \in C(X)$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in F$. Upon replacing $f \in C(X)$ by $\mathbb{0} \vee (f \wedge \mathbb{1}) \in C_b(X)$ if necessary —where $\mathbb{0}$ denotes the "zero function" $0 \cdot \mathbb{1}$— one may impose the condition $f \in C_b(X)$ on functions witnessing the complete regularity of $\underline{X}$.

8 Proposition. *Let $\underline{X}$ be completely regular. Given $x \in X$ and a closed $F \subseteq X \setminus \{x\}$ there exist open sets $G \ni x$, $H \supseteq F$ such that $\overline{G} \cap \overline{H} = \emptyset$.*

Proof. Let $f \in C(X)$ satisfy $f(x) = 0$ and $f{\restriction}F = 1$. The sets $G := f^{-1}\left(-\infty, \frac{1}{3}\right)$, $H := f^{-1}\left(\frac{2}{3}, \infty\right)$ have the required property (since G, H are open neighborhoods of x, F, and $f^{-1}\left(-\infty, \frac{1}{3}\right] \supseteq \overline{G}$ is disjoint from $f^{-1}\left[\frac{2}{3}, \infty\right) \supseteq \overline{H}$). □

Exercise 4

A completely regular space is Hausdorff (mod $\asymp$) (i.e., a completely regular space is in fact Tychonoff (mod $\asymp$)).

[*Hint:* This is a corollary of Proposition 8.]

A topological space $\underline{X} = \langle X, \mathcal{T}\rangle$ is a *Tychonoff* (or "$T_{3½}$") *space* if it is both completely regular and Hausdorff. The Hausdorff requirement may be weakened to T_0 by Exercise 7 below.

0.6 Čech-complete spaces

Recall that a G_δ*-subspace* of a topological space X is a subspace $Y \subseteq X$ that is an intersection $Y = \bigcap_{n<\omega} G_n$ of countably many opens $G_0, G_1, \dots, G_n, \dots \subseteq X$. (More precisely, the topological subspace $\underline{Y} = \langle Y, \mathcal{T}{\restriction}Y\rangle$ is the G_δ-subspace of $\underline{X}$.)

A Tychonoff space is called *Čech-complete* if it is a G_δ-subspace of some compact Hausdorff topological space $\underline{K}$. Equivalently, a Čech-complete space $\underline{X}$ is a Tychonoff space which is a G_δ-subspace of its Stone-Čech compactification βX.

Clearly, compact Hausdorff spaces are Čech-complete.

9 Proposition. *1. A G_δ-subspace of a Čech-complete space is Čech-complete.*

2. A locally compact Hausdorff space is Čech-complete.

Proof. 1. It is clear that a G_δ-subspace of a G_δ-subspace of a (say, compact Hausdorff) space $\underline{K}$ is a G_δ-subspace of $\underline{K}$.

2. A locally compact Hausdorff space X is an open subspace of its one-point compactification X^*.

3. By Theorem 17, if $\underline{X} = \langle X, \mathrm{d} \rangle$ is a complete metric space, then $\mathrm{tp}_{\mathrm{d}} : X \to \mathfrak{T}^\flat$ is a (continuous) surjection.

□

0.7 Baire spaces

A *nowhere dense* subset $N \subseteq X$ of a topological space $\underline{X}$ is one whose closure has empty interior $\overline{N}^o = \emptyset$; the complement of a nowhere dense will be called *openly dense*. (An open*ly* dense set D is one which includes an *open* dense subset U, but D need not itself be open.) A *meager* subset $M \subseteq X$ of a topological space $\underline{X}$ is any countable union $M = \bigcup_{n<\omega} N_n$ of nowhere dense subsets $N_n \subseteq X$. A *comeager* subset $Y \subseteq X$ is one whose complement $X \setminus Y$ is meager; equivalently, $Y = \bigcap_{n<\omega} D_n$ is a countable intersection of subsets $D_n \subseteq X$ each openly dense in X. Evidently, every subset of a meager set is meager, and every superset of a comeager set is comeager.

A topological space $\underline{X} = \langle X, \mathcal{T} \rangle$ *has the Baire property* (or *is a Baire space*) if every meager subset $M \subseteq X$ is codense, i.e., its complement $X \setminus M$ is dense in X; equivalently, $\underline{X}$ is Baire if every comeager subset $Y \subseteq X$ is dense in X.

In a topological sense, nowhere dense subsets of X are "small", while openly dense subsets are "large". In general spaces $\underline{X}$, these notions of "small" and "large" subspace are incompatible with countable processes: for instance, a space X may be a countable union of nowhere dense subspaces. (No reasonable sense of "smallness" would assert that the full space X is a "small" subspace of itself.) For instance, $\mathbb{Q}$ with its usual topology is the union of all (countably many!) singletons $\{r\}$ ($r \in \mathbb{Q}$), which are nowhere dense. Therefore, meager subsets of $\mathbb{Q}$ do *not* capture a notion of "smallness". Baire spaces $\underline{X}$ are, roughly speaking, those for which meager subspaces are "small" (and comeager subspaces are "large"), and the same holds for every open subspace $Y \subseteq X$.

10 Theorem (The Baire Category Theorem)**.** *Every Čech-complete space is Baire.*

Proof. Let $\underline{X}$ be Čech-complete, say $X = \bigcap_{m<\omega} G_m \subseteq \beta X$, and let $(D_n)_{n<\omega}$ be a sequence of open subsets of βX such that $D_n \cap X$ is dense in X. Given any open $A_0 \subseteq \beta X$ with $A_0 \cap X \neq \emptyset$, we show that $\bigcap_n D_n \cap (A_0 \cap X) \neq \emptyset$. Since βX is compact Hausdorff, it is Tychonoff.

Successively, for each $n < \omega$, we construct open sets $\beta X \supseteq A_0 \supseteq \cdots \supseteq A_n \supseteq \ldots$ such that $A_n \cap X \neq \emptyset$ and $\overline{A_n} \subseteq \bigcap_{m<n}(D_m \cap G_m \cap A_m)$. Certainly, A_0 has the stated property by assumption. Having constructed such $A_0 \supseteq \cdots \supseteq A_n$, by Proposition 8, the density of D_n and the definition of X, there is $x_n \in A_n \cap D_n \cap X$ and an open neighborhood A_{n+1} of x_n in βX such that the closure $\overline{A_{n+1}}$ of A_{n+1} in βX satisfies $\overline{A_{n+1}} \subseteq \bigcap_{m\leq n}(D_m \cap G_m \cap A_m)$ and $A_{n+1} \cap X \neq \emptyset$.

By compactness, some element $x^* \in \beta X$ satisfies $x^* \in \bigcap_{n<\omega} \overline{A_n} \subseteq A_0 \cap \bigcap_{n<\omega}(G_n \cap D_m) = (A_0 \cap X) \cap \bigcap_n D_n$. □

0.8 Types over a set of functions

As in the preceding section, X is any nonempty set, and $\mathcal{F} \subseteq \mathbb{R}^X$ any subset.

The *$\mathcal{F}$-type of a point* $x \in X$ (also called the *type of x over $\mathcal{F}$*) is the collection $\mathrm{tp}_{\mathcal{F}}(x) := (f(x) : f \in \mathcal{F}) \in \mathbb{R}^{\mathcal{F}}$. The point $\mathrm{tp}_{\mathcal{F}}(x)$ is called the $\mathcal{F}$-type *realized by x*. The *$\mathcal{F}$-type space* is the closure $\mathfrak{T}_{\mathcal{F}} \subseteq \mathbb{R}^{\mathcal{F}}$ (in the product topology) of the set $\{\mathrm{tp}_{\mathcal{F}}(x) : x \in X\}$ of realized types; all elements $\mathfrak{t} \in \mathfrak{T}_{\mathcal{F}}$ are called (possibly unrealized) *$\mathcal{F}$-types*.

(The notions of $\mathcal{F}$-type and $\mathcal{F}$-type space generalize those of (metric) d-type and d-type space, which correspond to the situation where $\mathcal{F} := \{\mathrm{d}(\cdot, y) : y \in X\} \subseteq \mathbb{R}^X$ when X is the underlying pointset of a space with metric d.)

If $\mathcal{F}$ consists of *bounded* functions, say each $f \in \mathcal{F}$ takes values in some closed bounded interval $[-r_f, r_f] \subseteq \mathbb{R}$, then $\mathfrak{T}_{\mathcal{F}}$ is compact because it is a (by definition, closed) subset of the product space $\prod_{f\in\mathcal{F}}[-r_f, r_f]$, which is compact by Tychonoff's Theorem.

We remark that the initial topology by $\mathcal{F}$ is the "pullback" topology on X via the $\mathcal{F}$-type map $\mathrm{tp}_{\mathcal{F}} : X \to \mathbb{R}^X$.

Exercise 5

If $\mathcal{F} \subseteq \mathbb{R}^X$ generates a unital vector lattice $\widetilde{\mathcal{F}}$, then the restriction map

$$\begin{aligned} \mathfrak{T}_{\widetilde{\mathcal{F}}} &\to \mathfrak{T}_{\mathcal{F}} \\ \mathfrak{t} &\mapsto \mathfrak{t}{\restriction}\mathcal{F} \end{aligned}$$

is a homeomorphism.

By Exercise 5, type spaces over $\mathcal{F}$ and over the unital lattice $\tilde{\mathcal{F}}$ it generates are essentially the same thing; therefore, whenever necessary, we may regard types to be over function lattices without loss of generality.

Exercise 6

Let $\underline{X} = \langle X, \mathcal{T} \rangle$ be completely regular, and let $\mathcal{C} = \mathcal{C}(X)$, $\mathcal{B} = \mathcal{B}(X)$ be the lattices of continuous (resp., and bounded) real functions on X.

Part I

Show that $x \asymp y$ iff $\mathrm{tp}_{\mathcal{C}}(x) = \mathrm{tp}_{\mathcal{C}}(y)$ iff $\mathrm{tp}_{\mathcal{B}}(x) = \mathrm{tp}_{\mathcal{B}}(y)$ (i.e., topological indistinguishability is precisely the relation "$f(x) = f(y)$ for all $f \in \mathcal{C}$", or even just for all $f \in \mathcal{B}$).

Part II

Show that $\mathcal{T} = \mathcal{T}_{\mathcal{C}} = \mathcal{T}_{\mathcal{B}}$ (i.e., the topology on X is initial by $\mathcal{C}$, or by $\mathcal{B}$).

** Part III*

Show that the type-restriction map $\mathfrak{T}_{\mathcal{C}} \to \mathfrak{T}_{\mathcal{B}} : \mathfrak{t} \mapsto \mathfrak{T}\restriction\mathcal{B}$ is continuous and injective, but not necessarily surjective nor open.

[*Hint:* Consider any noncompact (i.e., infinite) discrete topological space, say $\mathbb{N}$ for concreteness. Then $\mathcal{C}$ (resp., $\mathcal{B}$) is the set of all (resp., bounded) functions $f : \mathbb{N} \to \mathbb{R}$. Show that $\mathfrak{T}_{\mathcal{C}}(\mathbb{N}) \simeq \mathbb{N}$ consists of only realized types. (The identity function $\imath := \mathrm{id}_{\mathbb{N}}$ is an element of $\mathcal{C}$; if $\mathfrak{t} \in \mathfrak{T}_{\mathcal{C}}$, then $\mathfrak{t}_\imath$ is necessarily equal to a natural number n such that $\mathfrak{t} = \mathrm{tp}_{\mathcal{C}}(n)$.) On the other hand, the space $\mathfrak{T}_{\mathcal{B}}(\mathbb{N})$ (i.e., the Stone-Čechcompactification of $\mathbb{N}$ —see Section 0.9 below) is compact, so it extends —necessarily properly— the subspace of realized $\mathcal{B}$-types, which is homeomorphic to $\mathbb{N}$.]

0.9 Stone-Čech compactification

11 Theorem (Stone-Čech compactification)**.** *Given any topological space $\underline{X} = \langle X, \mathcal{T} \rangle$ there exist:*

- *a compact Hausdorff space $\underline{\check{X}} = \langle \check{X}, \check{\mathcal{T}} \rangle$; and,*
- *a continuous map $\imath : X \to \check{X}$;*

having the following universal property: for every continuous $f : X \to K$ from $\underline{X}$ into a compact space $\underline{K}$, there exists a unique continuous $\check{f} : \check{X} \to K$ such that

$$\begin{tikzcd} X \arrow[r, "\imath"] \arrow[dr, "f"'] & \check{X} \arrow[d, "\check{f}"] \\ & K \end{tikzcd}$$

commutes.

The space $\underline{\check{X}}$ is unique up to homeomorphism, and called "the" Stone-Čech compactification of $\underline{X}$. *The image $\imath(X)$ is necessarily dense in $\check{X}$.*

If $\underline{X}$ is Tychonoff (resp., completely regular) then $\imath$ is an embedding (resp., $\imath$ induces an embedding $X_{\equiv} \to \check{X}$).

12 Remarks.

- The notation βX for the Stone-Čech compactification (space) $\check{X}$ above is quite standard. The corresponding topological space $\langle \beta X, \check{\mathcal{T}} \rangle$ will be denoted $\underline{\beta X}$ as usual.
- If the original space X is Tychonoff and identified with a subspace of $\check{X}$ via the embedding $\imath$, then $\underline{\check{X}}$ is characterized by the properties (*i*) $\underline{\check{X}}$ is compact Hausdorff, (*ii*) $X \subseteq \check{X}$ is a dense subspace, and (*iii*) every $f \in C_{\mathrm{b}}(X)$ extends to some $\check{f} \in C_{\mathrm{b}}(\check{X})$ (uniquely so, by density).

The Exercise below shows one explicit construction of the space βX using types, at least when $\underline{X}$ is completely regular.

Exercise 7

If $\underline{X}$ is completely regular, then $\mathrm{tp}_{\mathcal{B}} : X \to \mathfrak{T}_{\mathcal{B}}$ induces an embedding of $X_{\equiv}$ onto the subset $\mathrm{tp}_{\mathcal{B}}(X) := \{\mathrm{tp}_{\mathcal{B}}(x) : x \in X\}$ of realized $\mathcal{B}$-types. (If $\underline{X}$ is Tychonoff, then $\mathrm{tp}_{\mathcal{B}} : X \to \mathrm{tp}_{\mathcal{B}}(X)$ is a homeomorphism.)

Exercise 7 shows that the image $\mathrm{tp}_{\mathcal{B}}(X)$ of a Tychonoff space $\underline{X}$ is densely embedded in its $\mathcal{B}$-type space $\mathfrak{T}_{\mathcal{B}}$, which is compact Hausdorff. *We take βX to mean $\mathfrak{T}_{\mathcal{B}}$ henceforth.*

Note that an arbitrary topological space $\underline{X} = \langle X, \mathcal{T} \rangle$ has an expansion $\langle \underline{X}, \mathfrak{T}_{\mathcal{B}}, \mathrm{tp}_{\mathcal{B}}, \mathrm{ev} \rangle$. If $\underline{X}$ is not completely regular, then the pullback topology by $\mathrm{tp}_{\mathcal{B}}$ is strictly coarser than $\mathcal{T}$; if not Hausdorff, then $\mathrm{tp}_{\mathcal{B}}$ is not injective.

The study of structures in analysis strictly through real-valued functions implies that all properties of intrinsic interest are of topological completely regular nature, and compatible with topological indistinguishability; therefore (up to reduction), structural analytic matters are properties of Tychonoff spaces. (Specific Tychonoff spaces of interest, of course.)

Chapter 1

Introduction

The material covered in these notes is classical, but our philosophy and approach are not. The tenet dictating our approach is the following:

Tenet ("Real-Structural" Perspective) *Properties directly relevant in real (functional) analysis are* only *those ultimately captured by real-valued quantities, and furthermore* only *in manners compatible with the topology of the reals.*

The above tenet is —perhaps— uncontroversial. However, thoroughly embracing it implies a "structural" viewpoint of analysis requiring a slight shifting of the usual perspective.

1.1 Metric spaces: a paradigmatic example

Recall the usual definition of metric space: a pair

$$\underline{X} := \langle X, \mathrm{d}(\cdot,\cdot)\rangle$$

where X is any set (which we shall always assume is nonempty), and $\mathrm{d}(\cdot,\cdot) : X \times X \to [0,+\infty)$ (the *metric on* X) is a nonnegative real-valued function satisfying, for all $x, y, z \in X$:

- [Symmetry] $\mathrm{d}(x,y) = \mathrm{d}(y,x)$.
- [Triangle Inequality] $\mathrm{d}(x,z) \leq \mathrm{d}(x,y) + \mathrm{d}(y,z)$.
- [Intrinsic/Extrinsic Equality]:
  - [($=\to\asymp$): "Distance null if equal"] If $x = y$, then $\mathrm{d}(x,y) = 0$;
  - [($\asymp\to=$): "Distance null only if equal"][1] If $\mathrm{d}(x,y) = 0$, then $x = y$.

We introduce the binary relation "$x \asymp y$" of *structural indistinguishability* to mean "$\mathrm{d}(x,y) = 0$", i.e., the elements $x, y \in X$ are indistinguishable from a structural (i.e., presently, metric) viewpoint.

13 Remark. Later we shall introduce "real-valued" structures endowed with a multitude of real-valued functions. Each such structures carries an indistinguishability relation $\asymp$ typically much

[1] We call (d0→=) the axiom that d is a *strict* metric rather than a *pseudo*metric. (Cf., the discussion of pseudometric spaces below.)

more restrictive than the zero-distance relation above. In metric (and pseudo-metric) spaces, we will use the name "zero distance (d0)" for the indistinguishability relation.[2]

We have phrased the axioms of Intrinsic/Extrinsic Equality in the verbose form given (rather than, simply, as the statement "$x \eqcirc y$ iff $x = y$") to make the following points clearer:

1. The property "$x = y$" is purely set-theoretical and therefore not the competence of analysis: it is structurally unnatural to axiomatize a relationship between the *discrete* notion of *set-theoretic equality* between objects x, y—which are ultimately themselves sets— and *analytic* (i.e., *topological*) properties which are captured in terms of real-valued quantities, e.g., the property "$x \eqcirc y$" (which means $\mathrm{d}(x, y) = 0$).

2. The property ($=\to\eqcirc$) is innocuous when rephrased, without any reference to set-theoretic equality, in the form

   - [($=\to\eqcirc$)] $\mathrm{d}(x, x) = 0$ for all $x \in X$.

3. The property ($\eqcirc\to=$) is *not* a structural notion, i.e., it is extrinsic to the study of $\underline{X}$ *solely* through the metric d.

There is a further point to make, a corollary of adopting a perspective compatible with the topology of $\mathbb{R}$, and yet entirely compatible with the standard viewpoint. Symmetry, for instance, is captured by the equality $\mathrm{d}(x, y) = \mathrm{d}(y, x)$, which ultimately hinges on the (set-theoretic) notion of equality of elements of $\mathbb{R}$. That is alright! The set $\mathbb{R}$ and its elements play a distinguished role in analysis. From our perspective, reals $r \in \mathbb{R}$ become "logic constants" which ultimately serve as interpretations of each and every property germane to analysis. Moreover, $\mathbb{R}$ is inherently and permanently endowed with its usual topology; therefore, ("analytic") properties captured by real values $r \in \mathbb{R}$ are themselves topologized in the sense that varying the value of the real quantity capturing a certain property φ results, of course, in a different property φ', but one that is only "slightly changed" from φ.

1.2 Pseudometric spaces

A *pseudometric space* $\underline{X} = \langle X, \mathrm{d}\rangle$ is one whose (pseudo)metric d is symmetric, and satisfies ($=\to\eqcirc$) as well as the triangle inequality —but not necessarily the extrinsic property ($\eqcirc\to=$). Therefore, the class of pseudometric spaces is *intrinsic* as its axioms only speak about properties of points of X directly captured by the real-valued metric d.

In $\underline{X} = \langle X, \mathrm{d}\rangle$, the indistinguishability relation "$x \eqcirc y$" also has the meaning "$\mathrm{d}(x, y) = 0$" as above.

In a pseudometric (or metric, for that matter) space $\underline{X}$, each real $\varepsilon > 0$ gives a neighborhood $(-\varepsilon, \varepsilon)$ of $0 \in \mathbb{R}$, and the intrinsic (real-valued) property "$\mathrm{d}(x, y) < \varepsilon$" is an (open) "topological condition". If, say, x is fixed, the conditions "$\mathrm{d}(x, y) < \varepsilon$" as $\varepsilon > 0$ varies (over arbitrarily small positive reals) state that y is increasingly "more and more" like x, i.e., $\mathrm{d}(x, y)$ measures how "different" y is from x so that $\mathrm{d}(x, y) = 0$ means that x and y are *not* different, i.e., are *equal*. (Although

[2] In general structures, the indistinguishability relation —although structural— is *implicit* rather *explicitly* captured by a "single" structural property such as "$\mathrm{d}(x, y) = 0$".

the axioms for *metric* spaces impose the set-theoretic condition $x = y$ if $\mathrm{d}(x, y) = 0$, the pseudometric perspective is better: set-theoretic inequality of x, y is irrelevant to the question of whether they are *structurally* the same —meaning that $\mathrm{d}(x, y) = 0$).

Metric quotients

Given a pseudometric space $\underline{X} = \langle X, \mathrm{d}\rangle$, one may always perform a set-theoretic construction (which we call *metric reduction*) resulting in a canonical *metric* space $\underline{X}^\sharp = \langle X^\sharp, \mathrm{d}^\sharp\rangle$. The construction is the familiar one: From the axioms, it follows that the "zero-distance (d0)" relation $\eqcirc$ is an *equivalence* relation on X. Let $X^\sharp$ be the identification (quotient space) $X/\eqcirc$ (elements of $X^\sharp$ are of the form $x^\sharp := \{y \in X : \mathrm{d}(x, y) = 0\}$, i.e., each is the $\eqcirc$-class of some element $x \in X$). It also follows easily from the definition of indistinguishability and the pseudometric axioms that the real-valued function $\mathrm{d}^\sharp : (x^\sharp, y^\sharp) \mapsto \mathrm{d}(x, y)$ is well defined, and $\langle X^\sharp, \mathrm{d}^\sharp\rangle$ is a *bona fide* metric space (because $\mathrm{d}^\sharp(x^\sharp, y^\sharp) = 0$ implies $\mathrm{d}(x, y) = 0$; hence, $x \eqcirc y$, so $x^\sharp = y^\sharp$).

From this perspective, a metric space is, for all intents and purposes, simply a *reduced* pseudometric space.

Although the construction of the metric $\underline{X}^\sharp$ from the pseudometric $\underline{X}$ is canonical, a structural viewpoint of pseudometric spaces should make *no* distinction between $\underline{X}$ and $\underline{X}^\sharp$, and there is no need to require (nor intrinsic way to enforce) the "strict" metric axiom ($\eqcirc \to =$).

Exercise 8

Show that the following properties:

1. $\underline{X}$ is complete (i.e., every Cauchy sequence in X has a limit in X);
2. $\underline{X}$ is bounded (i.e., $\sup_{x,y\in X} \mathrm{d}(x, y) < \infty$);
3. $\underline{X}$ is separable (i.e., some subset of X is at most countable, and dense in the metric topology of X);
4. $\underline{X}$ is sequentially compact (i.e., every sequence in X has a convergent subsequence);
5. $\underline{X}$ is compact[3] (i.e., every cover of X by open sets has a finite subcover);

of any pseudometric space $\underline{X} = \langle X, \mathrm{d}\rangle$ are equivalent to the respective properties of its canonical metric quotient $\underline{X}^\sharp = \underline{X}/\eqcirc$ (i.e., $\underline{X}$ has the property iff $\underline{X}^\sharp$ does). Thus, properties 1.–5. are structural.

1.3 Metric types

(Henceforth, when used in a general sense, the adjective "metric" shall be understood in the sense of "*pseudo*metric".)

[3] We adhere to the contemporary convention that "compact" means "every open cover has a finite subcover". In "traditional" topological nomenclature, such property is called "quasicompactness", while (traditionally) "compact" means "quasicompact and Hausdorff". The Hausdorff property (any two points have disjoint neighborhoods) holds for *metric* spaces, but evidently fails for *pseudometric* spaces that are not metric. Therefore, the Hausdorff property is *not* a structural one from our viewpoint.

Let $\underline{X} = \langle X, \mathrm{d}\rangle$ be a pseudometric space. We introduce the first example of the pivotal notion of "type", which shall accompany us throughout these lectures. The definition of type is meant to capture what points $x \in X$ "are", in the most intrinsic possible manner (from our tenet viewpoint).

A point x of the pseudometric space $\underline{X} = \langle X, \mathrm{d}\rangle$ is, from a structural perspective, nothing more and nothing less than the collection of its real distances to each of the elements of the underlying pointset X of $\underline{X}$.

The "metric type" of a point x of a metric space $\underline{X}$ precisely captures the above notion of what x "is" (*vis-à-vis* $\underline{X}$). Thus, we define the *metric type ("*d*-type") of* x as the real-valued function

$$\begin{aligned} \mathrm{tp}_{\mathrm{d}}(x) := \mathrm{d}(x,\cdot) : X &\to [0,+\infty) \\ y &\mapsto \mathrm{d}(x,y). \end{aligned}$$

Exercise 9

Show that $\mathrm{tp}_{\mathrm{d}}(x) = \mathrm{tp}_{\mathrm{d}}(y)$ iff $x \doteqdot y$.

Topology on metric types

As defined above, each type $\mathrm{tp}_{\mathrm{d}}(x)$ is a function $X \to [0,+\infty) \subseteq \mathbb{R}$. Thus, effectively, $\mathrm{tp}_{\mathrm{d}}(x)$ is a point in the product space $\mathbb{R}^X$; in fact, it is more appropriate to think of $\mathrm{tp}_{\mathrm{d}}(x)$ in this manner, i.e., simply as a collection $\mathfrak{t} = (\mathfrak{t}_x)_{x\in X} \subseteq \mathbb{R}$ (indexed by elements of X).

Caveat. *The domain X of the type* $\mathrm{tp}_{\mathrm{d}}(x)$ *is the* point-set X. *It is an* equivocation *to think of the domain X of a type as a "metric space".*

We will eventually introduce a variety of notions of "type" in structures studied in functional analysis —e.g., Hilbert spaces and Banach spaces. All such notions of type will always refer to tuples $(\mathfrak{t}_i)_{i\in I} \subseteq \mathbb{R}$ indexed by some set I, hence $(\mathfrak{t}_i) \in \mathbb{R}^I$. For now, the relevant context is that any such product space $\mathbb{R}^I$ is a topological space, specifically endowed with the *product* topology (which shall *always* be the topology on types). The product topology is also called the topology of "pointwise" convergence in the sense that a sequence (or net) $(\mathfrak{t}^{(n)}) \subseteq \mathbb{R}^I$ converges iff, for each $i \in I$ (i.e., "one 'point' at a time"), the sequence or net $(t_i^{(n)}) \subseteq \mathbb{R}$ converges.[4]

(When I is at most countable, notions of topology and convergence in product spaces $\mathbb{R}^I$ may be studied purely in terms of sequences; however, when I is uncountable, the more general notion of net —or ultrafilter— is needed.)

Exercise 10

Part I

Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence. Prove that, if (x_n) is Cauchy, then the sequence of types $(\mathrm{tp}_{\mathrm{d}}(x_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}^X$ converges (in the product topology).

** Part II*

Show that there exists $\langle X, \mathrm{d}\rangle$ and a sequence $(x_n) \subseteq X$ that is *not* Cauchy, but $(\mathrm{tp}_{\mathrm{d}}(x_n))_{n\in\mathbb{N}}$ converges (thus, the converse to Part I fails).

[*Hint:* Take X to be an infinite set endowed with the discrete metric, and let $(x_n) \subseteq X$ be any sequence of distinct points in X.]

[4] Readers familiar with ultrafilters: An ultrafilter $\mathcal{U}$ on $\mathbb{R}^I$ converges iff each of the projection maps $\pi_i : \mathbb{R}^I \to \mathbb{R}$ ($i \in I$) yields a convergent ultrafilter $\pi_i^*(\mathcal{U})$ on $\mathbb{R}$.

Realized and unrealized types

Througout this subsection, $\underline{X} = \langle X, \mathrm{d}\rangle$ is any pseudometric space.

Any type $\mathrm{tp_d}(x) \in \mathbb{R}^X$ for $x \in X$ is called a *metric type realized by* x. However, any accumulation point in $\mathbb{R}^X$ of realized types is also called a *metric type*. (In general, metric types are possibly *un*realized.) The set of (all) metric types will be denoted $\mathfrak{T}$ —or $\mathfrak{T}(X)$ when specifying X is necessary— and regarded as a topological subspace of $\mathbb{R}^X$; thus,

$$\mathfrak{T}(X) := \overline{\mathrm{tp_d}(X)}$$

is the closure in $\mathbb{R}^X$ of the set

$$\mathrm{tp_d}(X) := \{\mathrm{tp_d}(x) : x \in X\} \subseteq \mathbb{R}^X$$

of realized types.

More generally, if $Y \subseteq X$ is any subset, we obtain a set $\mathrm{tp_d}(Y) := \{\mathrm{tp_d}(y) : y \in Y\}$ of *realized types of Y (over X)*. The *space of* d*-types of Y (over X)* is the closure

$$\mathfrak{T}(Y/X) := \overline{\mathrm{tp_d}(Y)} \subseteq \mathfrak{T}(X);$$

its elements are called (possibly unrealized) d*-types of Y (over X)*. Note that types of Y are still tuples $\mathfrak{t} = (\mathfrak{t}_x)_{x\in X} \in \mathfrak{T}$ indexed by (all) points of X, whence the qualifier "over X" above, and the notation "Y/X". (Of course, $\mathfrak{T}(X/X)$ is the same as the full type space $\mathfrak{T} = \mathfrak{T}(X)$.)

14 Proposition. *Every* d*-type $\mathfrak{t} \in \mathfrak{T}(X)$ is of some bounded subset $Y \subseteq X$. In other words,*

$$\mathfrak{T}(X) = \bigcup_{\text{bounded } Y \subseteq X} \mathfrak{T}(Y/X).$$

In fact, it suffices to take bounded sets Y that are open balls $B_{x_0}(r)$ (or closed balls $B_{x_0}[r]$) of any fixed center x_0 for r varying over any unbounded set of $[0,\infty)$ *(e.g., $r = 1, 2, 3, \dots$).*

Proof. Fix $x_0 \in X$. Let $\mathfrak{t} \in \mathfrak{T}$ be any type, and let $r > \mathfrak{t}_{x_0}$ be arbitrary. Necessarily, $\mathfrak{t} \in \mathfrak{T}(Y/X)$ where $Y := B_{x_0}(r)$ (a bounded set); the same holds for the closed ball $Y := B_{x_0}[r]$ *a fortiori*. □

15 Proposition. *If $Y \subseteq X$ is bounded, the space of types of Y is compact.*

Proof. Fix $y_0 \in Y$. Let $C := \sup_{y_1,y_2\in Y} \mathrm{d}(y_1, y_2)$ be the (finite) diameter of Y. Every realized type $\mathrm{tp}(y)$ belongs to the compact Hausdorff product space $K = \prod_{x\in X}[0, r_x]$ where $r_x := C + \mathrm{d}(x, y_0)$. By definition, $\mathfrak{T}(Y/X) = \overline{\{\mathrm{tp}(y) : y \in Y\}} \subseteq K$ is closed, and hence compact. □

16 Corollary. *The type space $\mathfrak{T}(X)$ is σ-locally compact (i.e., a countable union of open sets with compact closure), and Čech-complete.*

Proof. Fix $x_0 \in X$. For $n = 1, 2, 3, \dots$, each set

$$T_n := \left\{\mathfrak{t} \in \mathfrak{T}(X) : \mathfrak{t}(x_0) < n\right\},$$

is open and included in $\overline{T_n} \cap \mathfrak{T}(X) = \mathfrak{T}\big(X/B_{x_0}(n)\big)$, which is compact by Proposition 15. Hence, $\mathfrak{T}(X)$ is locally compact, and thus Čech-complete by Proposition 9. The sets T_n (have compact closure and) cover $\mathfrak{T}(X)$, by Proposition 14, so $\mathfrak{T}(X)$ is σ-locally compact. □

Exercise 11

Show that the "d-type map"[5]

$$\begin{aligned} X &\to \mathfrak{T} \\ x &\mapsto \mathrm{tp}_{\mathrm{d}}(x) \end{aligned}$$

is an embedding from the d-metric topology on X to the topology (of pointwise convergence) on $\mathfrak{T}$.

What are (unrealized) types?

Each realized type $\mathrm{tp}_{\mathrm{d}}(x)$ captures the essence of a given point x of $\langle X, \mathrm{d}\rangle$. We already remarked that $\mathrm{tp}_{\mathrm{d}}(x) = \mathrm{tp}_{\mathrm{d}}(y)$ whenever x, y are, structurally speaking, the same point (even if, possibly, $x \neq y$ set-theoretically). *A priori*, types (realized or unrealized) are points $\mathfrak{t} = (\mathfrak{t}_x)_{x\in X} \in \mathbb{R}^X$. The following exercises shed some light on the meaning of "unrealized" types.

Exercise 12 below aims to make precise the notion (arising from their definition) that unrealized types $\mathfrak{t}$ behave in "similar ways" to realized types $\mathrm{tp}_{\mathrm{d}}(x)$. (Properties (1)–(3) therein are evidently satisfied when $\mathfrak{t} = \mathrm{tp}_{\mathrm{d}}(z)$ is any realized type.)

Exercise 12

Prove that all (realized or unrealized) types $\mathfrak{t} \in \mathfrak{T}$ satisfy the following for all $x, y \in X$:

1. $\mathfrak{t}_x \geq 0$.
2. $\mathfrak{t}_x + \mathfrak{t}_y \geq \mathrm{d}(x, y)$.
3. $\mathfrak{t}_y + \mathrm{d}(x, y) \geq \mathfrak{t}_x$.
4. $\left|\mathfrak{t}_x - \mathfrak{t}_y\right| \leq \mathrm{d}(x, y)$.
5. $\left|\mathfrak{t}_y - \mathrm{d}(x, y)\right| \leq \mathfrak{t}_x$.

A metric type $\mathfrak{t} \in \mathfrak{T}$ is called *proximal (to X)* if $\inf_{x\in X} \mathfrak{t}_x = 0$. Evidently, every realized type is proximal. A non-proximal type is called *distal*: $\inf_{x\in X} \mathfrak{t}_x > 0$; Let

$$\mathfrak{T}^\flat := \left\{\mathfrak{t} \in \mathfrak{T} : \inf_{x\in X} \mathfrak{t}_x = 0\right\}, \mathfrak{T}^\sharp := \left\{\mathfrak{t} \in \mathfrak{T} : \inf_{x\in X} \mathfrak{t}_x > 0\right\},$$

be the set of proximal and distal types, respectively. Thus,

$$\mathrm{tp}_{\mathrm{d}}(X) \subseteq \mathfrak{T}^\flat \subseteq \mathfrak{T} = \mathfrak{T}^\flat \sqcup \mathfrak{T}^\sharp.$$

[5] If X is a metric space, the map $\mathrm{tp}_{\mathrm{d}}(\cdot)$ is injective and may be used to identify X with (i.e., "include it as") a subset of $\mathfrak{T}$. In general, when X is pseudometric, this "inclusion" proceeds via $X/\!\asymp$ rather than via X.

Exercise 13

Part I

Show that $\mathrm{tp_d}(\cdot) : x \mapsto \mathrm{tp_d}(x)$ is injective iff $\underline{X}$ is a (reduced) metric space.

Part II

Show that $\mathrm{tp_d}(X) \subseteq \mathfrak{T}^\flat$, i.e., every realized type is proximal to X.

Part III

Show that $\underline{X}$ is complete iff every type proximal to X is realized (i.e., the inclusion $\mathfrak{T}^\flat \subseteq \mathrm{tp_d}(X)$ —reverse to Part I— holds, and thus $\mathfrak{T}^\flat = \mathrm{tp_d}(X)$).

Part IV

Show an example of a metric space $\langle X, \mathrm{d}\rangle$ having *non*-proximal types, i.e., such that $\mathfrak{T}^\flat \subsetneq \mathfrak{T}$. [*Hint:* Exercise 10 (2).]

Exercise 14 Metric Completion

Part I

Introduce the notation $X^* := \mathfrak{T}^\flat$. Show that

$$\mathrm{d}^* : (\mathfrak{t}, \mathfrak{u}) \mapsto \sup_{x\in X} \left|\mathfrak{t}_x - \mathfrak{u}_x\right|$$

is a metric on X^*, and $\underline{X}^* = \langle X^*, \mathrm{d}^*\rangle$ is a complete metric space.

Part II

Show that

$$\mathrm{d}^*(\mathrm{tp_d}(x), \mathrm{tp_d}(y)) = \mathrm{d}(x, y) \qquad \text{for } x, y \in X.$$

Deduce that, if $\langle X, \mathrm{d}\rangle$ is a metric space, then $\mathrm{tp_d} : X \to \mathfrak{T}$ is an isometric embedding of $\underline{X}$ as a subspace $\mathrm{tp_d}(X) := \{\mathrm{tp_d}(x) : x \in X\}$ of $\langle X^*, \mathrm{d}^*\rangle$.

Part III

Show that $\mathrm{tp_d}(X)$ is dense in $\langle X^*, \mathrm{d}^*\rangle$.

From Exercise 14, we deduce:

17 Theorem. *Every metric space $\underline{X} = \langle X, \mathrm{d}\rangle$ has a metric completion $\langle X^*, \mathrm{d}^*\rangle$, which may be constructed as follows:*

- $X^* := \mathfrak{T}^\flat = \{\mathfrak{t} \in \mathfrak{T} : \inf_{x\in X} \mathfrak{t}_x = 0\}$ *is the set of proximal* d*-types of X, and*
- $\mathrm{d}^*(\mathfrak{t}, \mathfrak{u}) := \sup_{x\in X} \left|\mathfrak{t}_x - \mathfrak{u}_x\right|$,

provided one identifies X with a subset of $\mathfrak{T}^\flat$ via the distance-preserving map $\mathrm{tp_d}(\cdot)$.

18 Remarks. 1. If $\underline{X}$ is a pseudometric space (rather than a metric space proper), then the type map $\mathrm{tp_d}(\cdot)$ is not injective. This is both unsurprising and unimportant. Unsurprising, because $\langle X^*, \mathrm{d}^*\rangle$ satisfies ($\eqsim\to=$), hence cannot accommodate set-theoretic distinct points $x \neq y$ of X such that $\mathrm{d}(x, y) = 0$. Unimportant, because the incidental fact that x, y are distinct is a purely set-theoretic matter not germane to the viewpoint of a metric space as a structure of real analysis (which ought to be studied purely in regard to real quantities such as values of the metric d to begin).

2. The space $\langle X^*, \mathrm{d}^* \rangle$ is a canonical metric completion of the metric space $\underline{X}/\!\asymp$ (of $\underline{X}$ itself, when a proper metric space). In general, X^* (realized as $\mathfrak{T}^\flat$) is a *proper* subset of the full type space $\mathfrak{T}$; i.e., there may exist distal types. From a structural perspective, what X^* does is "add any necessary points" to realize all types $\mathfrak{t} \in \mathfrak{T}$ (and only those types) such that $\inf_{x \in X} \mathfrak{t}_x = 0$. In doing so, X^* "becomes" the metric completion of X by realizing more types than those realized in X, i.e., by elements of X (unless X was already complete).

   Any distal types $\mathfrak{t} \in \mathfrak{T}^\flat$ of $\underline{X}$ remain unrealized in the metric completion $\underline{X^*}$. The following exercise explains the meaning of such unrealized types in one specific situation.

19 Proposition. *Let $\underline{X} = \langle X, \mathrm{d} \rangle$ be a pseudometric space. The space of proximal types $\mathfrak{T}^\flat \subseteq \mathfrak{T}$ is a G_δ.*

Proof. For each $n \in \omega$, the set

$$G_n := \bigcup_{x \in X} \left\{ \mathfrak{t} \in \mathfrak{T} \,:\, \mathfrak{t}_x < \frac{1}{n} \right\} \subseteq \mathfrak{T}$$

is open, so $\mathfrak{T}^\flat = \bigcap_{n<\omega} G_n$ is a G_δ. □

20 Corollary. *Any complete metric space is Čech-complete.*

Proof. Assume $\underline{X} = \langle X, \mathrm{d} \rangle$ is a complete metric space. By Proposition 19, $\mathfrak{T}^\flat$ is a G_δ of the locally compact Hausdorff type space $\mathfrak{T}$, and therefore Čech-complete; moreover, $\underline{X}$ is homeomorphic to $\mathfrak{T}^\flat$ (via the d-type map). □

* Exercise 15

Let $\underline{X} = \langle X, \mathrm{d} \rangle$ be a pseudometric space.

Part I

Show that, if $\underline{X}$ is compact, then $\mathrm{tp_d}(X) = \mathfrak{T}$.

[*Hint:* $\mathrm{tp_d} : \underline{X} \to \mathrm{tp_d}(X) \subseteq \mathfrak{T}$ is an embedding by Exercise 11. The claim follows from the following general topological facts: (1) any continuous image of a compact space into a Hausdorff space is compact, and (2) any compact subspace of a Hausdorff space is closed.]

Part II

Show that, if closed bounded subsets of X are compact (i.e., if closed balls of finite radius are compact), then $\mathrm{tp_d}(X) = \mathfrak{T}$.

(Compact metric spaces are necessarily bounded, so Part II strengthens Part I.)

[*Hint:* Let $\mathfrak{t} \in \mathfrak{T}$ and $\delta := \inf_{y \in X} \mathfrak{t}_y$. Fix $\delta' > \delta$ and $y' \in X$ with $\mathfrak{t}_{y'} < \delta'$. Then, $\mathfrak{t}$ is an accumulation point of types $\mathrm{tp_d}(x)$ such that $x \in B := B_{y'}(\delta')$. The closed ball $\bar{B} = B_{y'}[\delta']$ is compact by assumption. By Part I: $\mathfrak{t} \in \overline{\mathrm{tp_d}(B)} \subseteq \mathrm{tp_d}(\bar{B}) \subseteq \mathrm{tp_d}(X)$ (since the set $\mathrm{tp_d}(\bar{B})$ is closed as seen from Part I —more specifically, from the two topological facts in the hint thereof.).]

Part III

Show that, conversely, if $\mathrm{tp_d}(X) = \mathfrak{T}$, then closed bounded subsets of X are compact.

[*Hint:* Consider any closed bounded ball $\overline{B} = B_{x_0}[C]$ for some $C > 0$ and $x_0 \in X$. Let $r_x := C + \mathrm{d}(x, x_0)$ for $x \in X$; thus, $\mathrm{d}(x, y) \le r_x$ for all $y \in \overline{B}$ and $x \in X$. More generally, we have

$$\mathfrak{T}_{\overline{B}} := \overline{\mathrm{tp}_{\mathrm{d}}\big(\overline{B}\big)} \subseteq \mathfrak{K} := \prod_{x \in X} [0, r_x].$$

$\mathfrak{K}$ is a Hausdorff space, compact by Tychonoff's Theorem. Thus, $\mathfrak{T}_{\overline{B}} \subseteq \mathfrak{T} \cap \mathfrak{K}$ is closed and hence compact.

If $\overline{B}$ is not compact, it is not totally bounded. Hence, for some $\varepsilon > 0$, no finitely many open balls $B_y(2\varepsilon)$ with $y \in \overline{B}$ suffice to cover $\overline{B}$. It follows that no finitely many balls $B_x(\varepsilon)$ with $x \in X$ suffice to cover $\overline{B}$. Given an arbitrary tuple $\bar{x} = (x_1, \dots, x_n) \subseteq X$, the set

$$U_{\bar{x}} := \left\{ y \in \overline{B} \,:\, \mathrm{d}(y, x_i) \ge \varepsilon, 1 \le i \le n \right\}$$

is nonempty, hence

$$\mathfrak{U}_{\bar{x}} := \left\{ \mathfrak{t} \in \mathfrak{T}_{\overline{B}} \,:\, \mathfrak{t}_{x_i} \ge \varepsilon, 1 \le i \le n \right\} \supseteq \mathrm{tp}_{\mathrm{d}}(U_{\bar{x}})$$

is a nonempty and closed (hence compact) subset of the compact Hausdorff space $\mathfrak{T}_{\overline{B}}$. By compactness, there exists $\mathfrak{t} \in \mathfrak{T}_{\overline{B}}$ such that $\mathfrak{t}_x \ge \varepsilon$ for all $x \in X$; therefore, $\mathfrak{T} \supsetneq \mathrm{tp}_{\mathrm{d}}(X)$.]

Discrete metric spaces

Any pointset X (i.e., a "bare" set) may be regarded as (the underlying set of) a metric space by "expanding" X to $\underline{X}_\delta := \langle X, \delta \rangle$ where δ is the *discrete metric on* X (namely, $\delta(x, y) = 0$ if $x = y$, $\delta(x, y) = 1$ if $x \ne y$, where "$x = y$", "$x \ne y$" are understood in the set-theoretic sense). Evidently, any such discrete space $\langle X, \delta \rangle$ is necessarily complete. By Exercise 14, tp_{d} is a *bijection* from X *onto* $X^* \subseteq \mathfrak{T}$. The type $\mathrm{tp}_{\mathrm{d}}(x)$ of $x \in X$ is the function $\mathrm{tp}_{\mathrm{d}}(x) : y \mapsto \delta(x, y)$.[6]

The exercise below sheds some light on the nature of distal types in this particular (discrete) setting.

Exercise 16

Let $\underline{X}_\delta = \langle X, \delta \rangle$ be any discrete metric space. In such setting, the set of proximal types $X^* \subseteq \mathfrak{T}$ is precisely the set $\mathrm{tp}_{\mathrm{d}}(X)$ of realized types. Prove the following:

- if X is finite, then $\mathfrak{T} = X^*$.
- if X is infinite, then $\mathfrak{T} = X^* \cup \{\mathfrak{w}\}$, where $\mathfrak{w} = (1)_{x \in X} \in \mathbb{R}^X$ is the type that is constant-equal-to-1.

21 Example. In the setting of Exercise 16, in case X is infinite, let $Y \supsetneq X$ be any proper superset of X, and let $\underline{Y}_\partial = \langle Y, \partial \rangle$ be the corresponding extension of $\underline{X}_\delta$ (thus, ∂ is the discrete metric on Y, so $\delta = \partial \restriction (X \times X)$). The type spaces of X and Y are denoted $\mathfrak{T}_X$ and $\mathfrak{T}_Y$, respectively. Every type $\mathfrak{u} \in \mathfrak{T}_Y \subseteq \mathbb{R}^Y$ gives a type $\mathfrak{u} \restriction X \in \mathfrak{T}_X \subseteq \mathbb{R}^X$ by ("pointwise") restriction:

$$\mathfrak{u} \restriction X := (\mathfrak{u}_x)_{x \in X}.$$

[6]Apart from reversing the role of the numbers 0 and 1, the type $\mathrm{tp}_{\mathrm{d}}(x)$ is the so-called "characteristic function" $\chi_{\{x\}}$ of the singleton subset $\{x\}$ of X.

(The map $\mathfrak{u} \mapsto \mathfrak{u}\restriction X$ is continuous.)

If $y \in Y \setminus X$ is any of the "new" points of Y, then $\partial(y, x) = 1$ for all $x \in X$; therefore, $\mathrm{tp}_{\mathrm{d}}(y)\restriction X = (1) = \mathfrak{w}$ (from Exercise 16 above). In other words, the type $\mathfrak{w} = (1) \in \mathfrak{T}_X$, although unrealized in X, is realized as (the restriction to X of) a type of $\underline{Y}_\partial$, i.e., $\mathfrak{w} = \mathrm{tp}_{\mathrm{d}}(y)\restriction X$.

The example above hints at a general fact, which we discuss only informally for now. In full generality (not just for *discrete* metric or pseudometric spaces $\underline{X}$), types are "potentially" realizable in the sense that they are (the restriction to X of) types realized in some extension $\underline{Y}$ of $\underline{X}$. All proximal types are necessarily realized in (any) metric completion $\underline{X}^*$ of $\underline{X}$.[7] For other (distal) types, the situation is more delicate. The Example above shows that the distal type $\mathfrak{w} \in \mathfrak{T}_X$, although realized in Y, need not have (in general) a unique realization: Any points $y \neq z$ of $Y \setminus X$ have different type $\mathrm{tp}_{\mathrm{d}}(y) \neq \mathrm{tp}_{\mathrm{d}}(z)$ (because $\partial(y, z) = 1 = \mathrm{tp}_{\mathrm{d}}(y)(z) \neq 0 = \mathrm{tp}_{\mathrm{d}}(y)(y)$) but realize (by restriction) the same type $\mathfrak{w} \in \mathfrak{T}_X$ since $\mathrm{tp}_{\mathrm{d}}(y) \restriction X = \mathfrak{w} = \mathrm{tp}_{\mathrm{d}}(z)\restriction X$ (because $\partial(y, x) = 1 = \partial(z, x)$ for all $x \in X$).

[7] One may even show that proximal types of $\underline{X}$ are *uniquely* realized in *any* completion $\hat{X} \supseteq X$

Chapter 2

Real-valued structures

2.1 Motivation

The class of pseudometric spaces in the introduction is the foremost example of a class of "real-valued structures".

The notion of real-valued structure epitomizes the tenet that objects and properties germane to analysis are those, and only those, that can be captured through real-valued quantities.

Operations and relations on the real line

Real numbers play a pivotal role in analysis, and therefore also in the definition of real-valued structure. Since all intrinsic properties we shall study are captured in terms of real numbers, the real line $\mathbb{R}$ becomes the set of "interpretation values" of quantities of interest (e.g., of a metric d). However, as a "bare" pointset, $\mathbb{R}$ is of little use until endowed with additional structure:

The real line $\mathbb{R}$ shall always be regarded as endowed with the structure of a complete ordered field.

In addition to its ordering and arithmetic operations, the above principle implies that the real line is explicitly endowed with a real-valued function "sup" on the set $\mathcal{P}_{\leq}(\mathbb{R})$ of nonempty bounded-above subsets of $\mathbb{R}$. For our purposes, it is rather more appropriate, for each $r \in \mathbb{R}$, to endow the real line with the restriction of sup to the set $\mathcal{P}_{\leq r}(\mathbb{R})$ of nonempty subsets bounded above by r. We make the same convention, *mutatis mutandis,* for infima.

For convenience, we also take $\mathbb{R}$ as endowed with the (binary) "maximum" operation $(r, s) \mapsto \max(r, s)$, and similarly for minimum "min".[1] (In particular, $\mathbb{R}$ is a "vector lattice" in the sense of Section 3.2 below.)

22 Remark. The structure bestowed upon $\mathbb{R}$ above is far from exhaustive! For instance, we regard $\mathbb{R}$ as carrying arithmetic as well as "lattice" operations (max/min), but *no* others (such as, e.g., transcendental functions). We have chosen the structure above because it is appropriate for uses of real numbers as interpretation ("logical") values.

[1] Although the binary operation max is "implicitly defined" by the ordering of $\mathbb{R}$, the structural approach requires capturing such implicit structure in an explicit manner.

Real-valued predicates

A metric d on X is the foremost example of a "real-valued predicate" on X. Such metric d is a *binary* predicate because it takes two arguments $x, y \in X$.

The terminology "predicate" requires some explanation. We assume readers are familiar with the notion of *relation* R on a set A. Such a relation R always takes a fixed number $n \in \mathbb{N}$ of arguments (called the *arity of* R):

- single-argument relations $R \subseteq A$ are *unary*;
- two-argument relations $R \subseteq A \times A$ are *binary*;
- three-argument relations $R \subseteq A \times A \times A$ are *ternary*; ...
- n-argument relations $R \subseteq A \times A$ are *n-ary*; ....

However, the notions of mathematical interest about relations are the *statements* implied by relations in the sense above. In typical use, a relation $R \subseteq A^n$ is said to "hold for $(a_1, a_2, \dots, a_n) \in A^n$" precisely when $(a_1, a_2, \dots, a_n) \in R$, a property notationally abbreviated "$R(a_1, a_2, \dots, a_n)$", which becomes a *statement* about the n-tuple $\bar{a} = (a_1, a_2, \dots, a_n)$. (We shall use bars to denote such tuples in what follows.) From such perspective, a relation $R \subseteq A^n$ induces a *predicate*[2]

$$\begin{aligned} \mathrm{Pred}_R : A^n &\to \{\top, \bot\} \\ \bar{a} &\mapsto \begin{cases} \top, & \bar{a} \in R; \\ \bot, & \bar{a} \notin R. \end{cases} \end{aligned}$$

as a function on A^n taking (only) the "(discrete) logical values" $\top$ ("true") and $\bot$ ("false"). Thus, an *(n-ary) predicate of discrete logic* is a definite property that each tuple $\bar{a} \in A^n$ may (or may not) possess.

The distinction between the relation $R \subseteq A^n$ and the function Pred_R is so superficial as to seem pedantic (one may always recover R as $\mathrm{Pred}_R^{-1}(\top) = \{\bar{a} \in A^n : \mathrm{Pred}_R(\bar{a}) = \top\}$). However, it follows from our tenet that properties captured in terms of set-theoretic relations (such as set equality or membership) are not of direct intrinsic interest in analysis, but only properties that (*i*) can be captured in terms of real values, and (*ii*) are compatible with the topological notions (of proximity and approximation) implied by the use of such real values.

In sum, the notion of ($\{\top, \bot\}$-valued) predicate of discrete logic in the sense above is transformed per our tenet into the following definition:

A real-valued n-ary predicate on a set A is a function $P : A^n \to \mathbb{R}$.

Our tenet does not specify the mathematical "meaning" or interpretation of real-valued predicates, although eventual interpretations of real-valued predicates should feel completely natural and compatible with standard use of real-valued notions in analysis. Philosophically, predicates P should only be used in manners compatible with the topology of $\mathbb{R}$ in the sense that "proximity" of the values $P(\bar{a})$, $P(\bar{b})$ means that, at least insofar as the property P expresses or captures, $\bar{a}$ is close to $\bar{b}$.

[2]The logical symbols $\top$ ("top"), $\bot$ ("bottom") entail the meaning of "true" and "false", respectively.

The interpretation of a metric

Going back to a given pseudometric space $\underline{X} = \langle X, \mathrm{d}\rangle$, the metric d is a binary (real-valued) predicate on X. Intuitively, d is a topological ("soft", or "graded") notion of "equality", where the property "$\mathrm{d}(x, y) = 0$" relaxes the (set-theoretic) binary relation of equality "$x = y$" (literally —and very unpleasantly— captured by the "diagonal relation" $E = \{(x, x) : x \in X\}$; in other words, "=" is the predicate Pred_E). However, the notion of "sameness" carried by "$\mathrm{d}(x, y) = 0$" is softer and topolgical: a value $\mathrm{d}(x, y)$ near 0 means that x, y are "nearly" equal.

The property $\mathrm{d}(x, x) = 0$ dictates that (in this context), 0 is the interpretation value one should use as "$\top$" (true). Symmetry $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ reflects the symmetry of equality. The transitivity of equality needs to be captured "softly" as a property of the real-valued predicate d: this is captured by the triangle inequality $\mathrm{d}(x, y) \le \mathrm{d}(x, z) + \mathrm{d}(z, y)$ which implies (in a quantitative yet simple manner) that when x is near z, and z near y, then x is near y. In fact, when x is near y, then the real values ("properties") $\mathrm{d}(x, w)$, $\mathrm{d}(w, y)$ of x, y with respect to any other point w are equally near because $|\mathrm{d}(x, y) - \mathrm{d}(y, w)| \le \mathrm{d}(x, y)$ (regardless of the absolute magnitudes of the individual values $\mathrm{d}(x, y)$, $\mathrm{d}(y, w)$).

2.2 Real-valued structures

A *vocabulary* for real-valued structures is a pair $V = (\mathtt{F}, \mathtt{P})$, where

- $\mathtt{F}$ is a collection of formal symbols $\mathtt{f}$, called *function symbols*, each of which has associated a natural $n_{\mathtt{f}} \in \mathbb{N}$, called the *arity of f*;
- $\mathtt{F}$ is a collection of formal symbols $\mathtt{p}$, called *predicates symbols*, each of which has associated a natural $n_{\mathtt{p}} \in \mathbb{N}$, called the *arity of p*.

Each of the collections $\mathtt{F}, \mathtt{P}$ is otherwise arbitrary.

23 Example. The vocabulary V_{met} for metric (or pseudometric) spaces has empty function symbol collection $\mathtt{F} = \{\}$ and predicate symbol collection $\mathtt{P} = \{\mathtt{d}\}$, where $\mathtt{d}$, a symbol of arity $n_{\mathtt{d}} = 2$ (i.e., a *binary* predicate symbol) is a formal "name" (eventually used to denote the metric —real binary predicate— of pseudometric spaces —see Definition 24 and Example **??** (1) below).

24 Definition (Real-valued structure)**.**

Let $V = (\mathtt{F}, \mathtt{P})$ be any vocabulary for real-valued structures. A *real-valued structure with vocabulary V* (or *real V-structure*) is a triple

$$\mathcal{M} = \langle M, F, P\rangle,$$

where

- M is a nonempty set, called the *universe* (or *underlying set*) of $\mathcal{M}$,
- F is a family $(\mathtt{f}^{\mathcal{M}} : \mathtt{f} \in \mathtt{F})$ indexed by $\mathtt{F}$ that, to each function symbol $\mathtt{f} \in \mathtt{F}$, assigns a function $\mathtt{f}^{\mathcal{M}} : M^n \to M$ of arity $n = n_{\mathtt{f}}$.

- P is a family $(\mathtt{p}^{\mathcal{M}} : \mathtt{p} \in \mathtt{P})$ indexed by $\mathtt{P}$ that, to each predicate symbol $\mathtt{p} \in \mathtt{P}$, assigns a real-valued predicate (i.e., real-valued function) $\mathtt{p}^{\mathcal{M}} : M^n \to \mathbb{R}$ of arity $n = n_{\mathtt{p}}$.

F, P are called, respectively, the *family of functions*, and *of predicates*, of $\mathcal{M}$.

We often prefer calling functions $\mathtt{f}^{\mathcal{M}} \in F$ the *operations of* $\mathcal{M}$. For emphasis, we also say that the operations $\mathtt{f}^{\mathcal{F}}$ (and predicates $\mathtt{p}^{\mathcal{M}}$) are "distinguished" in the sense that each is the interpretation of a specific symbol $\mathtt{f} \in \mathtt{F}$ (or $\mathtt{p} \in \mathtt{P}$). From an "external" (non-structural) viewpoint, the set M admits a plethora of operations $g : M^m \to M$ and real-valued functions $q : M^m \to \mathbb{R}$; however, all such are operations and predicates are undistinguished ("external") as they generally bear no direct relation to the distinguished structural components (operations and predicates) of $\mathcal{M}$; properties of M, or even of the structur $\mathcal{M}$, that are captured only through external operations and predicates are not germane to the structural perspective.

Henceforth, "structure" means "real-valued structure" in the sense of Definition 24 unless explicitly said otherwise.

Examples of real-valued structures

Pseudometric spaces

For $V_{\mathrm{met}} = (\{\}, \{\mathtt{d}\})$ the vocabulary of metric spaces, a real V-structure is of the form $\mathcal{M} = \langle M, (), (\mathrm{d})\rangle$ where $\mathtt{d}^{\mathcal{M}}$ is an (otherwise arbitrary) function $\mathrm{d} : M \times M \to \mathbb{R}$. In particular, any metric (or pseudometric) space $\underline{X} = \langle X, \mathrm{d}\rangle$ may be regarded as a V-structure $\mathcal{X} = \langle X, (), (\mathrm{d})\rangle$. Since $\underline{X}$ and $\mathcal{X}$ are identical in all essential aspects, we regard (pseudo)metric spaces as V_{met}-structures. (Of course, *any* $\mathrm{d} : X \times X \to \mathbb{R}$ yields a V_{met}-structure $\langle X, (), (\mathrm{d})\rangle$ without regard to any properties possessed by d.)

Discrete real vector spaces

Real vector spaces are of the (usual) form $\underline{X} = \langle X, 0, +, \cdot\rangle$ having X as underlying set (whose element x we call "vectors"), $0 \in X$ as the "zero vector", and carrying the binary operation of addition $+$ and the operation of multiplication $\cdot : \mathbb{R} \times X \to X$ by real scalars.

From the structural perspective:

- specifying the "zero" element of X amounts to a *nullary* operation on X (i.e., $n_{\mathtt{0}} = 0$), for which we introduce the symbol $\mathtt{0}$.[3]

- multiplication by *each fixed real scalar* $r \in \mathbb{R}$ is a unary function $X \to X$ for which we introduce the symbol $\mathtt{r}\cdot$ (one for each $r \in \mathbb{R}$).

- addition on X is a binary operation which will be denoted by the formal symbol $+$.

[3]By definition, a nullary operation is a function $f : X^0 \to X$. However, $X^0 = \{\bullet\}$ is a singleton whose only element is the "null tuple" $\bullet = ()$ consisting of *no* elements of X; therefore, the function f is characterized by its single value $f(\bullet) \in X$. In practice, thus, a nullary function on X amounts to nothing more and nothing less than specifying some element $a \in X$.

- the set-theoretic relation of equality "$x = y$" must be included as the interpretation of a binary predicate on X, for which we introduce the symbol $\mathtt{d}$ (our preferred symbol for a metric, for reasons that will become clear). Specifically, $\mathrm{d} : X \times X \to \{0, 1\}$ is taken as the function

$$\mathrm{d}(x, y) = \begin{cases} 0, & (x = y) \\ 1, & (x \neq y), \end{cases}$$

Thus, real vector spaces in the sense above are axiomatically "discrete".

From a structural perspective, the vocabulary for real vector spaces is $V^{\mathrm{d}}_{\mathrm{vec}} = (\mathtt{F}_{\mathrm{vec}}, \{\mathtt{d}\})$, where $\mathtt{d}$ is a binary relation symbol, and with function symbol collection

$$\mathtt{F}_{\mathrm{vec}} = \{\mathtt{0}, \mathtt{+}\} \cup \{\mathtt{r}\cdot : r \in \mathbb{R}\},$$

of arities $n_0 = 0$, $n_+ = 2$, and $n_{\mathtt{r}\cdot} = 1$ for $r \in \mathbb{R}$.

To a given vector space $\underline{X} = \langle X, 0, +, \cdot \rangle$ (in the usual sense) there corresponds a $V^{\mathrm{d}}_{\mathrm{vec}}$-structure

$$\mathcal{X} = \langle X, (0, +, r\cdot : r \in \mathbb{R}), (\mathrm{d}) \rangle,$$

where the meaning of $\mathtt{0}^{\mathcal{X}}$ is the zero element of X, of $+^{\mathcal{X}}$ is the addition $+$, of $\mathtt{r}\cdot^{\mathcal{X}}$ is the function $r\cdot : x \mapsto rx$, and d is the discrete metric as explained above.

The passage from $\underline{X}$ to $\mathcal{X}$ is innocuous, so we may regard vector spaces in the usual sense as $V^{\mathrm{d}}_{\mathrm{vec}}$-structures.

A structure $\mathcal{X} = \langle X, (0, +, r\cdot : r \in \mathbb{R}), (\mathrm{d}) \rangle$ is a vector space if the interpretation of operations in $\mathtt{F}_{\mathrm{vec}}$ and of the relation d obey the usual vector-space axioms relative to set-theoretic equality interpreted as d, namely, for all $x, y, z \in X$ and $r, s \in \mathbb{R}$:

- d is a pseudometric;[4]
- [d is discrete] $\mathrm{d}(x, y) = 0$ or $\mathrm{d}(x, y) = 1$;

Henceforth, we write $x \asymp y$ *to mean* $\mathrm{d}(x, y) = 0$.

- [Addition and scalar multiplication are well defined modulo $\asymp$] $x \asymp y$ implies:
  - $x + z \asymp y + z$ and $z + x \asymp z + y$),
  - $r \cdot x \asymp r \cdot y$;
- [0 is neutral for addition] $x + 0 \asymp x$ and $x \asymp x + 0$;
- [Addition is commutative] $x + y \asymp y + x$;
- etc.

The most important classical example of a metric on a vector space which is compatible with the vector operations is when the metric is associated to a norm $\|\cdot\|$; this leads to the definition of *normed vector space* in Section 3 below.

[4]Strictly speaking, d is only a discrete pseudometric. We will not assume d satisfies d0$\Rightarrow$=. The remaining axioms will use the indistinguishability relation $\asymp$ of d in place of set-theoretic equality.

Exercise 17

1. Complete the list of axioms for discrete real vector spaces above.
2. Why is it necessary to require that operations be well defined modulo $\doteqdot$?
3. Assume that $\mathcal{X} = \langle X, (\dots), (\mathrm{d})\rangle$ is a structure in the language $V_{\mathrm{vec}}^{\mathrm{d}}$ which satisfies all the axioms for discrete vector spaces *except perhaps* the axiom "d is discrete" (where "$x \doteqdot y$" is still interpreted as the relation "$\mathrm{d}(x, y) = 0$" in the remaining axioms). Define the discrete binary predicate $\mathrm{d}' : X \times X \to \{0, 1\}$ by

$$\mathrm{d}'(x, y) = \begin{cases} 0 & (\mathrm{d}(x, y) = 0), \\ 1 & (\mathrm{d}(x, y) \neq 0). \end{cases}$$

   Show that $\mathcal{X}'$ obtained from $\mathcal{X}$ replacing d by d' is a discrete vector space.

Exercise 17.3 shows that merely replacing discrete equality by a ("continuous") pseudometric (while keeping the axioms of a vector space otherwise unchanged) does not lead to a richer class of structures. This motivates the need to introduce (non-discrete) notions of vector spaces that are axiomatized differently from the discrete ones; such is the main goal of the following chapter.

Discrete real vector lattices

The *vocabulary for real vector lattices*[5] is an expansion $V_{\mathrm{Riesz}} = \langle \mathtt{F}_{\mathrm{Riesz}}, (\mathtt{d})\rangle$ of $V_{\mathrm{vec}}^{\mathrm{d}}$, where

$$\mathtt{F}_{\mathrm{Riesz}} = (\mathtt{0}, +, \mathtt{r}\cdot, \vee, \wedge : r \in \mathbb{R}),$$

with $\vee, \wedge$ binary function symbols (denoting the operations of *join* and *meet*, respectively) expanding the function collection $\mathtt{F}_{\mathrm{vec}}$ for vector spaces. A *discrete real vector lattice* is a V_{Riesz}-structure

$$\underline{X} = \langle X, 0, +, r\cdot, \vee, \wedge, \mathrm{d} : r \in \mathbb{R}\rangle$$

where

- d is a discrete metric on X (takes only values $0, 1$);

The relation "$x \doteqdot y$" henceforth means "$\mathrm{d}(x, y) = 0$".

- the reduct $\langle X, 0, +, r\cdot, \mathrm{d} : r \in \mathbb{R}\rangle$ to the vocabulary $V_{\mathrm{vec}}^{\mathrm{d}}$ is a discrete vector space as in Section 2.2 above;
- the reduct $\langle X, \vee, \wedge, \mathrm{d}\rangle$ is a lattice: for all $x, y, z \in X$,
  - $\vee$ and $\wedge$ are well defined $\pmod{\doteqdot}$;
  - [idempotence] $x \wedge x \doteqdot x \doteqdot x \vee x$;
  - [absorption] $x \wedge (x \vee y) \doteqdot x \doteqdot x \vee (x \wedge y)$;

[5] A real vector lattice —not necessarily discrete— is also called a *Riesz space*.

  - [commutativity] $x \wedge y \eqcirc y \wedge x$, and $x \vee y \eqcirc y \vee x$;
  - [associativity] $x \wedge (y \wedge z) \eqcirc (x \wedge y) \wedge z$, and $x \vee (y \vee z) \eqcirc (x \vee y) \vee z$;

- the lattice operations are compatible with the vector structure:
  - [Positive homogeneity] $r(x \wedge y) \eqcirc (rx) \wedge (ry)$, and $r(x \vee y) \eqcirc (rx) \vee (ry)$ for $r \geq 0$;
  - [Negative homogeneity] $r(x \wedge y) \eqcirc (rx) \vee (ry)$, and $r(x \vee y) \eqcirc (rx) \wedge (ry)$ for $r \leq 0$;
  - [Translation invariance] $(x+z) \wedge (y+z) \eqcirc (x \wedge y) + z$, and $(x+z) \vee (y+z) \eqcirc (x \vee y) + z$.

(In particular, either of the lattice operations may be defined in terms of the other, e.g., $x \wedge y = -((-x) \vee (-y))$, so either one of them is redundant.)

A discrete real vector lattice is called *distributive* if it satisfies:

- [distributivity] $x \wedge (y \vee z) \eqcirc (x \wedge y) \vee (x \wedge z)$, and $x \vee (y \wedge z) \eqcirc (x \vee y) \wedge (x \vee z)$.

The definitions of discrete vector space and discrete vector lattice are not very natural from a real-valued perspective. As soon as one allows the pseudometric to be nondiscrete, the axioms given in terms of the strictly discrete relation $x \eqcirc y$ become topologically incompatible (discontinuous) with respect to the d-metric topology.

Exercise 18

Let $\langle X, 0, +, r\cdot, \wedge, \vee, \mathrm{d}\rangle$ be a vector lattice.

Part I

Show that $x \wedge y \eqcirc x$ holds iff $x \vee y \eqcirc y$. These equivalent properties will be denoted "$x \leq y$".

Part II

Show that "$x \leq y$" is (*i*) a partial ordering of X, (*ii*) has symmetric part $\eqcirc$ in the sense that $x \leq y \leq x$ iff $x \eqcirc y$, and (*iii*) possesses the following properties (universally for all $x, y, z \in X$ and $r \in \mathbb{R}$):

1. [Translation invariance] $x \leq y$ implies $x + z \leq y + z$;
2. [Homogeneity]
   - $x \leq y$ and $r \geq 0$ imply $rx \leq ry$;
   - $x \leq y$ and $r \leq 0$ imply $rx \geq ry$;
3. [Least upper bounds] Given $x, y \in X$ there is $z \in X$ such that $z \leq w$ whenever $w \geq x$ and $w \geq y$. (*Hint:* Let $z := x \vee y$.)
4. [Greatest lower bounds] Given $x, y \in X$ there is $z \in X$ such that $z \geq w$ whenever $w \leq x$ and $w \leq y$.

25 Remark. Exercise 18 suggests that one can re-define a (discrete) vector lattice as a structure in the vocabulary $(\mathrm{F}_{\mathrm{Riesz}} \setminus \{\wedge, \vee\}, \{\leq, \mathrm{d}\})$ that drops the join and meet operations, adding instead a binary discrete predicate $\leq$. The vector space axioms together with the properties in Part II may be taken as axioms for vector lattices in this alternate vocabulary. Joins may be defined using the

Least Upper Bound axiom (the witness element z therein is unique modulo $\asymp$ and may be taken as the definition of $x \vee y$). Meets are similarly defined using greatest lower bounds. (The Axiom of Choice is necessary to select one such witness z given x, y.)[6]

Since we prefer to have explicit structure rather than implicit, we have defined vector lattices as possessing join/meet operations instead of an ordering predicate $\leq$.

Function lattices Fix a set D (the "domain"). One may endow the set $\mathbb{R}^D$ of all real-valued functions $f : D \to \mathbb{R}$ with the structure of a discrete Riesz space (real vector lattice), namely

$$\underline{\mathbb{R}^D} := \langle \mathbb{R}^D, 0, +, r\cdot, \wedge, \vee, \mathrm{d} \rangle$$

with the usual element (constant function) $0 : x \mapsto 0$, pointwise addition $+$ and pointwise scalar multiplication $r\cdot$, pointwise minimum as meet $\wedge$ and pointwise maximum as join $\vee$. The predicate d is the discrete metric ($\mathrm{d}(f, g) = 0$ iff $f(x) = g(x)$ for all $x \in D$; $\mathrm{d}(f, g) = 1$ otherwise).

More generally, any subset $\mathcal{F} \subseteq \mathbb{R}^D$ containing 0 that is closed the vector lattice operations yields a Riesz subspace

$$\underline{\mathcal{F}} := \langle \mathcal{F}, 0, +, r\cdot, \wedge, \vee, \mathrm{d} \rangle$$

of $\underline{\mathbb{R}^D}$ (where the operations $+, r\cdot, \wedge, \vee$ are the restrictions to $\mathcal{F}$ of those on $\mathbb{R}^D$).

Substructures $\underline{\mathcal{F}}$ of some structure $\underline{\mathcal{R}^D}$ will be called *concrete vector lattices*. Although *discrete* vector lattices are not of great intrinsic interest, various notions of *topological* vector lattices are of paramount importance in analysis. Topological vector lattices (under various assumptions) are isomorphic to concrete ones. These matters will be discussed in Section *** below.

26 Remark. Although concrete vector sublattices $\underline{\mathcal{F}}$ of $\underline{\mathbb{R}^D}$ consist of (some) functions $f : D \to \mathbb{R}^D$, we stress that the vocabulary V_{Riesz} does *not* capture the "functional" nature of elements $f \in \mathcal{F}$ in any direct manner: V_{Riesz} does not include real-valued predicates giving the evaluation values $f(x)$ for $x \in D$ —indeed, the vocabulary does not allow "speaking" about the set D nor its points at all.

Although (topological) vector lattices are quintessential structures of functional analysis, the choice of a vocabulary that intentionally abstracts away ("hides") the "underlying" functional nature of lattice elements f is quite typical. In practice, "functional" analysis is the study of (topological) *vector spaces* (including vector lattices) rather than literally the study of functions or function spaces.

Our structural viewpoint and reliance on real-valued types *de facto* represent a back-to-roots approach to functional analysis where elements are studied through real-valued functions —their types.

Exercise 19

Show that $\underline{\mathbb{R}^D}$ and $\underline{\mathcal{F}}$ above are discrete real vector lattices as per Section 2.2.

The notions of discrete vector space and discrete vector lattice above are structural, but are real-valued in name only rather than in an essential manner because of the assumption that d is discrete

[6]In this discrete setting (only!), it is even possible to drop the metric d as well as the operations $\wedge, \vee$ —keeping only $\leq$. The axioms for $\leq$ imply that the expansion whereby d is defined by $\mathrm{d}(x, y) = 0$ iff $x \leq y \leq x$ (and $\mathrm{d}(x, y) = 1$ otherwise) is a discrete vector lattice in the vocabulary with predicates $\{\leq, \mathrm{d}\}$.

(or, as shown in Exercise 17, because the axioms ultimately only depend on the relation "$x \eqcirc y$" —literally, $\mathrm{d}^{-1}(0)$— which effectively trivializes the topology on the set $\mathbb{R}$ of logical values). Therefore, they are meant only as motivation going forward when we shall introduce structures whose axioms capture the topology of $\mathbb{R}$ in meaningful ways when predicates are nondiscrete.

Chapter 3

Normed vector spaces

Metric spaces and real vector spaces are structures of fundamental interest in analysis. Section 2.2 above introduced the vocabulary $V^{\mathrm{d}}_{\mathrm{vec}} = (\mathtt{F}_{\mathrm{vec}}, \{\mathtt{d}\})$ for real vector spaces, regarded *de facto* as discrete structures. However, the discrete axioms for vector spaces above are unnatural for studying them from the perspective of analysis.

The problem is that the axioms are expressed in terms of the *discrete* relation $\asymp$ defined as $\mathrm{d}^{-1}(0)$. All properties $\mathrm{d}(x, y) = r$ for $r \neq 0$ effectively become $x \not\asymp y$. If d takes only the values 0 but no other values in a neighborhood of 0 (e.g., if d is discrete) then the discrete axioms are appropriate. However, if one wishes to impose a topology on X compatible with interpretations of d possibly taking values that accumulate at 0, then the discrete axioms are *dis*continuous with respect to values of d, violating our tenet.

The notion of pseudometric space stands in stark contrast to the definition of discrete vector spaces. The pseudometric axioms are all captured in terms of equalities (and inequalities) between *real* values of the distance predicate d; in fact, one of the most elementary properties of the metric topology on one such $\underline{X}$ is the (1-Lipschitz) continuity of the map $\mathrm{d} : X \times X \to \mathbb{R}$. Such continuity is not an axiom of pseudometric spaces, but follows immediately from the triangle inequality.

In many ways, the historical genesis of functional analysis is about finding ways (many different ones, because there is no unique or best approach) to study vector spaces in richer vocabularies that the above vocabulary $V^{\mathrm{d}}_{\mathrm{vec}}$ (which is only useful for the study of vector spaces from a purely discrete perspective sans analysis).

3.1 Some general topological considerations

Analysis and topology seem to find their closest points of contact in the setting of functional analysis. Adhering to our tenet to study structures exclusively through real-valued predicates, we will *not* consider "abstract" topological spaces at any point. After all, the notion of topological space is quite set-theoretical, while our focus is to study properties of sets and their elements *only* through real predicates. Of course, as soon as one has real-valued functions (i.e., predicates) on (Cartesian powers of) a set M, our philosophy (that properties are slightly perturbed when predicate values are slightly perturbed) implies a topology on M.

In Section ***, we shall introduce a canonical topology $\mathcal{T}_{\mathcal{M}}$ on each structure $\mathcal{M}$. Presently, we make only some remarks:

- each n-ary predicate $p^{\mathcal{M}} : M^n \to \mathbb{R}$ is $\mathcal{T}_{\mathcal{M}}$-continuous, as is each φ in a larger collection of real-valued k-ary functions on M (called real-valued “first-order predicates” or “formulas”);
- $\mathcal{T}_{\mathcal{M}}$ is the initial topology by the collection of all unary first-order predicates $\varphi : M \to \mathbb{R}$ (i.e., the coarsest of all topologies making every unary φ continuous).

Consequently, from the real-valued perspective, we shall deal primarily with topologies that are initial with respect to real-valued functions, sometimes, with non-initial topologies (such as metric topologies) that are still characterized in terms of reals, but not directly with topologies in the most general set-theoretic sense.

3.2 Normed vector and Banach spaces and lattices

Metric vector spaces

By way of motivation, we begin the discussion of normed vector spaces with exercises introducing an *ad hoc* notion of “metric vector space” (to be later superseded by the notion of “normed vector space”).

Exercise 20

Throughout this Exercise, a *metric vector space*[1] is a $V^{\mathrm{d}}_{\mathrm{vec}}$-structure $\mathcal{X} = \langle X, 0, +, r\cdot, \mathrm{d} : r \in \mathbb{R}\rangle$ satisfying the following properties for all $x, y, z \in X$ and $r, s \in \mathbb{R}$:

- $\langle X, \mathrm{d}\rangle$ is a pseudometric space, whose implied zero-distance relation will be denoted $\bumpeq$;
- “Vector Space” axioms:
  - $x + 0 \bumpeq x$;
  - $x + y \bumpeq y + x$;
  - $x + (y + z) \bumpeq (x + y) + z$;
  - $1 \cdot x \bumpeq x$;
  - $r \cdot (s \cdot x) \bumpeq (rs) \cdot x$;
  - $(r + s) \cdot x \bumpeq (r \cdot x) + (s \cdot x)$;
  - $r \cdot (x + y) \bumpeq (r \cdot x) + (r \cdot y)$.
- “Metric” axioms:
  - [Translation invariance] $\mathrm{d}(x + z, y + z) = \mathrm{d}(x, y)$;
  - [Homogeneity] $\mathrm{d}(rx, 0) = |r|\,\mathrm{d}(x, 0)$;
  - [Sub-additivity] $\mathrm{d}(x + y, 0) \le \mathrm{d}(x, 0) + \mathrm{d}(y, 0)$;

(The simplified notations $rx, rx + sy, -x, x - y$ for $r \cdot x, (rx) + (sy), (-1)x, x + (-y)$ will be used henceforth.)

Prove the following:

[1] Strictly speaking, d induces a metric only on $X/\bumpeq$: d is merely a pseudometric on X.

1. For each $x \in X$, the function $\mathbb{R} \to X : r \mapsto rx$ ("dilation of x") is continuous.

2. Let the product space $X^2 = X \times X$ be endowed with the real-valued predicate

$$\mathrm{d}^2 : ((x_1, y_1), (x_2, y_2)) \mapsto \max\{\mathrm{d}(x_1, x_2), \mathrm{d}(y_1, y_2)\}.$$

Show that d^2 is a pseudometric on X^2, and addition is a continuous function $X^2 \to X$.

In sum, the notion of metric vector space in Exercise 20 seems natural enough, and implies desired continuity properties of vector space operations. (Note that the notion is really a *pseudo*metric one, since the axioms rely only on the intrinsic pseudometric equivalence $\eqsim$, not on set-theoretic equality.)

Exercise 21

Let $\mathcal{X} = \langle X, 0, \dots, \mathrm{d}\rangle$ be a metric vector space as in Exercise 20 above. Define the *norm* $\|\cdot\| = \|\cdot\|$ obtained from the metric d on $\mathcal{X}$ as the real-valued predicate $x \mapsto \|x\| := \mathrm{d}(x, 0)$.

Part I

Show that d may be recovered from $\|\cdot\|$ using the identity $\mathrm{d}(x, y) = \|y - x\|$.

Part II

A (general) *norm on* $\mathcal{X}$ is any real-valued $N : X \to \mathbb{R}$ satisfying the following properties for all $x, y \in X$ and $r \in \mathbb{R}$:

1. [Positivity] $N(x) \geq 0$;

2. [Homogeneity] $N(rx) = |r|\, N(x)$;

3. [Sub-additivity] $N(x + y) \leq N(x) + N(y)$.

Show that the norm $\|x\| := \mathrm{d}(x, 0)$ of a metric vector space is a norm in this sense (i.e., $\|\cdot\|$ is positive, homogeneous and sub-additive).

Part III

Show that any norm $N : X \to [0, +\infty)$ on, say, a discrete vector space $\langle X, \dots, \delta\rangle$ yields a metric vector space $\langle X, \dots, \mathrm{d}_N\rangle$ when δ is replaced by the metric $\mathrm{d}_N(x, y) := N(y - x)$.

(In other words, a norm N on a "discrete" vector space naturally relaxes the discrete δ-topology to a metric topology d_N, typically non-discrete.)

Part IV

Prove that $\mathrm{d}_N = \mathrm{d}$ when $N = \|\cdot\|$ is obtained from a metric vector space $\langle X, \dots, \mathrm{d}\rangle$. Moreover, show that d_N is discrete if and only if N is identically zero. (Any nontrivial norm N gives a metric d_N easily seen —by homogeneity— to take *all* values in $[0, \infty)$, hence d_N is as non-discrete/"continuous" as possible.)

Exercise 21 shows that the notion of (real-valued) "norm" on a vector space is materially equivalent to the metric d (satisfying the Metric Vector Space axioms, of course). This motivates the next section.

Normed vector spaces

The *vocabulary for normed vector spaces* is $V_{\mathrm{nrm}} = (\mathtt{F}_{\mathrm{vec}}, \{\mathtt{N}\})$ where $\mathtt{F}_{\mathrm{vec}}$ is the set of symbols for operations of vector spaces, and $\mathtt{N}$ is a symbol for a unary real-valued predicate. (Exercise 21 illustrates the very common practice of using the nomenclature $\|\cdot\|$ for the norm–predicate $\mathtt{N}^{\mathcal{X}}$ of a V_{nrm}-structure $\mathcal{X}$.)

27 Definition (Normed Vector Spaces)**.** A *normed vector space* is a V_{nrm}-structure

$$\mathcal{X} = \langle X, 0, +, r\cdot, \|\cdot\| \,:\, r \in \mathbb{R}\rangle$$

satisfying the following requirements for all $x, y, z \in X$ and $r, s \in \mathbb{R}$:

- [Norm Axioms] The (real-valued unary predicate) $\|\cdot\|$ is a norm:
    1. $\|0\| = 0$;
    2. $\|x\| \geq 0$ (nonnegativity),
    3. $\|rx\| = |r|\,\|x\|$ (homogeneity),
    4. $\|x + y\| \leq \|x\| + \|y\|$ (sub-additivity —the “Triangle Inequality”).
- [Vector Axioms] (Recall the standard abbreviations $-u := (-1)u$ and $u - v := u + (-v)$.) The relation $x \eqcirc y$ defined by $\|y - x\| = 0$ satisfies:
    - $x + (y + z) \eqcirc (x + y) + z$;
    - $x + y \eqcirc y + x$;
    - $1 \cdot x \eqcirc x$;
    - $r \cdot (s \cdot x) \eqcirc (rs) \cdot x$.
    - $(r + s) \cdot x \eqcirc (r \cdot x) + (s \cdot x)$;

The *norm-induced metric* (or $\|\cdot\|$*-metric*) *on* $\mathcal{X}$ is the predicate $\mathrm{d} = \mathrm{d} = \mathrm{d}_{\|\cdot\|}$ defined by

$$\mathrm{d}(x, y) := \|y - x\| \quad (= \|y + ((-1) \cdot x)\|).$$

(Although, *sensu stricti*, not a distinguished predicate of $\mathcal{X}$ since it is not named in the vocabulary V_{nrm}, we will effectively treat d as distinguished when given the meaning above.)

Exercise 22

Let $\mathcal{X} = \langle X, \dots, \|\cdot\|\rangle$ be a normed vector space.

Part I

Show that $\mathrm{d} = \mathrm{d}_{\|\cdot\|}$ is a pseudometric on X.

Part II

Let $X^\sharp := X/\eqcirc$ be the quotient of X (seen as metrized by $\mathrm{d}_{\|\cdot\|}$) by the zero-distance relation $\eqcirc$. (Elements of $X^\sharp$ are d0-equivalence classes of the form $x^\sharp = \{z \in X \,:\, \|z - x\| = 0\}$ for $x \in X$.) Show that the operations (and the predicate $\|\cdot\|$) of $\mathcal{X}$ induce corresponding (well-defined) operations (as well as a predicate $\|\cdot\|^\sharp$) on $X^\sharp$ in the natural way:

- the zero of $X^\sharp$ is $0^\sharp$;
- addition on $X^\sharp$ is $+^\sharp : (x^\sharp, y^\sharp) \mapsto (x+y)^\sharp$;
- multiplication by a scalar $r \in \mathbb{R}$ on $X^\sharp$ is $r\cdot^\sharp : x^\sharp \mapsto (rx)^\sharp$;
- the norm on $X^\sharp$ is $\|\cdot\|^\sharp : x^\sharp \mapsto \|x\|$.

Moreover, $\mathcal{X}^\sharp := \langle X^\sharp, O^\sharp, r\cdot^\sharp, \|\cdot\|^\sharp : r \in \mathbb{R}\rangle$ is a normed vector space that is *reduced* in the metric sense: $x^\sharp = y^\sharp$ only if $x^\sharp \doteqdot y^\sharp$. (In particular, $0^\sharp$ is the unique zero-norm element of $X^\sharp$.)

Caveat: *The standard nomenclature for the notion of normed vector space introduced above is "*semi-*normed vector space". The term "normed vector space" is commonly reserved for normed vector spaces that are* reduced *in the sense of Exercise 22 II. Since the zero-distance relation $x \doteqdot y$ is structurally meaningful, while literal set-theoretic equality $x = y$ of elements of X is not, we prefer "normed vector spaces" as the default (more general) nomenclature, with the adjective "reduced" added only when necessary. We will use the adjective* faithful *for the norm $\|\cdot\|$ on a reduced space (when the zero-distance relation is set-theoretic equality —this is standard nomenclature).*

Exercise 23

Let $\mathcal{X} = \langle X, \dots, \|\cdot\|\rangle$ be a normed vector space (say, reduced, for simplicity). Replace the norm $\|\cdot\|$ with the *discrete* metric

$$\delta(x,y) := \begin{cases} 0, & \text{if } \|x-y\| = 0; \\ 1, & \text{if } \|x-y\| > 0. \end{cases}$$

Prove that $\mathcal{X}_\delta := \langle X, \dots, \delta\rangle$ is a *discrete* vector space.

28 Convention. We will tacitly assume that normed vector spaces are *reduced* in the sense that $\|x\| = 0$ implies $x = 0^{\mathcal{X}}$ (set-theoretically). (Without this convention, one must explicitly assume that operations are, at the very least, well defined modulo $\doteqdot_X$.)

Normed vector lattices

The *vocabulary for normed vector lattices* is $V_{\|\mathrm{Riesz}\|} = \langle \mathtt{F}_{\mathrm{Riesz}}, \{\mathtt{N}\}\rangle$, where $\mathtt{N}$ is a unary predicate symbol (i.e., the binary metric symbol $\mathtt{d}$ in the signature for vector lattices is replaced by a "norm" unary symbol $\mathfrak{N}$).

A *normed vector lattice* is a $V_{\|\mathrm{Riesz}\|}$-structure

$$\mathcal{X} = \langle X, 0, +, r\cdot, \wedge, \vee, \|\cdot\| : r \in \mathbb{R}\rangle$$

such that:

- the V_{nrm}-reduct $\langle X, 0, +, r\cdot, \|\cdot\| : r \in \mathbb{R}\rangle$ is a normed vector space (henceforth regarded as a pseudometric space by the $\|\cdot\|$-metric $\mathrm{d}_{\|\cdot\|}$ and with zero-distance relation $\doteqdot$);
- $\vee$ and $\wedge$ are 1-Lipschitz $\|\cdot\|$-continuous functions;

- the reduct $\langle X, \vee, \wedge, \|\cdot\|\rangle$ is a *distributive* lattice: the idempotence, absorption, commutativity, associativity and distributivity laws (modulo $\doteq$) are satisfied.
- the lattice operations are compatible with the vector structure, i.e., homogeneous and translation invariant (modulo $\doteq$).

For $x \in X$, define its *positive part* $x^+ := x \vee 0$, its *negative part* $x^- = (-x)^+ = (-x) \vee 0 = -(x \wedge 0)$, and its *"pointwise" magnitude* $|x| := x \vee (-x)$.

Exercise 24

Let $\mathcal{X}$ be a Banach lattice.

Part I

Show that the *(discrete) ordering relation "$x \leq y$" on X* defined by "$x \wedge y \doteq x$" is a partial ordering for which the join and meet operations are lattice operations in the usual sense (i.e., $z \geq x$ and $z \geq y$ iff $z \geq x \vee y$, and similarly for $\leq$ and $\wedge$).

Part II

The *positive cone* of a Banach lattice X is the set $X^+ = \{x \in X : x \geq 0\}$. Show that X^+ is closed under addition, multiplication by positive scalars, and under finite convex combinations (i.e., if $x_i \in X^+$ and $r_i \in [0, 1]$ such that $\sum_i r_i = 1$, then $\sum_{i<n} r_i x_i \in X^+$.)

* Exercise 25

Let $\mathcal{X} = \langle X, 0, +, r\cdot, \wedge, \vee, \|\cdot\| : r \in \mathbb{R}\rangle$ be a normed vector lattice. Prove the identities (for all $x, y, z \in X$):

1. $x \doteq x^+ - x^-$;
2. $|x| \doteq x^+ + x^-$;
3. $|x - y| \doteq |x^+ - y^+| + |x^- - y^-|$;
4. $|x - y| \doteq |(x \vee z) - (y \vee z)| + |(x \wedge z) - (y \wedge z)|$.

29 Remarks. Although the notion of normed vector lattice is quite standard and of great importance in functional analysis, there are several different definitions. The most common (classical) definition (e.g., as per Lindenstrauss-Zafriri's *Classical Banach Spaces*) includes the *discrete* relation "$x \leq y$" as an ingredient of the definition. Instead of imposing a Lipschitz condition on the lattice operations, the classical definition imposes a compatibility requirement between the discrete relation $\leq$ and the lattice operations which, *a posteriori,* implies that the lattice operations are Lipschitz (using identity (4) in Exercise 25). (Worse even, some authors only require a Lipschitz condition without specifying a constant. *A posteriori*, the latter is typically not a significant issue because a lattice admitting any Lipschitz constant is continuously homeomorphic to one with Lipschitz constant 1.) From a real-valued structural perspective, it is more natural (and ultimately equivalent) to require 1-Lipschitz continuity axiomatically and dispense with the ordering relation $\leq$ altogether, at least as part of the vocabulary.

Banach spaces

30 Definition. A normed vector space $\mathcal{X} = \langle X, 0^{\mathcal{X}}, +, r\cdot, \|\cdot\| : r \in \mathbb{R}\rangle$ is called a *Banach space* if $\langle X, \mathrm{d}\rangle$ is a complete metric space (where d is the $\|\cdot\|$-pseudometric). Explicitly,

- [faithfulness] $\|x\| = 0$ only if $x = 0^{\mathcal{X}}$, and
- [completeness] every sequence $(x_n) \subseteq X$ that is Cauchy (in the $\|\cdot\|$-metric d) converges.

With "$x \mathrel{\underline{\underline{\smash{\times}}}} y$" the zero-distance relation $\|y - x\| = 0$, we say that $\mathcal{X}$ is *Banach modulo zero-distance* (or (mod $\underline{\underline{\smash{\times}}}$)*)* if $X/\underline{\underline{\smash{\times}}}$ is Banach.

31 Remarks. Many natural normed vector spaces $\mathcal{X}$ are Banach spaces, or at least Banach modulo zero-distance. For that reason, Banach spaces play a central role in functional analysis.

From our perspective, normed vector space are more fundamental structures. We often approach completeness from the perspective that proximal metric types are realized. While proximal types are important, properties captured by distal types are also very important. In our view, *a priori* classical notions of completeness in various senses (including in the metric sense) are best understood as approximations to the more general phenomenon of *saturation* first introduced in nonstandard analysis, which in our structural approach to (standard) analysis is captured via types.

Banach lattices

A vector lattice whose V_{nrm}-reduct is a Banach space is called a *Banach lattice*.

By continuity of the lattice operations, any vector lattice may be embedded into a Banach lattice (in the sense of "isomorphic embedding", i.e., in particular, the norm is preserved in the embedding).

3.3 Finite-dimensional ℓ^p-spaces

("Normed space" means "normed vector space" henceforth.)

For any natural $n \in \mathbb{N}$, the set

$$\mathbb{R}^n := \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}\ (i = 1, 2, \dots, n)\}$$

of n-tuples of real numbers has zero element $0 = (0, 0, \dots, 0)$ as well as the standard "coordinate-wise" operations of addition and scalar multiplication. (Note: $\mathbb{R}^0$ has as sole element "zero" the empty tuple ().) Although $\mathbb{R}^n$ may be regarded as a discrete vector space $\langle \mathbb{R}^n, 0, +, r\cdot, \delta\rangle$ under such operations (which is the manner in which elementary linear algebra books define the "vector spaces" $\mathbb{R}^n$), such viewpoint is divorced from the topology of $\mathbb{R}$ and therefore unnatural.

The functional analytic viewpoint replaces the discrete metric δ on $\mathbb{R}^n$ with one of various classical norms which we introduce in a series of exercises.

Exercise 26 $\quad \ell^p_{(n)}$

Given $n \in \mathbb{N}$, prove that each of the V_{nrm}-structures below is a Banach space with underlying pointset $\mathbb{R}^n$, and that the respective norm-topologies on $\mathbb{R}^n$ are all the same (usual) topology given by the Euclidean norm $\|\cdot\|_2$. (Theorem 42 below explains such topological coincidence in general.)

Part I — $\ell^1_{(n)}$

$$\ell^1_{(n)} := \langle \mathbb{R}^n, \dots, \|\cdot\|_1 \rangle,$$

where the "ℓ^1-norm" of a vector $x \in \mathbb{R}^n$ is defined by

$$\|x\|_1 := \sum_{i=1}^{n} |x_i| .$$

* *Part II* — $\ell^p_{(n)}$

Fix any real $p \in [1, +\infty)$. ($p = 1$ is the simplest case, covered in Part I above.) Let

$$\ell^p_{(n)} := \langle \mathbb{R}^n, \dots, \|\cdot\|_p \rangle,$$

where the "ℓ^p-norm" is defined by

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} .$$

Note that $\|\cdot\|_2$ is the Euclidean distance. [*Hint:* The proof of sub-additivity requires Minkowski's inequality —which reduces to Cauchy-Schwarz when $p = 2$.]

Part III — $\ell^\infty_{(n)}$

$$\ell^\infty_{(n)} := \langle \mathbb{R}^n, \dots, \|\cdot\|_\infty \rangle$$

where the "ℓ^∞-norm" (also called "supremum norm" because of its definition below) is defined by

$$\|x\|_\infty := \sup_{1 \le i \le n} |x_i| .$$

(Of course, the supremum of the magnitudes of the n coordinates is simply the maximum thereof.)

Part IV

Show that, for fixed $x \in \mathbb{R}^n$, the function

$$\|x\|_\cdot : p \mapsto \|x\|_p$$

is continuous on the interval $[1, +\infty]$.

Exercise 27

Show that each of the structures $\ell^p_{(n)}$ admits an expansion to a Banach lattice when endowed with the operations of pointwise maximum and minimum as join and meet. What is the Lipschitz constant for the join and meet operations?

3.4 ℓ^p spaces

Finite-dimensional "ell-p" spaces $\ell^p_{(n)}$ were introduced in Exercise 26. As explained in Section 3.3, normed spaces of the same finite dimension n, although perhaps cosmetically different-looking, are all isomorphic —if not *isometric*— to (say) $\ell^\infty_{(n)}$ as proved in Theorem 42.

The situation for infinite dimensional normed spaces is drastically different. We shall now introduce the infinite-dimensional spaces ℓ^p (little-ell p).

Below, κ shall denote an arbitrary ordinal (typically, but not necessarily, a cardinal). The ordinal $\kappa = \omega$ (the cardinality of the set $\mathbb{N}$ of naturals) will be the case of primary interest, but we choose to give the general definition in order to have examples of (intuitively speaking) spaces of arbitrarily large, plus-quam-countable dimension.

32 Convention. We adopt the von Neumann convention that any ordinal κ *is* the set $\{i : i < \kappa\}$ of its predecessors. When κ is used as index set (as we will) it is always in the von Neumann sense. In particular, $\omega = \{0, 1, 2, \dots\}$ is the (same as the) set $\mathbb{N}$ of naturals from this viewpoint. On the other hand, a natural n, regarded as a cardinal and used as an index set, becomes $\{0, 1, \dots, n-1\}$.

Caveat: For finite ordinals $\kappa = n$, this has the quirky effect that n-tuples become of the form $(x_0, x_1, \dots, x_{n-1})$ indexed by $i < n$, rather than by $1 \le i \le n$.

At any rate, $\mathbb{R}^\kappa$ is the set of "κ-long" sequences $\bar{x} = (x_i : i < \kappa) \subseteq \mathbb{R}$ (i.e., functions $\bar{x} : \kappa \to \mathbb{R}$). $\mathbb{R}^\omega$ is the usual set of countable, i.e., ω-sequences $(x_0, x_1, \dots, x_n, \dots)$.

33 Definition. Let κ be any ordinal and $p \in [1, \infty)$. The *ℓ^p-gauge* of $x = (x_i) \in \mathbb{R}^\kappa$ is defined by

$$[\![x]\!]_p = \sum_{i<\kappa} |x_i|^p \in [0, \infty]$$

(i.e., as the supremum of sums $\left|x_{i_1}\right|^p + \cdots + \left|x_{i_n}\right|^p$ of entries given by finitely many indexes $i_1 < i_2 < \cdots < i_n$).

The normed space $\underline{\ell}^p_{(\kappa)} = \langle \ell^p_{(\kappa)}, 0, +, r\cdot, \|\cdot\|_p : r \in \mathbb{R}\rangle$ has

- underlying set
$$\ell^p_{(\kappa)} := \{x \in \mathbb{R}^\kappa : [\![x]\!]_p < \infty\},$$
- zero element $0 = (0)_{i<\kappa}$,
- pointwise addition $x + y := (x_i + y_i)_{i<\kappa}$,
- pointwise scalar multiplication $rx := (rx_i)_{i<\kappa}$, and
- norm
$$\|x\|_p = [\![x]\!]_p^{1/p} = \left(\sum_{i<\kappa} |x_i|^p\right)^{1/p},$$

The ℓ^∞-norm on $\mathbb{R}^\kappa$ is
$$\|x\|_\infty = \sup_{i<\kappa} |x_i| \in [0, \infty].$$

The normed space $\underline{\ell}^\infty_{(\kappa)} = \langle \ell^\infty_{(\kappa)}, 0, +, r\cdot, \|\cdot\|_\infty : r \in \mathbb{R}\rangle$ has underlying set

$$\ell^\infty_{(\kappa)} := \{x \in \mathbb{R}^\kappa : \|x\|_\infty < \infty\},$$

zero $0 = (0)_{i<\kappa}$ and the (pointwise) operations formally defined in identical manner to the spaces $\ell^p_{(\kappa)}$ for $p < \infty$ above.

* Exercise 28 $\ell^p_{(\kappa)}$ spaces

Part I

Prove that $\ell^p_{(\kappa)}$ is a Banach space for any ordinal κ and all $p \in [1, \infty]$.

[*Hint:* Minkowski's Inequality ensures that $\|\cdot\|_p$ is sub-additive. Individual ("i-th") entries of a Cauchy sequence $(x^{(n)})_{n<\omega}$ in $\ell^p_{(\kappa)}$ form a Cauchy sequence $(x_i^{(n)})_{n<\omega}$ in $\mathbb{R}$, which thus has a "pointwise" limit $y = (\lim_n x_i^{(n)})_{i<\kappa} \in \mathbb{R}^\kappa$. The final (simple) task is to show that $(x^{(n)})$ converges to y in norm.]

Part II

Show that, up to isomorphism (i.e., isometry), $\ell^p_{(\kappa)}$ depends only on the cardinality $|\kappa|$ of κ.

[*Hint:* If $|\lambda| = |\kappa|$, then any bijection $f : \lambda \to \kappa$ induces a norm-preserving linear map $\tilde{f} : \ell^p_{(\kappa)} \to \ell^p_{(\lambda)} : (x_i) \mapsto (x_{f(i)})$.]

34 Convention. When the ordinal κ is omitted, it is understood to be ω. Henceforth, $\underline{\ell}^p$ *shall denote the Banach space* $\underline{\ell}^p_{(\omega)}$ *with underlying set* $\ell^p = \ell^p_{(\omega)} \subseteq \mathbb{R}^\omega$.

From a structural perspective, $\ell^p_{(\kappa)}$ is characterized up to isomorphism (isometry) by the cardinality $|\kappa|$. (The ordering of κ was not used in defining the space $\ell^p_{(\kappa)}$ at any rate.) Going forward, we relax the requirement that κ be an ordinal and, for any set D, we introduce the structures $\ell^p(D) \subseteq \mathbb{R}^D$ in the obvious (same) manner.

Exercise 29 c_{00} and c_0

This exercise introduces the subspace $c_{00} \subseteq \ell^p$ for $p \in [1, \infty]$.

c_{00} is the subset of ℓ^p consisting of "finitely supported" sequences $x = (x_0, x_1, \dots)$, i.e., such that $x_i \neq 0$ for (only) finitely many $i < \omega$. Evidently, c_{00} is a subset of ℓ^p closed under vector operations, so it is a subspace of ℓ^p as well as the underlying pointset of a normed space $\underline{c}^p_{00} = \langle c_{00}, \dots, \|\cdot\|_p\rangle$. (From a structural viewpoint, there is one such normed space $\underline{c}^p_{00}$ for each p.)

Part I

For $1 \le p < \infty$, show that c_{00} is dense in ℓ^p (i.e., $\ell^p = \overline{c_{00}}$ is the $\|\cdot\|_p$-norm closure of c_{00}.)

Part II

Let $c_0 = \overline{c_{00}}$ be $\|\cdot\|_\infty$-closure of c_{00} in ℓ^∞. Thus, c_0 is the underlying set of a *bona fide* Banach space $\underline{c}_0 = \langle c_0, \dots, \|\cdot\|_\infty\rangle$.

Show that c_0 is the set of ("zero-limit") sequences $x \in \mathbb{R}^\omega$ such that $\lim_{i\to\infty} x_i = 0$. Deduce that $c_{00} \subsetneq c_0 \subsetneq \ell^\infty$.

Exercise 30

Show that each space $\underline{\ell^p}_{(\kappa)}$ admits an expansion to a Banach lattice when endowed with the operations $\vee, \wedge$ of pointwise maximum and minimum, respectively. What are the Lipschitz constants for $\vee, \wedge$

* Exercise 31

Part I

For $p \in (0,1)$, one may define a V_{nrm}-structure $\underline{\ell^p_{(\kappa)}}$ using Definition 33. Show that the real-valued predicate $\|\cdot\|_p$ is, however, *not* a norm, so the structure is not a normed space.

Part II

Show that, for $p \in (0,1)$,

$$\mathrm{d}_p(x,y) := [\![y - x]\!]_p := \sum_{i<k} |y_i - x_i|^p$$

is a metric on the set $\ell^p_{(\kappa)} = \{x \in \mathbb{R}^\kappa : [\![x]\!]_p < \infty\}$. In this manner, one obtains a $V^{\mathrm{d}}_{\mathrm{vec}}$-structure $\underline{\ell^p_{(\kappa)}} = \langle \ell^p_{(\kappa)}, \dots, \mathrm{d}_p \rangle$. Show that the operations of addition and scalar multiplication are d_p-continuous. ($\ell^p_{(\kappa)}$ is "almost" a metric vector space in the sense of Section 3.2 —only homogeneity fails.)

3.5 Operators, functionals and Banach duals

35 Definitions. Let $\mathcal{X} = \langle X, 0^{\mathcal{X}}, +^{\mathcal{X}}, r\cdot^{\mathcal{X}}, \|\cdot\|_{\mathcal{X}} : r \in \mathbb{R} \rangle$ and $\mathcal{Y} = \langle Y, 0^{\mathcal{Y}}, +^{\mathcal{Y}}, r\cdot^{\mathcal{Y}}, \|\cdot\|_{\mathcal{Y}} : r \in \mathbb{R} \rangle$ be normed spaces. A map $T : X \to Y$ is called *(discretely) linear* if it is well defined modulo the zero-distance relation $\asymp$ for $\|\cdot\|_{\mathcal{Y}}$ and respects the linear structure proper. Explicitly (in addition to being well defined modulo $\asymp$), for $x, x_1, x_2 \in X$ and $r \in \mathbb{R}$:

0. [Zero-preserving] $T(0^{\mathcal{X}}) \asymp 0^{\mathcal{Y}}$;[2]

1. [Additivity] $T(x_1 +^{\mathcal{X}} x_2) \asymp T(x_1) +^{\mathcal{Y}} T(x_2)$;

2. [Homogeneity] $T(r \cdot^{\mathcal{X}} x) \asymp r \cdot^{\mathcal{Y}} T(x)$.

A *bounded linear transformation* $T : X \to Y$ is a linear map such that, for some real $C \geq 0$:

$$\|T(x)\|_{\mathcal{Y}} \leq C\,\|x\|_{\mathcal{X}} \quad \text{for all } x \in X.$$

(By homogeneity, the condition above is equivalent to $\|T(x)\|_{\mathcal{Y}} \leq C$ whenever $|x| \leq 1$.) The word *operator* will be used as a synonym of *bounded linear transformation*.

The *operator norm* of a linear map $T : X \to Y$ is

$$\|T\| := \sup_{\|x\|_{\mathcal{X}} \leq 1} \|T(x)\|_{\mathcal{Y}}.$$

[2] It is very easy to show that an additive and homogeneous transformation must preserve zero; however, it seems appropriate to explicitly require that a linear map preserve zero so that it is (by definition) compatible with all operations of the vocabulary V_{vec}.

By homogeneity, one sees that T is bounded (i.e., is an operator) iff T has finite operator norm $\|T\| < \infty$.

Operators $T_1, T_2 : X \to Y$ are called *(modulo-zero) equivalent* (denoted: $T_1 \eqcirc T_2$) if $T_1(x) \eqcirc T_2(x)$ for all $x \in X$.

Operators $T : X \to Y$, $U : Y \to X$ are called (modulo-zero) *inverses* if $U \circ T \eqcirc \mathrm{id}_X$ and $T \circ U \eqcirc \mathrm{id}_Y$; each of T, U is said to be a (modulo-zero) *inverse* to the other. An operator $T : X \to Y$ is called an *isomorphism* if it has a (modulo-zero) inverse. Such space are called *isomorphic*, an (equivalence) relation denoted $X \cong Y$.

A *discrete- (resp., continuous-)linear functional on* $\mathcal{X}$ is a discrete (resp., continuous) linear map $\varphi : X \to \mathbb{R}$. Here, the codomain $\mathbb{R}$ is seen as the underlying set of a normed vector space $\underline{\mathbb{R}} = \langle \mathbb{R}, 0, +, r\cdot, |\cdot| : r \in \mathbb{R} \rangle$ whose norm is the absolute value predicate $r \mapsto |r|$.

Given any normed vector space $\mathcal{X} = \langle X, \dots, \|\cdot\| \rangle$ let X^* be the set of continuous linear functionals on $\mathcal{X}$. The *dual space* of $\mathcal{X}$ is the normed vector space

$$\mathcal{X}^* = \langle X^*, 0, +, r\cdot, \|\cdot\|^* \rangle$$

where $\|f\|^* := \sup_{\|x\| \leq 1} |f(x)|$ is the norm of f regarded as a linear operator $f : X \to \mathbb{R}$.

Exercise 32 Dual of a vector lattice

Let $\mathcal{X} = \langle X, \dots, \vee, \wedge, \|\cdot\| \rangle$ be a normed vector lattice whose normed vector space reduct has dual $\mathcal{X}^* = \langle X^*, \dots, \|\cdot\|^* \rangle$ (as a normed vector space). Define the join and meet operations on X^* "pointwise", first on arguments $x \geq 0$, by

$$\begin{aligned}(f \vee g)(x) &:= \sup_{0 \leq y \leq x} (f(y) + g(x-y)) \\ (f \wedge g)(x) &:= \inf_{0 \leq y \leq x} (f(y) + g(x-y)),\end{aligned}$$

and extended to all elements $x = x^+ - x^- \in X$ by $(f \vee g)(x) = (f \vee g)(x^+) - (f \vee g)(x^-)$ and $(f \wedge g)(x) = (f \wedge g)(x^+) - (f \wedge g)(x^-)$. When so expanded, show that $\mathcal{X}^* = \langle X^*, \cdots \vee, \wedge, \|\cdot\|^* \rangle$ is a Banach lattice such that $f^+(x^+) \geq 0$ for all $f \in X^*$ and $x \in X$ (i.e., the "evaluation pairing" respects the lattice structures).

Caveat. An isomorphism T between normed spaces $\mathcal{X} = \langle X, \dots, \|\cdot\|_X \rangle$ and $\mathcal{Y} = \langle Y, \dots, \|\cdot\|_Y \rangle$ in the sense of operator theory (as defined above) —having (say) modulo-zero inverse $S : Y \to X$— need *not* preserve the norms, i.e., only inequalities of the form $\|T(x)\|_Y \leq C\,\|x\|_X$ and $\|S(y)\|_X \leq C\,\|y\|_Y$ are required to hold.

If T preserves norm in the sense that $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$, then T is called an *isometric embedding of* $\mathcal{X}$ *into* $\mathcal{Y}$. If S is a modulo-zero inverse to T which is also an isometric embedding (of $\mathcal{Y}$ into $\mathcal{X}$), then T is called an *isometry between* $\mathcal{X}$ *and* $\mathcal{Y}$. (By the Bounded Inverse Theorem 45, a surjective isometric embedding of a Banach space is an isometry.)

36 Remarks. • We have chosen to write conditions 0.–2. in the precise form above to emphasize that linear maps intertwine the operations of $\mathcal{X}$ and $\mathcal{Y}$ (i.e., are homomorphisms). Going forward, the conditions will be written in the simpler-looking forms: $T(0) \eqcirc 0$, $T(x_1 + x_2) \eqcirc T(x_1) + T(x_2)$ and $T(rx) \eqcirc rT(x)$.

- In Banach spaces (or any reduced spaces for that matter), "modulo zero" becomes "equal" in the set-theoretic sense.
- Since a Lipschitz map is obviously well defined modulo zero-distance, Theorem 37 below implies that operators are necessarily well defined modulo $\doteq$.
- Discrete linearity is a very fragile notion, unnatural from a structural perspective; it does not play well with the norm-topology because it does not impose any continuity requirements.
- The canonical dual-pair expansion is generally *not* a commutative procedure. If one starts with the dual normed vector space $\mathcal{X}^* = \langle X^*, \dots, \|\cdot\|^* \rangle$, then its dual $\langle X^{**}), \dots, \|\cdot\|^{**} \rangle$ need *not* be isomorphic to $\mathcal{X}$. There is an isometric canonical embedding of $\mathcal{X}$ as a subspace of $\mathcal{X}^{**}$, but generally the embedding is not surjective. The embedding is $X \mapsto X^{**} : x \mapsto \mathrm{ev}_x$, where $\mathrm{ev}_x : f \mapsto f(x)$ is the "evaluation at x" map (a Lipschitz linear functional on X^*). Spaces for which $\mathcal{X}^{**}$ is isomorphic to $\mathcal{X}$ called *reflexive*. We have seen that ℓ^p spaces are reflexive for $p \in (1, \infty)$. However, ℓ^1 is not reflexive. Exercise 37 shows that the canonical embedding $\ell^1 \subseteq (\ell^\infty)^*$ is not surjective, but this is not enough! However, one can show that the map $\mathcal{U} \mapsto \mathcal{U}^*$ is injective —in fact, $\|\mathcal{U}^* - \mathcal{V}^*\| = 2$ for all ultrafilters $\mathcal{U} \neq \mathcal{V} \in \beta\mathbb{N}$. Since ℓ^1 has a countable dense subset (which, e.g., may be taken as a subset of c_{00}), the collection $\{\mathcal{U}^* : \mathcal{U} \in \beta\mathbb{N}\}$ cannot isometrically embed in ℓ^1.

37 Theorem. *For a linear map $T : X \to Y$ between normed spaces, the following properties are equivalent:*

1. *T is continuous;*
2. *T is continuous at $0^{\mathcal{X}}$;*
3. *T is bounded;*
4. *T is Lipschitz (in the norm-metrics $\mathrm{d}_{\mathcal{X}}$, $\mathrm{d}_{\mathcal{Y}}$).*

If (1)–(4) hold, the operator norm $\|T\|$ is the infimum of Lipschitz constants for T.

(A map $f : \langle X, \mathrm{d}_X \rangle \to \langle Y, \mathrm{d}_Y \rangle$ between pseudometric spaces is *Lipschitz* if, for some $C \geq 0$, the inequality $\mathrm{d}_Y(f(x_1), f(x_2)) \leq C\, \mathrm{d}_X(x_1, x_2)$ holds for all $x_1, x_2 \in X$. Such C is called a *Lipschitz constant* for f.)

Exercise 33

Prove Theorem 37.

38 Corollary. *Let $\mathcal{X} = \langle X, \dots, \|\cdot\| \rangle$ be a normed vector space, and let $N : X \to [0, \infty)$ be any (other) norm on $\mathcal{X}$. The following properties are equivalent:*

1. *N is continuous in the $\|\cdot\|$-metric of $\mathcal{X}$.*
2. *N is $\|\cdot\|$-bounded: there exists $C \geq 0$ such that $N(x) \leq C\, \|x\|$ for all $x \in X$.*

Moreover, such C are precisely Lipschitz constants for N with respect to the $\|\cdot\|$-metric on X.

Proof. The asserted equivalence is that of (2) and (3) in Theorem 37 for the identity map $T = \mathrm{id}$ from $\langle X, \dots, \|\cdot\|\rangle$ to $\langle X, \dots, N\rangle$. $\square$

39 Definitions. Let $\mathcal{X} = \langle X, \dots, \|\cdot\|\rangle$ be a normed space, and $S \subseteq X$ be any subset. The implied zero-distance relation "$\|y - x\| = 0$" is denoted "$x \doteq y$". (This is literal equality "$x = y$" when $\mathcal{X}$ is reduced; in particular, in any Banach space).

A *linear combination* of a finite tuple $\bar{x} = (x_1, \dots, x_n) \subseteq X$ is any element $x := \sum_{i=1}^{n} r_i x_i$ for some real scalars $r_1, \dots, r_n$. (Any element at zero distance from such linear combination is structurally *de facto* a linear combination.) A *finite linear combination* of elements of S is any linear combination of a finite tuple $(x_1, \dots, x_n) \subseteq S$. The *span (modulo $\doteq$)* of an n-tuple $\bar{x}$ or a set S is the set all elements of X at zero distance from some (finite) linear combination of $\bar{x}$ or S, respectively.[3] (When S is infinite, one may consider "infinite" linear combinations which will be discussed later.)

A finite tuple $(x_1, \dots, x_n) \subseteq X$ is *linearly independent (modulo $\doteq$)* if, for all scalars $r_1, \dots, r_n$, the "null-combination" condition $\sum_{i=1}^{n} r_i x_i \doteq 0$ implies $r_1 = \cdots = r_n = 0$. S is called a set of *finitely (linearly) independent* elements if every finite tuple $(x_i) \subseteq S$ is linearly independent.[4]

Henceforth, "span" and "linear independence" are meant in the "modulo zero-distance" senses above, unless otherwise specified.

$\mathcal{X}$ is *finite-dimensional* if there is a finite tuple $\bar{x} = (x_1, \dots, x_n) \subseteq X$ whose linear combinations span X modulo zero-distance. The least such n (if any) is called the *dimension of $\mathcal{X}$*, and any corresponding spanning tuple $\bar{x}$ is a *basis* (or *base*) of $\mathcal{X}$ (abusing nomenclature, "of X").

A non-finite dimensional space is called *infinite-dimensional*; such spaces are spanned (modulo $\doteq$) by *no* finite tuple of elements.

Caveat. The definition of "basis" above is for finite-dimensional spaces. The notion of basis for infinite-dimensional spaces is quite delicate. We warn the reader that the notions of basis allowing only *finite* linear combinations (a so-called *Hamel basis*, of limited use in analysis) and of basis allowing *infinite* linear combinations (subject to some convergence condition) are quite different.

Functional analysis focuses on *infinite*-dimensional vector spaces carrying "good" topologies, of which normed vector spaces are the foremost example. With the above in mind, finite-dimensional results are relatively trivial and of little intrinsic interest in functional analysis.

Some facts about finite-dimensional spaces are summarized in the following:

40 Theorem. *Let $\mathcal{X} = \langle X, \dots, \|\cdot\|\rangle$ be a normed vector space of finite dimension n. (By Definition 39 above, n is the least length of a tuple of elements spanning X.)*

1. *Every independent m-tuple extends to a basis, and therefore has length $m \leq n$.*

2. *Every spanning m-tuple has a sub-tuple that is a basis, and therefore has length $m \geq n$.*

3. *Bases of $\mathcal{X}$ are precisely of the following (equivalent) forms:*

    - *Minimal tuples (x_i) spanning X.*

[3] The empty tuple $() \subseteq X$ is allowed; it has length $n = 0$ and its span consists of only the —"empty"— combination $0 \in X$.

[4] In particular, the empty tuple $() \subseteq X$ is ("vacuously") linearly independent, as is the empty set $S = \{\}$.

- *Maximal linearly independent tuples* (x_i).

In particular, a length-n tuple spans X iff it is independent.

For the exercises below, we introduce some definitions. A function $f : X \to Y$ between metric spaces $\mathcal{X} = \langle X, \mathrm{d}_X \rangle$ and $\mathcal{Y} = \langle Y, \mathrm{d}_Y \rangle$ is called

- *well-defined modulo zero* (or "modulo zero(-distance)") if $x_1 \eqcirc_X x_2$ implies $f(x_1) \eqcirc_Y f(x_2)$;
- *injective modulo 0* if $\mathrm{d}_Y(y_1, y_2) = 0$ implies $\mathrm{d}(x_1, x_2) = 0$;
- *surjective modulo 0* if the image $f(X)$ is dense in Y, i.e., if for all $\varepsilon > 0$ and $y \in Y$ there is $x \in X$ such that $\mathrm{d}_Y(y, f(x)) < \varepsilon$;
- *bijective modulo 0* if it is both injective and surjective modulo zero;

When X, Y are normed spaces, the notions above refers to the norm-metrics on X, Y.

41 Remark. There are various ways —none canonical— of mirroring the set-theoretic notion of "injective function" in real-valued structures. Injectivity modulo zero-distance (let alone set-theoretic injectivity) is an *ad hoc* (and fragile) property, because a positive value $\mathrm{d}_Y(y_1, y_2) > 0$ implies nothing about $\mathrm{d}(x_1, x_2)$, i.e., there is no parameter ε built into the definition to "soften" the notion as defined. Set-theoretic surjectivity is also very delicate and unnatural from a real-valued perspective. By contrast, the notion of surjectivity modulo zero-distance is completely natural.

Exercise 34

Let $\mathcal{X} = \langle X, \dots, \|\cdot\|_{\mathcal{X}} \rangle$ be a normed vector space.

Part I

Show that every finite tuple $\bar{x} = (x_1, \dots, x_n) \subseteq X$ induces a linear map

$$\begin{aligned} T_{\bar{x}} &: \ell^{\infty}_{(n)} \to \mathcal{X} \\ \bar{r} = (r_i)_{i=1}^n &\mapsto \sum_{i=1}^n r_i x_i \end{aligned}$$

with operator norm $\|T_{\bar{x}}\| \leq \sum_{i=1}^n \|x_i\|$.

Part

If $T_{\bar{x}}$ above is regarded as a map $\ell^1_{(n)} \to \mathcal{X}$ instead, show that it has operator norm $\|T_{\bar{x}}\| \leq \sup_{1 \leq i \leq n} \|x_i\|$.

Part II

Show that $T_{\bar{x}}$ is (*i*) injective (mod 0) iff $\bar{x}$ is linearly independent, and (*ii*) surjective (mod 0) iff $\bar{x}$ spans X.

42 Theorem. *Any two normed vector spaces of the same finite dimension are isomorphic.*

(Theorem 42 is a special case of a pivotal result in Banach space theory: the Open Mapping Theorem *** below.)

Up to isomorphism, the only zero-dimensional normed vector space is a one-point space $X = \{0\}$ with all trivial operations (and norm). We consider only spaces of finite dimension $n > 0$ for the remainder of the proof.

43 Lemma. *If $n > 0$ and $N : \mathbb{R}^n \to [0, +\infty)$ is any norm:*

1. *N is continuous in the $\|\cdot\|_\infty$-metric;*
2. *$\sup_{\|\bar{r}\|_\infty = 1} N(\bar{r}) < \infty$;*
3. *if N is "strictly positive" in the sense that $N(\bar{r}) = 0$ only when $\|\bar{r}\|_\infty = 0$ (i.e., only when $r_1 = \cdots = r_n = 0$), then $\inf_{\|\bar{r}\|_\infty = 1} N(\bar{r}) > 0$.*

Proof. For $1 \leq i \leq n$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the i-th standard basis vector of the underlying set $\mathbb{R}^n$ of $\ell^\infty_{(n)}$. Then we have

$$N(\bar{r}) = N\left(\sum_{i=1}^n r_i e_i\right) \leq \sum_{i=1}^n r_i N(e_i) \leq \left(\sum_{i=1}^n N(e_i)\right) \max_{1 \leq i \leq n} |r_i| = C \|\bar{r}\|_\infty ,$$

say, where $C := \sum_i N(e_i)$. Thus, N is clearly C-Lipschitz with respect to $\|\cdot\|_\infty$, whence (1) follows.

The set $\mathcal{D} = \{\bar{r} \in \mathbb{R}^n : \|\bar{r}\|_\infty = 1\}$ is nonempty, by the assumption $n > 0$, and compact in the $\|\cdot\|_\infty$-metric, which induces the usual topology of $\mathbb{R}^n$. Since N is continuous on $\mathcal{D}$, its image $\mathcal{K} := N(\mathcal{D})$ is a compact subset $\mathcal{K} \subseteq [0, \infty)$, whence (2) follows. When N is strictly positive, (3) follows from the fact that $\mathcal{K} \subseteq (0, \infty)$. □

Proof of Theorem 42. Let $\mathcal{X} = \langle X, \ldots, \|\cdot\|_{\mathcal{X}} \rangle$ be any normed space of finite dimension $n > 0$, with basis $\bar{x} = (x_1, \ldots, x_n)$. Without loss of generality, we may take $\mathcal{X}$ to be reduced.[5] We show that $T_{\bar{x}} : \ell^\infty_{(n)} \to X : \bar{r} \mapsto \sum_i r_i x_i$ (from Exercise 34 I) is an isomorphism. The function $N : \bar{r} \mapsto \left\|\sum_i r_i x_i\right\|$ is trivially a norm on (the underlying set $\mathbb{R}^n$ of) $\ell^\infty_{(n)}$, and Lemma 43 (2) implies that $T_{\bar{x}}$ is bounded, and hence continuous, by Theorem 40. Since $\bar{x}$ is a basis and $\mathcal{X}$ is reduced by assumption, $T_{\bar{x}}$ is a set-theoretic bijection. The set-theoretic inverse $U := T_{\bar{x}}^{-1} : \mathcal{X} \to \mathbb{R}^n$ is clearly (discrete) linear and satisfies $U \circ T_{\bar{x}} = \mathrm{id}_{\mathbb{R}^n}$, $T_{\bar{x}} \circ U = \mathrm{id}_X$; moreover, by Lemma 43 (3),

$$\|U\| = \sup_{\|\bar{y}\|=1} \|U(\bar{y})\|_\infty = \sup_{\left\|T_{\bar{x}}(\bar{r})\right\|=1} \|\bar{r}\|_\infty = \sup_{N(\bar{r})=1} \|\bar{r}\|_\infty = \left(\inf_{\|\bar{r}\|_\infty=1} N(\bar{r})\right)^{-1} < \infty. \qquad \square$$

Exercise 35

Show that Lemma 43 remains true for any of the norms $\|\cdot\|_p$ on $\mathbb{R}^n$. How is the Lipschitz constant C affected? [*Hint:* Exercise 34]

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[5] If $\mathcal{X}^\sharp = \langle X^\sharp, \ldots, \|\cdot\|^\sharp \rangle$ is the $\|\cdot\|$-reduction of X, the natural map $T : x \mapsto x^\sharp$ is evidently an isomorphism, which therefore preserves dimension. (Any set-theoretic left inverse T' to T —i.e., any choice $T'(\xi)$ of a representative for each equivalence class $\xi \in \mathcal{X}^\sharp$— is a modulo-0 inverse to T). By the transitivity of the relation of isomorphism, it suffices to show that such reduced $X^\sharp$ is isomorphic to $\ell^\infty_{(n)}$ as done next.

Exercise 36

Part I

Show that the dual $\mathcal{X}^*$ of an n-dimensional normed space $\mathcal{X}$ is also n-dimensional (and therefore isomorphic to $\mathcal{X}$).

* *Part II*

Show that, for $p \in [1, \infty]$, the dual of $\ell^p_{(n)}$ is isometric to $\ell^q_{(n)}$, where q is implicitly given by the relation $1/p + 1/q = 1$ (under which $p = 1$ corresponds to $q = \infty$, and vice versa).

[*Hint:* Use the standard dot product $x \cdot y := \sum_{i=1}^n x_i y_i$ to identify a point $y \in \mathbb{R}^n$ with a linear functional $y^* : x \mapsto x \cdot y$ on $\mathbb{R}^n$. Regard $y \mapsto y^*$ as a map $\ell^q_{(n)} \to (\ell^p_{(n)})^*$. By Hölder's inequality, $|x \cdot y| \le \|x\|_p \|y\|_q$, so $\|y^*\|^* \le \|y\|_q$. Show that any given $f \in (\ell^p_{(n)})^*$ is of the form $f = y^*$ where $y = (f(e_i))_{i<n}$ (here, $e_i \in \mathbb{R}^n$ is the i-th standard basis vector). When $\|y\|_q > 0$, there is x such that $\|x\|_p > 0$ and $|x \cdot y| = \|x\|_p \|y\|_q$: if $p > 1$, take $x = (x_i)_{i=1}^n$ where $x_i = \operatorname{sgn}(y_i) \left|y_i\right|^{q-1}$. (Here, $\operatorname{sgn}(r) \in \{0, 1, -1\}$ is the sign of $r \in \mathbb{R}$.) When $p = 1$, then for some i the entry y_i has magnitude $\|y\|_\infty$: take $x = e_i$ (standard basis vector).]

Exercise 37

(The whole exercise generalizes to $\ell^p_{(\kappa)}$ for any κ, essentially with the same proofs: we take $\kappa = \omega$ only for simplicity.)

Part I

For $p \in [1, \infty)$ show that the dual of $\ell^p = \ell^p_{(\omega)}$ is isometric to ℓ^q where $1/p + 1/q = 1$. (The argument in Exercise 36 applies *mutatis mutandis*.)

* *Part II*

Show that c_0^* —the dual of the Banach subspace $c_0 = \overline{c_{00}} \subseteq \ell^\infty$— is isometric to ℓ^1.

[*Hint:* From the normed pair $\langle(\ell^1, c_0), \dots, (\|\cdot\|_\infty, \|\cdot\|_1, \langle \cdot \mid \cdot \rangle)$ we obtain a continuous embedding $x \mapsto \langle x \mid \cdot \rangle : \ell^1 \to c_0^*$ easily seen to preserve norm (by evaluating at vectors $y \in c_0$ approximating the vector $(\operatorname{sgn}(x_i))_{i<\omega} \in \ell^\infty$ "in type", i.e., finitely many coordinates at a time). Given $f \in c_0^*$, let $x_i = f(e_i)$ where e_i is the "i-th unit vector" $(0, \dots, 0, 1, 0, \dots)$ of ℓ^1 (with single nonzero entry equal to 1 at the i-th place). Let $x = (x_i)_{i<\omega}$. Show that $x \in \ell^1$ and $f(y) = \langle x \mid y \rangle$ for all $y \in c_0$.]

* *Part III*

Let $\mathcal{U} \in \beta\mathbb{N}$ be any ultrafilter on ω ($= \mathbb{N}$). Then $\mathcal{U}$ defines a map

$$\begin{aligned} \mathcal{U}^* : \ell^\infty &\to \mathbb{R} \\ x &\mapsto \mathcal{U}\lim_i x_i. \end{aligned}$$

($\mathcal{U}^*$ is defined on ℓ^∞ since the entries x_i of any $x \in \ell^\infty$ are bounded, and hence included in a compact interval $[r, s] \subseteq \mathbb{R}$.)

Show that

1. $\mathcal{U}^*$ is a continuous functional with norm $\|\mathcal{U}^*\|^* = 1$.

2. If $\mathcal{U}$ is nonprincipal, then

- the restriction $\mathcal{U}^* \restriction c_0$ is the zero functional $x \mapsto 0$ on c_0;
- $\mathcal{U}^*$ is *not* of the form $y^* : \ell^\infty \to \mathbb{R} : x \mapsto x \cdot y$ for any $y \in \ell^1$.

(Since $c_0 \subsetneq \ell^\infty$ is a Banach subspace, it follows from the Hahn-Banach Theorem *** that functionals $y^* \in c_0^*$ for $y \in \ell^1$ do *not* exhaust all continuous functionals on $\ell_\infty \supsetneq c_0$. Part III above constructs some "missing" such functionals.)

3.6 The Open Mapping and Bounded Inverse Theorems

44 Theorem (Open Mapping Theorem)**.** *Let $\mathcal{X} = \langle X, \dots, \|\cdot\|_X \rangle$ and $\mathcal{Y} = \langle Y, \dots, \|\cdot\|_Y \rangle$ be Banach spaces. If $T : X \to Y$ is a surjective operator, then it is an open map (the image of an open $U \subseteq X$ is an open set $T(U) \subseteq Y$.)*

Proof. By linearity, it suffices to show that $T(X(r)) \supseteq Y(1)$ for some r. Since T is surjective, the closed unit ball $Y[1] := \{y \in Y : \|y\|_Y \le 1\}$ is included in $T(X) = \bigcup_{n<\omega} T(X(n))$. Now, $Y[1] \subseteq Y$ is closed and hence a complete metric space, which has the Baire property. Since every relatively open $\emptyset \neq S \subseteq Y[1]$ includes an open $\emptyset \neq S' = S \cap Y(1)$, we see that $\overline{T(X(n))}^{o} \cap Y(1) \neq \emptyset$ for some $n < \omega$. By a linearity argument (combining translations an dilations), it follows that $\overline{T(X(C))} \supseteq Y(1)$ for $C = 2n/\varepsilon$ (and so $\overline{T(X(rC))} \supseteq Y(r)$ for any $r > 0$). In particular, fixing $y \in Y(1)$, there exists $x_0 \in X(C)$ such that $y_0 := T(x_0)$ satisfies $\|y_0 - y\| < 1/2$. Starting with x_0, recursively construct a sequence $(x_k)_{k<\omega} \subseteq X(C)$ such that $s_k := \sum_{i \le k} 2^{-i} x_i$ satisfies $\|T(s_k) - y\| < 2^{-(k+1)}$. (At the k-th step, choose $x_{k+1} \in X(C)$ such that $\|(y - T(s_k)) - 2^{-(k+1)} T(x_{k+1})\| < 2^{-(k+2)}$.) Clearly, (s_k) converges to an element s of norm $\|s\| < \sum_k 2^{-k} C = 2C$ such that $T(s) = y$ (by continuity of T). Therefore, $T(X(2C)) \supseteq Y(1)$. □

45 Theorem (Bounded Inverse Theorem)**.** *A bounded linear bijection $T : \mathcal{X} \to \mathcal{Y}$ between Banach spaces is an isomorphism, i.e., its set-theoretic inverse S is also bounded.*

Proof. Since T is bijective with inverse S, the inverse image $S^{-1}(U)$ of an open $U \subseteq X$ is precisely the set $T(U)$, which is open since T is open (by Theorem 44). □

3.7 Multinormed spaces and the Banach-Steinhaus Theorem

Multinormed spaces

Fix an arbitrary ordinal $\kappa \ge 1$. Let $\mathtt{P}^{(\kappa)}_{\mathrm{nrm}} = \{\mathtt{N}_i : i < \kappa\}$ be a collection of κ-many distinct unary predicate symbols $\mathtt{N}_i$ ("norm symbols"). The vocabulary of *κ-normed spaces* is $V^{(\kappa)}_{\mathrm{nrm}} = (\mathtt{F}_{\mathrm{vec}}, \mathtt{P}^{(\kappa)}_{\mathrm{nrm}})$.[6]

A *κ-normed space* (or just *multi-normed space* when κ is fixed by context) is a $V^{(\kappa)}_{\mathrm{nrm}}$-structure

$$\mathcal{Z} = \langle Z, 0, +, r\cdot, \|\cdot\|_i : r \in \mathbb{R}, i < \kappa \rangle$$

such that, for each fixed $i < \kappa$, the ("reduct") V_{nrm}-structure

$$\mathcal{Z}_i = \langle Z, 0, +, r\cdot, \|\cdot\|_i : r \in \mathbb{R} \rangle$$

[6] $V^{(1)}_{\mathrm{nrm}}$ is the vocabulary V_{nrm} of (single-)normed vector spaces in 3.2.

(obtained by dropping the interpretations of all but the single symbol $\mathtt{N}_i$) is a (single-)normed space. (It is customary to use the notation $\|\cdot\|_i$ for the interpretation $\mathtt{N}_i^{\mathcal{Z}}$ of a norm-symbol $\mathtt{N}_i$.)

Each predicate $\|\cdot\|_i : Z \to \mathbb{R}$ is called the *i-th norm of $\mathcal{Z}$*.

46 Theorem (Uniform Boundedness Principle)**.** *Let $\mathcal{Z} = \langle Z, 0, +, r\cdot, \|\cdot\|_i : r \in \mathbb{R}, i < \kappa\rangle$ be a multinormed space. Assume that the single-normed ("reduct") space $\mathcal{Z}_0 = \langle Z, \dots, \|\cdot\|_0\rangle$ is a Banach space, and that the family of norms of $\mathcal{Z}$ is:*

1. *Pointwise bounded:*
$$\sup_{i<\kappa} \|z\|_i < \infty \qquad \text{for all } z \in Z;$$
2. *Bounded modulo $\|\cdot\|_0$: for each $i < \kappa$ there is $C_i \geq 0$ such that $\|z\|_i \leq C_i \|z\|_0$ for all $z \in Z$.*

Then the family of norms is uniformly bounded modulo $\|\cdot\|_0$: there exists $C \geq 0$ such that
$$\|z\|_i \leq C \|z\|_0 \qquad \text{for all } z \in Z \text{ and } i < \kappa.$$

Proof. Let $\|\cdot\|_\kappa$ be the real-valued map $z \mapsto \sup_{i<\kappa} \|z\|_i$. By pointwise boundedness, $\|\cdot\|_\kappa : Z \to \mathbb{R}$ has (finite) real values, and a routine argument shows that $\|\cdot\|_\kappa$ is a norm on Z. Let $\mathcal{Z}_\kappa = \langle Z, \dots, \|\cdot\|\rangle$ be the corresponding normed space. For $i \leq \kappa$ and $r \geq 0$, we let $Z_i[r] = \{z \in Z : \|z\|_i \leq r\}$ be the origin-centered closed $\|\cdot\|_i$-ball of radius r in Z, $Z_i(r)$ the corresponding open ball, and $\overline{Z}_\kappa[r] \subseteq \overline{Z}$ the corresponding ball for $\|\cdot\|_\kappa$. Since each norm $\|\cdot\|_i$ is bounded for $i < \kappa$, one sees that $Z_\kappa[r] = \bigcap_{i<\kappa} Z_i[r]$ is closed. Since Z is complete (hence has the Baire property) and $Z = \bigcup_{n<\omega} Z_\kappa[n]$ is a countable union of closed sets, we deduce that for some $n < \omega$, $\varepsilon > 0$ and $z \in Z$ we have $Z_\kappa[n] \supseteq \{w \in Z : \|w - z\| \leq \varepsilon\}$. Using linearity, it follows easily that $Z_0[1] \subseteq Z_\kappa[C]$ for some $C > 0$,[7] and hence $\|\cdot\|_i \leq \|\cdot\|_\kappa \leq C \|\cdot\|_0$ for all i. □

The classical result below is a corollary of 46.

47 Theorem (Banach-Steinhaus)**.** *If $\mathcal{X} = \langle X, \dots, \|\cdot\|_X\rangle$ is a Banach space, $\mathcal{Y} = \langle Y, \dots, \|\cdot\|_Y\rangle$ any normed space, and $(T_i)_{i<\kappa}$ a family of bounded operators $T_i : X \to Y$ that are pointwise bounded in the sense that*
$$\sup_{i<\kappa} \left\|T_i(x)\right\|_Y < \infty \qquad \text{for all } x \in X,$$
then the family (T_i) is uniformly bounded: there exists $C \geq 0$ such that $\left\|T_i(x)\right\|_Y \leq C \|x\|_X$ for all $i < \kappa$.

Proof. Apply Theorem 46 to the family $\{\|\cdot\|_X\} \cup \left(\left\|T_i(\cdot)\right\|_Y\right)_{i<\kappa}$. □

[7] One may take, e.g., $C = 2n/\varepsilon$.

Chapter 4

Dual pairs and Hilbert spaces

The dual $\mathcal{X}^*$ of a normed space $\mathcal{X}$ stands as a normed space in its own right, hence removed from the fact that elements $f \in X^*$ "are" functionals on X. Structurally, $\mathcal{X}^*$ has lost any connection to the original space $\mathcal{X}$. With no significant exception, uses of dual spaces involve the original space; this speaks of a need to structurally consider *both* spaces simultaneously. Roughly speaking, the appropriate structure shall have two separate "underlying pointsets" X, X^*, each individually endowed with its own operations and norm, plus an additional predicate relating the sorts —called the *pairing on* $X^* \times X$. When X^* is the literal dual space of X, the pairing is the evaluation map $X^* \times X \to \mathbb{R} : (f, x) \mapsto f(x)$.

In order to carry out the above plan, we introduce *multisorted real-valued structures*. The word "sort" intuitively means "any specific pointset consisting of elements of a common 'kind' ". For dual pairs, there will be two sorts, say X and X^*: elements $x \in X$ are all of the "same kind" —i.e., are of "sort X"— which is completely distinct from the "other kind" of (continuous functionals) f of sort X^*.

4.1 Multisorted real-valued structures

Sort descriptors

A novelty in introducing multisorted structures is that we need to specify, at the outset, distinct formal names for the distinct sorts of the structure. A *collection of sort descriptors* (or *collection of sort names*) is simply a nonempty collection $\mathcal{M} \neq \emptyset$ each of whose elements $\mathtt{M} \in \mathcal{M}$ is called a *sort descriptor* (or *sort name*). (For dual pairs, we will eventually take $\mathcal{M} = \{\mathtt{X}, \mathtt{X}^*\}$ as the collection of two purely formal but distinct symbols $\mathtt{X}, \mathtt{X}^*$.)

Function and relation symbols

Function symbols

Let $\mathcal{M}$ be any collection of sort descriptors. A *collection of function symbols for* $\mathcal{M}$ is a set $\mathtt{F}$ each of whose elements is called a *function symbol*. Each symbol $\mathtt{f} \in \mathtt{F}$ is explicitly accompanied by (*i*) a natural $n = n_{\mathtt{f}} \in \mathbb{N}$ and (*ii*) a *function descriptor* $\mathtt{M}^{(\mathtt{f})}_{\cdot} = (\mathtt{M}_i)_{i \leq n}$ (an $(n+1)$-tuple of sort descriptors $\mathtt{M}_i \in \mathcal{M}$ —not necessarily distinct— ordered in a specific manner via explicit indexing). (Both $n_{\mathtt{f}}$

and $\mathtt{M}^{(\mathtt{f})}_{\cdot}$ are explicitly allowed to depend on the symbol $\mathtt{f}$.) The natural $n = n_{\mathtt{f}}$ is called the *arity* (one also says that the symbol $\mathtt{f}$ *is n-ary*), and the tuple $\mathtt{M}^{(\mathtt{f})}_{\cdot}$ is the *extended arity*, of $\mathtt{f}$. The arity and extended arity may be expressed using either of the following notations:

$$\mathtt{f} : \mathtt{M}_0 \times \mathtt{M}_2 \times \dots \mathtt{M}_{n-1} \to \mathtt{M}_n,$$
$$\mathtt{f} : \prod_{i<n} \mathtt{M}_i \to \mathtt{M}_n.$$

(We stress that the above are purely syntactic ways of saying that $\mathtt{f}$ has arity n and generalized arity $(\mathtt{M}_0, \mathtt{M}_1, \dots, \mathtt{M}_{n-1}; \mathtt{M}_n)$.) Abusing nomenclature, we may drop the qualifier "extended" and call $\mathtt{M}^{(\mathtt{f})}_{\cdot}$ the "arity of f".

Predicate symbols

Let $\mathcal{M}$ be any collection of sort descriptors. A *collection of predicate symbols for* $\mathcal{M}$ is a set $\mathtt{P}$ each of whose elements is called a *predicate symbol*. Each predicate symbol $\mathtt{p} \in \mathtt{P}$ is explicitly accompanied by (*i*) a natural $n = n_{\mathtt{p}} \in \mathbb{N}$ and (*ii*) a *predicate descriptor* $\mathtt{M}^{(\mathtt{p})}_{\cdot} = (\mathtt{M}_i)_{i<n}$ consisting of n sort descriptors. The natural $n = n_{\mathtt{p}}$ is called the *arity* (one also says that the symbol $\mathtt{p}$ *is n-ary*), and the tuple $\mathtt{M}^{(\mathtt{p})}_{i}$ is the *extended arity*, of $\mathtt{p}$. The arity and extended arity may be expressed using either of the following (purely syntactic) notations:

$$\mathtt{p} : \mathtt{M}_0 \times \mathtt{M}_1 \times \dots \mathtt{M}_{n-1} \to \mathbb{R},$$
$$\mathtt{p} : \prod_{i<n} \mathtt{M}_i \to \mathbb{R}.$$

(We may drop the qualifier "extended" and call $\mathtt{M}^{(\mathtt{p})}_{\cdot}$ the "arity of p".)

Multisorted vocabularies

Let $\mathcal{M}$ be a (nonempty) collection of sort names. A *vocabulary for (multisorted) structures with sort names* $\mathcal{M}$ is a triple

$$V = \langle \mathcal{M}, \mathtt{F}, \mathtt{P} \rangle$$

where $\mathtt{F}$ and $\mathtt{P}$ are collections of function symbols and predicate symbols for $\mathcal{M}$, each endowed with an extended arity $\mathtt{M}_{\cdot} \subseteq \mathcal{M}$.

Multisorted structures

Let $V = \langle \mathcal{M}, \mathtt{F}, \mathtt{P} \rangle$ be a vocabulary for multisorted structures.

A *(multisorted) real-valued structure with vocabulary* V (or V*-structure*) is a triple

$$\mathfrak{M} = \langle (\mathtt{M}^{\mathfrak{M}})_{\mathtt{M} \in \mathcal{M}}, (\mathtt{F}^{\mathfrak{M}})_{\mathtt{f} \in \mathtt{F}}, (\mathtt{P}^{\mathfrak{M}})_{\mathtt{p} \in \mathtt{P}} \rangle,$$

where

- for each $\mathtt{M} \in \mathcal{M}$, $\mathtt{M}^{\mathfrak{M}}$ is a nonempty set;

- for each $\mathtt{f} \in \mathtt{F}$, $\mathtt{f}^{\mathfrak{M}}$ is a function

$$\mathtt{f}^{\mathfrak{M}} : \prod_{i<n} \mathtt{M}_i^{\mathfrak{M}} \to \mathtt{M}_n^{\mathfrak{M}};$$

- for each $\mathtt{p} \in \mathtt{P}$, $\mathtt{p}^{\mathfrak{M}}$ is a real-valued function

$$\mathtt{p}^{\mathfrak{M}} : \prod_{i<n} \mathtt{M}_i^{\mathfrak{M}} \to \mathbb{R}.$$

To simplify the notation, when the structure $\mathfrak{M}$ is understood from context, $\mathtt{M}^{\mathfrak{M}}, \mathtt{f}^{\mathfrak{M}}, \mathtt{p}^{\mathfrak{M}}$ will be denoted M, f, p (simply shifting from `teletype` to *italic* font style).

Henceforth, "real-valued structure", or just "structure", shall always mean "multisorted real-valued structure".

4.2 Normed dual pairs

Let $\mathtt{X}, \mathtt{X}'$ be two distinct sort names. The *vocabulary for normed dual pairs* is

$$V_{\langle\cdot|\cdot\rangle}^{\mathrm{nrm}} = \langle \{\mathtt{X}, \mathtt{X}'\}, \mathtt{F}, \{\mathtt{N}, \mathtt{N}', \langle \cdot \mid \cdot \rangle\}\rangle,$$

where $\mathtt{F} = \mathtt{F}_{\mathrm{vec}} \sqcup \mathtt{F}'_{\mathrm{vec}} = \{\mathtt{0}, \mathtt{0}', +, +', \mathtt{r}\cdot, \mathtt{r}\cdot' : r \in \mathbb{R}\}$ consists of two disjoint copies of the function symbol collection $\mathtt{F}_{\mathrm{vec}}$ for vector spaces, the predicate symbols $\mathtt{N} : \mathtt{X} \to \mathbb{R}$ and $\mathtt{N}' : \mathtt{X}' \to \mathbb{R}$ are both unary, and the predicate symbol $\langle \cdot \mid \cdot \rangle : \mathtt{X} \times \mathtt{X}' \to \mathbb{R}$ is binary.

A *normed dual pair* is a $V_{\langle\cdot|\cdot\rangle}$-structure $\mathfrak{M} = \langle (X, X'), (0, 0', +, +', r\cdot, r\cdot' : r \in \mathbb{R}), (\|\cdot\|, \|\cdot\|', \langle \cdot \mid \cdot \rangle)\rangle$ such that

- the reducts $\langle X, (0, +, r\cdot : r \in \mathbb{R}), \|\cdot\|\rangle$ and $\langle X', (0', +', r\cdot' : r \in \mathbb{R}), \|\cdot\|'\rangle$ to the vocabulary V_{nrm} are normed spaces,[1] and
- $\langle \cdot \mid \cdot \rangle : X \times X' \to \mathbb{R}$ is bilinear, and

$$|\langle x \mid y\rangle| \le \|x\| \, \|y\|'$$

  for all $(x, y) \in X \times X'$.

For each $x \in X$ and $y \in X'$, define

$$\|x\|_{|X'\rangle} := \sup_{y \in X'[1]} |\langle x \mid y\rangle| \le \|x\|;$$
$$\|y\|_{\langle X|} := \sup_{x \in X[1]} |\langle x \mid y\rangle| \le \|y\|.$$

However, the inequalities may be strict. If both equalities hold, we call the dual pair *faithful*. (Proposition 48 provides some context for the inequalities above.)

[1] Up to renaming the sorts and symbols by removing "′" from their syntactic names.

The canonical dual-pair expansion

If $\mathcal{X} = \langle X, \dots, \|\cdot\|\rangle$ is any normed vector space, one obtains a canonical expansion of $\mathcal{X}$ to a normed dual pair

$$\mathfrak{M} = \langle (X, X^*), (0, 0', +, +', r\cdot, r\cdot' \,:\, r \in \mathbb{R}), (\|\cdot\|, \|\cdot\|^*, \langle\cdot \mid \cdot\rangle)\rangle,$$

where X^* is the set of all continuous real functionals on $\mathcal{X}$, endowed with the (pointwise) operations of addition $+'$, scalar multiplication $r\cdot'$, the zero functional $0'$, and the operator norm $\|\cdot\|^*$, together with the pairing by evaluation

$$\langle x \mid f\rangle := f(x).$$

48 Proposition. *Given a dual pair $\langle (X, X'), \dots, (\|\cdot\|, \|\cdot\|', \langle\cdot \mid \cdot\rangle)\rangle$, the map $y \mapsto \langle\cdot \mid y\rangle$ is an operator $X' \to X^*$ of norm ≤ 1.*

Of course, the map $X \to (X')^* : x \mapsto \langle x \mid \cdot\rangle$ is also an operator of norm ≤ 1.

Proof. Straightforward and omitted. □

49 Proposition. *If $\langle X, \dots, \|\cdot\|\rangle$ is any vector space, its dual $\langle X^*, \dots, \|\cdot\|^*\rangle$ is complete (in the norm $\|\cdot\|^*$).*

Proposition 49 explains the common name *Banach dual* for X^* (even when X itself is not necessarily Banach).

Proof. Let $(f_n)_{n<\infty} \subseteq X^*$ be a $\|\cdot\|^*$-Cauchy sequence. As x varies over any bounded ball $X[r]$, the sequences $(f_n(x))$ are uniformly Cauchy (because $\left|f_m(x) - f_n(x)\right| \leq r \left\|f_m - f_n\right\|^*$); therefore, $\lim_m f_m(x)$ exists for all $x \in X$. Let $g : X \to \mathbb{R} : x \mapsto \lim_m f_m(x)$. By linearity of all f_m and compatibility of limits with addition and multiplication in $\mathbb{R}$, we see that g is linear. Since (f_m) is $\|\cdot\|^*$-Cauchy, and hence $\left\|f_m\right\|^* \leq C$ for some C, we deduce that $\|g(x)\| \leq C\,\|x\|$ for all x, so $g \in X^*$. For all $x \in X[1]$ and $m < \omega$ we have $\left|g(x) - f_m(x)\right| \leq \sup_{n\geq m}\left|f_n(x) - f_m(x)\right| \leq \sup_{n\geq m}\left\|f_n - f_m\right\|^*$; therefore, $\left\|g - f_m\right\|^* \to 0$ as $m \to \infty$, so (f_m) $\|\cdot\|$-converges to $g \in X^*$. □

Note that $\|\cdot\|^* = \|\cdot\|^*_{\mathrm{w}}$ holds by definition of $\|\cdot\|^*$. In fact, the canonical dual expansion $\langle (X, X^*)\dots\rangle$ is faithful: the equality $\|\cdot\| = \|\cdot\|_{\mathrm{w}}$ also holds, as a consequence of the following pivotal result in the theory of normed spaces:

50 Theorem (Hahn-Banach)**.** *If $\langle X^*, \dots, \|\cdot\|^*\rangle$ is the dual of a normed vector space $\langle X, \dots, \|\cdot\|\rangle$, then for every $x_0 \in X$ there is $f \in X^*[1]$ such that $f(x_0) = \left\|x_0\right\|$. In particular, $\|\cdot\|_{|X^*\rangle} = \|\cdot\|$.*

Lemma. *If $Y \subsetneq X$ is a proper subspace, $f \in Y^*$ is a functional with $\|f\|^* := \sup_{y\in Y}|f(y)|$, and $z_0 \in X \setminus Y$, then there is an extension $\hat{f}$ of f to a functional on $Z := Y \oplus \mathbb{R}z_0$ with norm $\left\|\hat{f}\right\|^* = \|f\|^*$.*

Proof. By homogeneity, we may assume $\|f\|^* = 1 = \left\|z_0\right\|$. For $y_1, y_2 \in X$, we have

$$\begin{aligned} f(y_2) - f(y_1) = f(y_2 - y_1) \leq \|f\|^* \left\|y_2 - y_1\right\| = \left\|y_2 - y_1\right\| &= \left\|(y_2 + z_0) - (y_1 + z_0)\right\| \\ &\leq \left\|y_2 + z_0\right\| + \left\|y_1 + z_0\right\|; \end{aligned}$$

hence, $-\|y_1 + z_0\| - f(y_1) \le \|y_2 + z_0\| - f(y_2)$, so

$$u := \sup_{y\in Y}\left[-\|y + z_0\| - f(y)\right] \le \inf_{y\in Y}\left[\|y + z_0\| - f(y)\right] =: v.$$

Let $\lambda \in [u, v]$ be otherwise arbitrary; then, $|f(y) + \lambda| \le \|y + z_0\|$ for all $y \in Y$. Extend f to $\hat{f} : Z \to \mathbb{R} : y + rz_0 \mapsto f(y) + r\lambda$. Then,

$$\left|\hat{f}(y + z_0)\right| = |f(y) + \lambda| \le \|y + z_0\|.$$

By homogeneity, $\left|\hat{f}(z)\right| \le \|z\|$ for all $z \in Z$; therefore, $\left\|\hat{f}\right\|^* = 1 = \|f\|^*$. □

Proof of Theorem 50. By homogeneity, it suffices to prove the assertion given $x_0 \in X$ with $\|x_0\| = 1$. The functional $f_0 : \mathbb{R}x_0 \to \mathbb{R} : rx_0 \mapsto r$ evidently has norm 1 and satisfies $f_0(x_0) = 1 = \|x_0\|$. Using transfinite induction, construct a sequence $(f_\alpha)_{\alpha\le\kappa}$, starting with f_0, of linear functionals of norm $\|f\|_\alpha = 1$ on a chain of subspaces $\mathbb{R}x_0 = Y_0 \subseteq \ldots \subseteq Y_\alpha \subseteq \ldots$ in such manner that $f_{\alpha+1}$ extends f_α to a linear space $Y_{\alpha+1} = Y_\alpha \oplus \mathbb{R}y_{\alpha+1}$ (for some $y_{\alpha+1} \notin Y_\alpha$) whenever $Y_\alpha \subsetneq X$ and, for limit ordinals α, f_α is the unique real functional on $Y_\alpha := \bigcup_{\beta<\alpha} Y_\beta$ extending all functionals f_β. The induction must eventually terminate, yielding a functional f_κ after, say, κ-many steps, on the full space $Y_\kappa = X$. □

Exercise 38

Recast the proof of Theorem 50 (based on the same Lemma) as an application of Zorn's Lemma rather than using transfinite induction.

Normed lattice pairs

Exercise 32 suggests generalizing the notion of dual of a vector lattice to introduce the notion normed lattice pair.

Let $V^{\vee}_{\langle\cdot|\cdot\rangle}$ expand the vocabulary $V_{\langle\cdot|\cdot\rangle}$ for dual pairs with function symbols $\vee, \wedge, \vee', \wedge'$ for binary (lattice) operations on sorts $\mathtt{X}, \mathtt{X}'$, respectively.

A *normed lattice pair* is a $V^{\vee}_{\langle\cdot|\cdot\rangle}$-structure

$$\mathfrak{M} = \langle (X, X'), (0, 0', +, +', (r\cdot, r\cdot')_{r\in\mathbb{R}}, \vee, \vee', \wedge, \wedge'), (\|\cdot\|, \|\cdot\|', \langle\cdot \mid \cdot\rangle)\rangle$$

such that:

- the reduct of $\mathfrak{M}$ to the vocabulary $V_{\langle\cdot|\cdot\rangle}$ is a dual pair,
- the reducts of $\mathfrak{M}$ to the vocabularies $V_{\mathrm{Riesz}}, V'_{\mathrm{Riesz}}$ each is a normed vector lattice,
- $\langle x^+ \mid y^+\rangle \ge 0$ for all $x \in X$ and $y \in X'$ (where $x^+ = (-x) \vee x$ and $y^+ = (-y) \vee' y$).

A *Banach lattice pair* is one whose sorts are both norm-complete.

4.3 Weak topologies

Let

$$\mathfrak{M} = \langle (X,Y), (0, 0', +, +', r\cdot, r\cdot' : r \in \mathbb{R}), (\|\cdot\|, \|\cdot\|', \langle \cdot \mid \cdot \rangle) \rangle,$$

be a normed dual pair.

The *weak topology on sort* X *of a dual pair* $\langle (X,Y), \dots \rangle$ is the topology $\mathcal{T}_X^{\langle \cdot | Y \rangle}$ of ("pointwise") convergence for functionals $\langle \cdot \mid y \rangle$ ($y \in Y$); we will also call this the $\langle \cdot \mid Y \rangle$*-topology on* X. Equivalently, it is the topology with subbasic opens of the form $U_{y,(r,s)} = \{x \in X : r < \langle x \mid y \rangle < s\}$ for $y \in Y$ and $r < s \in \mathbb{R}$. In the same ("dual") manner, one defines the weak topology $\mathcal{T}_Y^{\langle X | \cdot \rangle}$ (the $\langle X \mid \cdot \rangle$-topology) on Y.

The continuity of $\langle \cdot \mid \cdot \rangle$ (in the norm topologies) implies that $\mathcal{T}_X^{\langle \cdot | Y \rangle} \subseteq \mathcal{T}_X^{\|\cdot\|}$ (and, dually, $\mathcal{T}_Y^{\langle X | \cdot \rangle} \subseteq \mathcal{T}_Y^{\|\cdot\|'}$), i.e., the weak topologies are coarser than the metric topologies.

Weak and weak-* topologies

When $\mathcal{X} = \langle X, \dots, \|\cdot\| \rangle$ is a given normed space, the *weak topology* $\mathcal{T}_X^{\mathrm{w}}$ is the topology $\mathcal{T}_X^{\langle \cdot | X^* \rangle}$ for the canonical dual $\langle (X, X^*), \dots \rangle$, i.e., the topology of pointwise convergence of all continuous functionals $f : X \to \mathbb{R}$. The *weak-* ("weak-star") topology* on the dual space $\mathcal{X}^* = \langle X^*, \dots \rangle$ is the weak topology $\mathcal{T}_{X^*}^{\langle X | \cdot \rangle}$. The weak-* topology on $\underline{X^*}$ is coarser —"weaker"— than the weak topology $\mathcal{T}_{X^*}^{\langle X^{**} | \cdot \rangle}$ since X^{**} includes all evaluation functionals $\mathrm{ev}_x : f \mapsto f(x)$ for $x \in X$, i.e., X^{**} extends X (at least structurally, if not as set-theoretic inclusion).

4.4 Hilbert spaces

The *vocabulary for Hilbert spaces* is $V_{\mathrm{Hilb}} = \langle \{\mathtt{X}\}, \mathtt{F}_{\mathrm{vec}}, \{\langle \cdot \mid \cdot \rangle\} \rangle$, where $\langle \cdot \mid \cdot \rangle : \mathtt{X} \times \mathtt{X} \to \mathbb{R}$ is a binary predicate symbol on the (sole) sort $\mathtt{X}$.

A V_{Hilb}-structure $\mathcal{X} = \langle X, (0, +, r\cdot)_{r \in \mathbb{R}}, \langle \cdot \mid \cdot \rangle \rangle$ is a *(real) pre-Hilbert space* (also called a *(real) inner product space*) if, for all $x, y, z \in X$ and $r \in \mathbb{R}$:

- $\langle 0 \mid 0 \rangle = 0$;
- $\langle \cdot \mid \cdot \rangle$ is nonnegative: $\langle x \mid x \rangle \geq 0$.
- $\langle \cdot \mid \cdot \rangle$ is symmetric (under exchange of its first and second arguments): $\langle x \mid y \rangle = \langle y \mid x \rangle$;
- $\langle \cdot \mid \cdot \rangle$ is bilinear (i.e., linear in each argument): $\langle x_1 + r x_2 \mid y \rangle = \langle x_1 \mid y \rangle + r \langle x_2 \mid y \rangle$ (and similarly —if redundantly, by symmetry— linear in the second argument);

The pairing $\langle \cdot \mid \cdot \rangle$ is called the *inner product* or *Hilbert pairing* of $\mathcal{X}$.

Two elements $x, y \in X$ *are orthogonal* (denoted: "$x \perp y$") if $\langle x \mid y \rangle = 0$.

The *norm* on a pre-Hilbert space is defined by $\|x\| := \sqrt{\langle x \mid x \rangle}$. It is obviously nonnegative and homogeneous unary predicate; indeed, it is a norm on X as follows from the following:

Exercise 39 Pre-Hilbert spaces are normed spaces

Let $\mathcal{X} = \langle X, \dots, \langle \cdot \mid \cdot \rangle \rangle$ be a pre-Hilbert space, expanded by defining the "norm predicate" $\|\cdot\| : x \mapsto \sqrt{\langle x \mid x \rangle}$.

Part I

Prove the *Cauchy-Schwarz Inequality*

$$|\langle x \mid y \rangle| \leq \|x\| \, \|y\| \qquad \text{for all } x, y \in X.$$

[*Hint:* By homogeneity, it suffices to prove $\langle x \mid y \rangle^2 \leq \|x\|^2$ when $\|y\| = 1$. The real-valued function $f : r \to \mathbb{R} : r \mapsto \|rx - y\|^2 = r^2 - 2\langle x \mid y \rangle r + \|y\|^2$ is a nonnegative quadratic form on $\mathbb{R}$, hence has discriminant ≥ 0.]

Part II

Show that a pre-Hilbert vector space $\mathcal{X}$ (as defined above), when regarded as a V_{nrm}-structure with $N^{\mathcal{X}} = \|\cdot\|$ (and stripped of the pairing $\langle \cdot \mid \cdot \rangle$), is a (possibly unreduced) normed vector space $\mathcal{X}_{\|\cdot\|}$, and that $\langle \cdot \mid \cdot \rangle$ is a continuous function on $\mathcal{X}_{\|\cdot\|}$ (locally uniformly Lipschitz in each of its two arguments in fact).

For reference, we state the *Parallelogram Law:*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad \text{for all } x, y \in X;$$

and the *Pythagorean Theorem:*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 \qquad \text{if } x \perp y;$$

whose proofs are routine.

More cryptically, a V_{Hilb}-structure $\mathcal{X}$ is a pre-Hilbert space if its expansion to a $V_{\langle\cdot|\cdot\rangle}$-structure $\langle (X, 0, +, r\cdot, \|\cdot\|)_{r\in\mathbb{R}}, (X, 0, +, r\cdot, \|\cdot\|)_{r\in\mathbb{R}}, \langle \cdot \mid \cdot \rangle \rangle$ (both of whose sorts $\mathtt{X}$, $\mathtt{X}'$ have the same underlying set X) is a normed dual pair with norm(s) given by $\|x\| := \sqrt{\langle x \mid x \rangle}$, and such that $\langle \cdot \mid \cdot \rangle$ is symmetric in both arguments.

A *Hilbert space* is a pre-Hilbert space (i.e., inner product space) that is complete with respect to the Hilbert norm $\|\cdot\|$-metric.

Exercise 40

For any ordinal κ, the space $\ell^2_{(\kappa)}$ is a Hilbert space when endowed with the inner product $\langle r \mid s \rangle := \sum_{i<\kappa} r_i s_i$.

(One can also show that $\ell^2_{(\kappa)}$ is isometric to $\ell^2_{(\lambda)}$ iff $|\kappa| = |\lambda|$, so spaces $\ell^2_{(\kappa)}$ for ordinals κ exhaust the class of Hilbert spaces, up to isometry.)

51 Theorem. *Any Hilbert space $\mathcal{X} = \langle X, \dots, \langle \cdot \mid \cdot \rangle \rangle$ has an orthonormal (ON) basis $B = (x_i)_{i<\kappa} \subseteq X$ consisting of κ-many elements (i.e., such that $\langle x_i \mid x_j \rangle = \delta(i, j)$).*

More precisely, the basis $\mathcal{B}$ gives an inner-product preserving linear isomorphism (hence an isometry)

$$\begin{aligned} \ell^2_{(\kappa)} &\to X \\ r = (r_i)_{i<\kappa} &\mapsto r \cdot B := \sum_{i<\kappa} r_i x_i, \end{aligned}$$

where the sum of the series is interpreted in the sense of convergence (in norm) over the net of finite subsets $(i_j : j < n) \subseteq \kappa$.

Note: Note that the notion of "basis" above is *not* in the usual (discrete) sense ("Hamel basis"): linear combinations of finitely many elements x_i do *not* exhaust X, but are merely *dense* in X. Every element x is the limit-in-norm of some series $\sum_i r_i x_i$; in fact, such series converges iff $\sum_i (\pm r_i) x_i$ converges for any choice of the signs (the basis is "unconditional") and, the sum is independent of any particular re-ordering of the indexes i (the basis is "symmetric"). These and many other regularity properties not shared by general normed (and Banach) spaces make infinite-dimensional Hilbert spaces the best-behaved relatives of finite-dimensional normed spaces.

Proof. The basis is constructed transfinitely inductively by an adaptation of the Gram-Schmidt orthogonalization process. The trivial subspace $X_{<0} = \{0\} \subseteq X$ has empty basis $B_{<0} := ()$. Inductively construct partial bases $B_{<\lambda} = (x_i)_{i<\lambda}$ for ordinals λ as follows. Given $B_{<\lambda}$, let $X_{<\lambda}$ be the $\|\cdot\|$-closure in X of the space of linear combinations of finitely many elements $x_i \in B_{<\lambda}$ at a time. If $X_{<\lambda} \subsetneq X$, then $B_{<\lambda}$ extends to a base $B_{<\lambda+1} = B \cup \{x_\lambda\}$ by constructing x_λ as follows. Take any vector $y \notin X_{<\lambda}$. As $i_. = (i_j)_{j<n}$ varies over finite collections $i_0 < \cdots < i_{n-1} \leq \lambda$ ordered by inclusion, the net of corresponding projections $y_{i_.} := \sum_{j<n} \left\langle x_{i_j} \mid y \right\rangle x_{i_j}$ is Cauchy (as a consequence of the Pythagorean Theorem) and therefore —by completeness of X— converges to an element $y' \in X_{<\kappa}$ (the "projection of y to $X_{<\lambda}$") such that $\langle x_i \mid y' - y\rangle = 0$ for all $i \leq \kappa$. Necessarily, $y^\perp := y - y'$ has $\|y^\perp\| > 0$—since $y' \in \overline{X_{<\lambda}} = X_{<\lambda}$, but $y \notin X_{<\lambda}$. Let $x_\lambda := y^\perp / \|y^\perp\|$. At limit ordinals λ, let $X_{<\lambda} = \overline{\bigcup_{\alpha<\lambda} X_{<\alpha}}$ (no new vector is added to the basis at limit stages).

The process ends after, say, κ-many steps when the finite span of $B = B_\kappa$ is dense in X. The details are left to the reader.

Using Exercise 40 and the orthonormality of the sets $B_{<\lambda}$ it is easy to show that $X_{<\lambda}$ is a Hilbert subspace isometric to $\ell^2_{(\lambda)}$ for each λ, so $X = X_{<\kappa}$ is isometric to $\ell^2_{(\kappa)}$. □

Exercise 41

Let $\mathcal{X} = \langle X, \dots, \langle\cdot \mid \cdot\rangle\rangle$ be a Hilbert space. Given any closed subspace $Y = \overline{Y} \subseteq X$, construct an operator $\pi_Y : X \to X$ of "orthogonal projection on Y" characterized by the properties:

- $\pi_Y \restriction Y = \mathrm{id}_Y$;
- $\pi_Y \restriction Y^\perp = 0$,

where used the notation

$$S^\perp := \{x \in X : \langle x \mid y\rangle = 0 \text{ for all } y \in S\},$$

for the *orthogonal complement* $S^\perp$ of an arbitrary subset $S \subseteq X$. Show also that the operator norm $\|\pi_Y\| = 1$ (except when $Y = \{0\}$, when π_Y is identically zero and hence of norm zero).

By Proposition 48, the map $x \mapsto \langle x \mid \cdot\rangle$ is linear and 1-Lipschitz $\mathcal{X} \to \mathcal{X}^*$ with respect to the norms $\|\cdot\|, \|\cdot\|^*$. (By symmetry, the embedding $y \mapsto \langle\cdot \mid y\rangle$ is the same.) By Proposition 49, any dual space $\mathcal{X}^*$ is always complete in its norm $\|\cdot\|^*$. The following is a central result in the theory of Hilbert spaces.

52 Theorem (Riesz-Frechét Representation Theorem)**.** *Let $\mathcal{X} = \langle X, \dots, \langle\cdot \mid \cdot\rangle\rangle$ be a (complete) Hilbert space. The map $x \mapsto \langle x \mid \cdot\rangle$ is a surjection $X \to X^*$, i.e., for all $f \in X^*$ there exists $x_f \in X$ such that $f(y) = \left\langle x_f \mid y\right\rangle$ for all $y \in X$ (i.e., f is represented in the form $\left\langle x_f \mid y\right\rangle$).*

Proof. By homogeneity, it suffices to consider the case $\|f\|^* = 1$. Let $B = (x_i)_{i<\kappa}$ be an ON basis of $\mathcal{X}$. Any sum $y_{i_\cdot} := \sum_{j<n} f(x_{i_j})x_{i_j}$ over finitely many $i_0 < \cdots < i_{n-1} < \kappa$ satisfies $\left\|y_{i_\cdot}\right\|^2 = \sum_{j<n} f(x_{i_j})f(x_{i_j}) = f(y_{i_\cdot}) \le \|f\|^* \left\|y_{i_\cdot}\right\| = \left\|y_{i_\cdot}\right\|$, hence $\left\|y_{i_\cdot}\right\| \le 1$. Therefore, the sum $x_f := \sum_{i<\kappa} f(x_i)x_i$ converges, and clearly $f = \langle x_f \mid \cdot \rangle$. □

53 Proposition. *The normed vector space $\langle X, \dots, \|\cdot\| \rangle$ canonically associated to a Hilbert space $\langle X, \dots, \langle \cdot \mid \cdot \rangle \rangle$ is isometric to its dual $\langle X^*, \dots, \|\cdot\|^* \rangle$.*

Another way to state Proposition 53 is that the expansion of a Hilbert space to a symmetric dual pair (in the sense explained after the definition of pre-Hilbert space) is isomorphic (in the sense of normed dual pairs) to the canonical expansion of the associated normed space by its dual.

Proof. In view of Theorem 52, it remains only to show that the map $x \mapsto x^* := \langle x \mid \cdot \rangle$ is norm-preserving and, by homogeneity, it suffices to show that $\|x^*\|_{\langle X|} =$ if $\|x\| = 1$. On the one hand, $\|x^*\|_{\langle X|} \le 1$ (since $|x^*(y)| = |\langle x \mid y \rangle| \le \|x\|\,\|y\| = \|y\|$ for all $y \in X$). On the other hand, $x^*(x) = \langle x \mid x \rangle = \|x\|^2 = 1$ implies $\|x^*\|_{\langle X|} \ge 1/\,\|x\| = 1$. □

4.5 Dual pair types

Let $\langle (X, Y), \dots, (\|\cdot\|_X, \|\cdot\|_Y, \langle \cdot \mid \cdot \rangle) \rangle$ be a normed dual pair. One may define types for the pairing predicate $\langle \cdot \mid \cdot \rangle$ as follows:

- The type of $x \in X$ is $\mathrm{tp}_{\langle \cdot|Y \rangle}(x) := \langle x \mid \cdot \rangle = (\langle x \mid y \rangle : y \in Y) \in \mathbb{R}^Y$;
- The type of $y \in Y$ is $\mathrm{tp}_{\langle X|\cdot \rangle}(y) := \langle \cdot \mid y \rangle = (\langle x \mid y \rangle : x \in X) \in \mathbb{R}^X$.

By Proposition 48, $\mathrm{tp}_{\langle \cdot|Y \rangle} : X \to Y^*$ and $\mathrm{tp}_{\langle X|\cdot \rangle} : Y \to X^*$ are bounded linear transformations of norm ≤ 1.

Types $\mathrm{tp}_{\langle \cdot|Y \rangle}(x)$, $\mathrm{tp}_{\langle X|\cdot \rangle}(y)$ are *realized* (by x and y, respectively). The corresponding type spaces are $\mathfrak{T}_{\langle \cdot|Y \rangle}(X) := \overline{\mathrm{tp}_{\langle \cdot|Y \rangle}(X)} \subseteq \mathbb{R}^Y$ and $\mathfrak{T}_{\langle X|\cdot \rangle}(Y) := \overline{\mathrm{tp}_{\langle X|\cdot \rangle}(Y)} \subseteq \mathbb{R}^X$. As always, these are closures in the product topology of $\mathbb{R}^Y$, $\mathbb{R}^X$, respectively. (Even though $\mathrm{tp}_{\langle \cdot|Y \rangle}(X) \subseteq Y^*$, the type space $\mathfrak{T}_{\langle \cdot|Y \rangle}(X)$ is *not* obtained taking closure in the norm-topology of Y^*, but in the much weaker —coarser— product topology of $\mathbb{R}^Y$ —generally, $\mathfrak{T}_{\langle \cdot|Y \rangle}(X) \nsubseteq Y^*$.)[2]

Remark. The $\langle \cdot \mid Y \rangle$-topology on X is precisely the initial topology induced by the map $\mathrm{tp}_{\langle \cdot|Y \rangle} : X \to \mathfrak{T}_{\langle \cdot|Y \rangle}(X)$. The $\langle X \mid \cdot \rangle$-topology on Y is correspondingly induced by $\mathrm{tp}_{\langle X|\cdot \rangle} : Y \to \mathfrak{T}_{\langle X|\cdot \rangle}(Y)$.

4.6 Alaoglu's Theorem

54 Proposition. *If $\langle (X, Y), \dots, (\|\cdot\|_X, \|\cdot\|_Y, \langle \cdot \mid \cdot \rangle) \rangle$ is a dual normed pair, every $\langle X \mid \cdot \rangle$-type of the closed unit ball $Y[1]$ is realized by an element $f \in X^*[1]$ in the canonical dual normed pair $\langle (X, X^*), \dots, (\|\cdot\|_X, \|\cdot\|_X^*, \langle \cdot \mid \cdot \rangle) \rangle$ of the normed space $\langle X, \dots \rangle$.*

[2]Types $\mathfrak{t} \in \mathfrak{T}_{\langle \cdot|Y \rangle}(X)$ are linear maps $X \to \mathbb{R}$, but not necessarily bounded: this is implicit in the proof of Theorem 55 below, where it is shown that "bounded" type spaces, say $\mathfrak{T}_{\langle \cdot|Y \rangle}(X[1])$, are necessarily spaces of types realized by *bounded* linear maps $X \to \mathbb{R}$.

Proof. The space of realized types of $Y[1]$ is a subspace of the compact product $\prod_{x\in X}[-\|x\|, \|x\|]$, hence its closure (the type space $\mathcal{T}_{\langle X|\cdot\rangle}(Y[1])$) is compact.[3] Note that the type $\mathrm{tp}_{\langle X|\cdot\rangle}(f) = (\langle x \mid f\rangle)_{x\in X} = (f(x))_{x\in X}$ of an element $f \in X^*$ is materially the same as f; therefore, it suffices to show that $\mathrm{tp}_{\langle X|\cdot\rangle}(Y[1]) \subseteq X^*[1]$. Let $\mathfrak{t} \in \mathcal{T}_{\langle X|\cdot\rangle}(Y[1])$. Since $\mathfrak{t}$ is pointwise approximated by realized types $\langle \cdot \mid y\rangle$ (with $\|y\|_Y \leq 1$), which satisfy $|\langle x \mid y\rangle| \leq \|x\|_X$ and $\langle x_1 + rx_2 \mid y\rangle = \langle x_1 \mid y\rangle + r\langle x_2 \mid y\rangle$ for all $x, x_1, x_2 \in X$ and $r \in \mathbb{R}$, we see that $\mathfrak{t} \in \mathbb{R}^X$ is a linear function with $\sup_{x\in X[1]} |\mathfrak{t}(x)| \leq \sup_{x\in X[1]} \|x\|_X \leq 1$, so $\mathfrak{t} \in X^*[1]$. $\square$

55 Theorem (Alaoglu)**.** *The unit ball* $X^*[1] := \{f \in X^* : \|f\|^* \leq 1\}$ *(of the dual* $\underline{X^*} = \langle X^*, \dots, \|\cdot\|^*\rangle$ *of a normed space* $\underline{X} = \langle X, \dots, \|\cdot\|\rangle$*) is compact in the weak-* topology (relative to* $\underline{X}$*).*

Proof. The space of realized types $\langle \cdot \mid f\rangle = f(\cdot)$ of $X^*[1]$ is a subspace of the compact product $\prod_{x\in X}[-\|x\|, \|x\|]$, hence its closure (the type space $\mathcal{T}_{\langle X|\cdot\rangle}(X^*[1])$) is compact. By Proposition 54 (when applied to the canonical pair $\langle (X, X^*) \dots\rangle$), every type is realized in X^*, hence $X^* = \mathcal{T}_{\langle X|\cdot\rangle}(X^*[1])$ is thus compact in the $\mathrm{tp}_{\langle X|\cdot\rangle}$-topology, which is the weak-* topology. $\square$

[3] This akin to Proposition 15.

Chapter 5

Banach lattice-algebras

The Banach lattices $\ell^p(X)$ (where X is any pointset) are quintessential function spaces. However, they "abstract away" the domain pointset X and the evaluation operation $\ell^p(X) \times X \to \mathbb{R}$; therefore they are no longer "function" spaces in a direct sense.

The space $\ell^\infty(X)$ is special: the operation of pointwise multiplication is continuous —in fact, $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$. Therefore, $\ell^\infty(X)$ is an example of a "(unital) Banach algebra" in the sense below.

5.1 Normed algebras and lattice-algebras

Every function lattice $\mathcal{F} \subseteq \mathbb{R}^D$ (in the sense of Section 2.2) is a very special kind of real vector lattice, because it can be endowed with the operation of pointwise multiplication of functions $(f, g) \mapsto fg$, and has an identity $\mathbb{1}$.

Since the product of any two bounded real functions on a set D is still bounded, the vector lattice $\ell^\infty(D)$ admits an expansion to a (commutative) real algebra by the operation of pointwise multiplication; moreover, the constant function $\mathbb{1} \in \ell^\infty(D)$ is unity for this product. When endowed with the vector lattice and algebra operations, $\ell^\infty(D)$ is a quintessential "real function space". This motivates the following definitions.

Normed algebras The *vocabulary for Banach algebras* is $V_{\text{alg}} = (\mathtt{F}_{\text{alg}}, \{\mathtt{N}\})$, where $\mathtt{F}_{\mathcal{L}^\infty}$ expands the function symbol collection $\mathtt{F}_{\text{vec}}$ of vector spaces with a new symbol $\cdot$ for a binary operation.

The *vocabulary for unital Banach algebras* is V_{alg}^1 expanding V_{alg} with a unary symbol $\mathtt{1}$.

A *(real) normed algebra* is a V_{alg}-structure

$$\mathcal{X} = \langle X, 0, +, \cdot, r\cdot, \|\cdot\| : r \in \mathbb{R} \rangle$$

whose V_{nrm}-reduct is a normed vector space, and satisfying:

- $\cdot : X \times X \to X$ is associative, bilinear, and distributes over addition (i.e., $\mathcal{X}$ is a ring whose multiplication is homogeneous with respect to scalar multiplication in each argument);
- $\|xy\| \leq \|x\| \, \|y\|$ for all $x, y \in X$.

A normed algebra is *commutative* if its product $\cdot$ is commutative.

A *(real) normed unital algebra* is a V_{alg}^{1}-structure $\mathcal{X}$ whose V_{alg}-reduct is a normed algebra, and such that $\mathtt{1}$ is a (two-sided) identity for the multiplication on X.

A *(unital) Banach algebra* is a normed (unital) algebra that is complete in its norm-metric.

Normed lattice-algebras The *vocabulary for normed lattice-algebras* is

$$V_{L^\infty} = (\{\mathtt{0}, +, (\mathtt{r}\cdot)_{r\in\mathbb{R}}, \cdot, \vee, \wedge\}, \{\mathtt{N}, \langle\cdot \mid \cdot\rangle\});$$

it expands the vocabulary V_{Riesz} of vector lattices with a symbol $\cdot$ for the binary (algebra) operation of multiplication.

The *vocabulary $V_{L^\infty}^{1}$ for normed unital lattice-algebras* expands V_{L^∞} with a unary symbol $\mathtt{1}$ denoting "unity".

We regard the vocabulary V_{L^∞} (resp. $V_{L^\infty}^{1}$) as the "joint" expansion of both V_{Riesz} and V_{alg} (resp., of both V_{Riesz} and V_{alg}^{1}) when the common ingredients (sort descriptor, vector space operation and norm symbols) are identified.

A *normed (unital) lattice-algebra* is a V_{L^∞}- (resp., $V_{L^\infty}^{1}$-)structure[1]

$$\mathcal{X} = \langle X, 0, +, \cdot, r\cdot, \wedge, \vee, \|\cdot\| \,:\, r \in \mathbb{R}\rangle$$

(resp., $\mathcal{X} = \langle X, 0, \mathbb{1}, +, \cdot, r\cdot, \wedge, \vee, \|\cdot\| \,:\, r \in \mathbb{R}\rangle$) whose

- $V_{\|\text{Riesz}\|}$-reduct is a *distributive* normed vector lattice;
- V_{alg}-reduct is a normed algebra (resp., V_{alg}^{1}-reduct is a normed unital algebra)

A *(unital) Banach lattice-algebra* is a complete normed (unital) lattice algebra.

Exercise 42

Part I

For any set D, show that $\ell^\infty(D)$ is a unital Banach lattice-algebra.

Part II

Show that $\ell^\infty(D)$ satisfies the following property (called the *Axiom of Idempotents*):

For every $f \in \ell^\infty(D)$ there is an idempotent $\chi \in \ell^\infty(D)$ (i.e., an element such that $\chi^2 = \chi$ —a "characteristic function" taking only the values 0, 1) such that

$$\chi f \doteq f^+.$$

Moreover, show that any two idempotents χ, χ' having such property are "equivalent modulo $|f|$" in the sense that $\chi\,|f| \doteq \chi'\,|f|$.

[*Hint:* Take $\chi = \chi_D$ as the characteristic function of the set $f^{-1}(0, \infty) \subseteq D$.]

[1] The notion of normed (and Banach) lattice-algebras has closely related —but slightly different— definitions (See Wickstead (2017) `http://dx.doi.org/10.1007/s11117-015-0387-8.`)). We consider only *distributive* lattices.

5.2 A representation theorem for real commutative Banach algebras

Generally speaking, a representation theorem shows that spaces in a certain abstract/general class are, up to isomorphism, instances of spaces in a more concrete/particular class (often, a subclass). We saw a first example of this in the Riesz-Frechét Theorem, which represents "abstract" functionals on a Hilbert space "concretely" realized in the form $\langle \cdot \mid y \rangle$. (From a slightly different perspective, it represents the canonical dual pair $\langle (X, X^*), \dots, \mathrm{ev} \rangle$ obtained from a Hilbert space $\langle X, \dots, \langle \cdot \mid \cdot \rangle \rangle$ as the pair $\langle (X, X), \dots, \langle \cdot \mid \cdot \rangle \rangle$ obtained by "doubling" X.)

The result below is (in essence) a real-valued version of the Stone Representation Theorem for conjugation-invariant closed subalgebras of the (complex) unital Banach algebra $\ell^\infty(X)$.

56 Theorem (Stone Representation Theorem (real version))**.** *If $\mathcal{F} = \langle F, \dots, \|\cdot\|_\infty \rangle$ is a (real) sub-algebra of $\ell^\infty(X)$ for some set X, then there are*

- *a compact Hausdorff space $\check{X}$; and*
- *maps $\iota : X \to \check{X}$ and $F \to \ell^\infty(\check{X}) : f \mapsto \check{f}$,*

such that

1. *ι has dense image;*
2. *the image $\check{F}$ is included in the sub-algebra $C_{\mathrm{b}}(\check{X}) \subseteq \ell^\infty(\check{X})$ of continuous bounded functions $\check{X} \to \mathbb{R}$;*
3. *$\check{\cdot}$ is an (isometric) embedding of algebras;*
4. *$\check{f}(\iota(x)) = f(x)$ for all $x \in X$.*

If F has a unity element $\mathbb{1}$, then $\check{\mathbb{1}}$ is a unity of $\check{F}$.

Proof. Consider the two-sorted structure

$$\mathfrak{M} = \langle (F, X), (0, \mathbb{1}, +, \cdot r, \cdot)_{r \in \mathbb{R}}, (\|\cdot\|_\infty, \langle \cdot \mid \cdot \rangle) \rangle,$$

for the vocabulary $V = V_{\mathrm{alg}}^{|\mathtt{X}\rangle}$ consisting of the operations of an algebra, a new sort symbol $\mathtt{X}$, and an additional binary predicate $\langle \cdot \mid \cdot \rangle : F \times X \to \mathbb{R} : (f, x) \mapsto f(x)$ (which is not a *bona fide* pairing: although linear on the first argument f, the second argument x is a point in a "structureless" sort). Each $x \in X$ has a *(realized) F-type*, namely the point $\iota(x) := \langle \cdot \mid x \rangle = (\langle f \mid x \rangle)_{f \in F} \in \prod_{f \in F} [-\|f\|_\infty, \|f\|_\infty] \subseteq \mathbb{R}^F$. The *type space* $\check{X} \subseteq \mathbb{R}^f$ is the (necessarily compact Hausdorff) closure of the set of realized types; in particular, $\iota(X) \subseteq \check{X}$ is dense. Each $f \in F$ yields a continuous bounded function $\check{f} := \pi_f : \check{X} \to [-\|f\|_\infty, \|f\|_\infty]$ (by restriction of the continuous "f-th coordinate" map on $\mathbb{R}^X$); therefore, $\check{F} := \{\check{f} : f \in F\} \subseteq C_{\mathrm{b}}(\check{X})$. Clearly, $\check{f}(\iota(x)) = \langle f \mid x \rangle = f(x)$ holds for $x \in X$. We claim that $F \to C_{\mathrm{b}}(\check{X}) : f \mapsto \check{f}$ preserves all algebra operations and the norm (thus, in particular, $\check{F}$ is a sub-algebra of $C_{\mathrm{b}}(\check{X})$). First, $\check{\mathbb{1}} = (\mathbb{1}(x))_{x \in X} = (1)_{x \in X}$ is the unity of $C_{\mathrm{b}}(\check{X})$. Next, given $f, g \in F$,

$$(f + g)\check{}(\iota(x)) = \langle f + g \mid x \rangle = \langle f \mid x \rangle + \langle g \mid x \rangle = \check{f}(\iota(x)) + \check{g}(\iota(x))$$

holds for all (realized types by) $x \in X$; therefore, $(f+g)\check{}(\mathfrak{x}) = \check{f}(\mathfrak{x}) + \check{g}(\mathfrak{x})$ holds for all $\mathfrak{x} \in \check{X}$ by continuity of $(f+g)\check{}$, $\check{f}$, $\check{g}$ on $\check{X}$; thus, $(f+g)\check{} = \check{f} + \check{g}$. The verification that $\check{F}$ is closed under all remaining operations is similar and omitted. Finally, given $f \in F$, on the one hand, since $\check{X}$ contains all principal types, we have $\left\|\check{f}\right\|_\infty \geq \|f\|_\infty$; on the other hand, by the definition of the type topology, $\check{f}(\mathfrak{x}) \in \overline{f(X)} \subseteq [-\|f\|_\infty, \|f\|_\infty]$ for all $\mathfrak{x} \in \check{X}$, so $\left\|\check{f}\right\|_\infty \leq \|f\|_\infty$ as well; therefore, $f \mapsto \check{f}$ is an isometric embedding. □

57 Corollary (Stone-Čech compactification)**.** *Let X be any Tychonoff space (completely regular Hausdorff), and let $F = C_{\mathrm{b}}(X)$ be the (unital) algebra of continuous bounded functions on X, regarded as a Banach algebra $\underline{F} = \langle F, (+, 0, r\cdot, \cdot), \|\cdot\|_\infty \rangle_{r\in\mathbb{R}}$. Then the map $\iota : X \to \check{X}$ in Theorem 56 is a topological embedding with dense image having the following universal property: for every continuous bounded $f \in C_{\mathrm{b}}(X)$, the function $\check{f} \in C_{\mathrm{b}}(\check{X})$ is unique with the property that $f = \check{f} \circ \iota$.*

Proof. By Urysohn's Lemma, the topology on the Tychonoff space X is precisely the initial topology by the family of functions $f \in C_{\mathrm{b}}(X) = F$. The topology on $\check{X}$ is precisely the initial topology by functions $\check{f} \in \check{F}$; since $f = \check{f} \circ \iota$, the topology on X is precisely the pullback topology by ι (i.e., the "subspace topology" if one identifies X with $\iota(X) \subseteq \check{X}$), so ι is an embedding. Since $\iota(X) \subseteq \check{X}$ is dense, the extension $\check{f}$ of f is unique. □

Exercise 43

Show the following more general universal property of $\check{X}$ holds in Corollary 57: any continuous $f : X \to K$ into a Hausdorff compact space K admits a unique extension $\check{f} : \check{X} \to K$ in the sense that $\check{f} \circ \iota = f$.

[*Hint:* Compact Hausdorff spaces are Tychonoff.]

Theorem 56 may be restated in a more structural form. A $V_{\mathrm{alg}}^{|\mathrm{X}\rangle}$-structure

$$\mathfrak{M} = \langle (F, X), (0, +, \cdot r, \cdot)_{r\in\mathbb{R}}, (\|\cdot\|_\infty, \langle \cdot \mid \cdot \rangle) \rangle,$$

will be called an *abstract algebra of bounded functions* if:

- the V_{alg}-reduct of $\mathfrak{M}$ is a commutative algebra,
- for each $x \in X$, the map $\langle \cdot \mid x \rangle : F \to \mathbb{R}$ is a homomorphism of unital vector lattice-algebras and, for all $f \in F$,
$$\sup_{x\in X} |\langle f \mid x \rangle| = \|f\|_\infty \quad (< \infty). \tag{5.1}$$

The following theorem is a structural version of 56 above.

58 Theorem (Stone Representation (structural version))**.** *Let*

$$\mathfrak{M} = \langle (F, X), (0, +, \cdot r, \cdot)_{r\in\mathbb{R}}, (\|\cdot\|_\infty, \langle \cdot \mid \cdot \rangle) \rangle,$$

be an abstract algebra of bounded functions. Then there is a structure-preserving embedding[2] *of* $\mathfrak{M}$ *into*

$$\check{\mathfrak{M}} = \langle (\check{F}, \check{X}), (0, \mathbb{1}, +, \cdot r, \cdot)_{r\in\mathbb{R}}, (\|\cdot\|_\infty, \langle \cdot \mid \cdot \rangle)\rangle,$$

where

1. $\check{X} \subseteq \mathbb{R}^F$ *is the compact Hausdorff space of* $\langle F \mid \cdot\rangle$*-types of* X*, on which* X *embeds densely via the "type map"* $x \mapsto \langle \cdot \mid x\rangle$*;*
2. $\check{F} := \{\check{f} := \pi_f : f \in F\} \subseteq C_{\mathrm{b}}(\check{X})$ *is the set of "coordinate projections" on* $\check{X}$*, endowed with the operations and supremum norm induced from* $C_{\mathrm{b}}(\mathcal{X})$*.*

Thus, the abstract algebra F is represented concretely as an algebra $\check{F}$ of functions on a compact space $\check{X}$.

Proof. This is essentially a restatement of Theorem 56; the same proof applies *mutatis mutandis*. □

Exercise 44

In the definition of abstract algebra of bounded functions above, show that the requirement that the V_{alg}-reduct is commutative may be omitted (i.e., any not-necessarily-commutative abstract algebra of bounded functions is commutative —modulo zero-distance, of course).

Exercise 45

Part I

Give a suitable definition of "abstract normed lattice-algebra of functions" on the vocabulary

$$V_{L^\infty}^{|\mathtt{X}\rangle} = (\{\mathtt{F}, \mathtt{X}\}, \{0, +, (\mathtt{r}\cdot)_{r\in\mathbb{R}}, \cdot, \vee, \wedge\}, \{\mathtt{N}, \langle \cdot \mid \cdot \rangle\})$$

expanding the vocabulary $V_{L^\infty} = (\mathtt{F}, \dots, \mathtt{N})$ of Banach lattice-algebras with a second sort symbol $\mathtt{X}$ and a binary predicate symbol $\langle \cdot \mid \cdot \rangle : \mathtt{F} \times \mathtt{X} \to \mathbb{R}$.

Part II

Show that every abstract algebra F of functions admits a unique expansion to an abstract normed lattice-algebra of functions such that every map $\langle \cdot \mid x\rangle$ respects the lattice operations on F and $\mathbb{R}$.

Part III

State and prove a version of Theorem 58 for abstract normed lattice-algebras of functions.

Exercise 46 Dual-pair expansions of bounded function algebras

Let

$$\mathfrak{M} = \langle (F, X), (0, +, \cdot r, \cdot)_{r\in\mathbb{R}}, (\|\cdot\|_\infty, \langle \cdot \mid \cdot \rangle)\rangle,$$

[2] The structural embedding is not necessarily injective on X in a set-theoretic sense, but is only injective modulo type equality (structural indistinguishability). In order to obtain a set-theoretic injection, points $x, y \in X$ with the same $\langle F \mid \cdot\rangle$-type (i.e., not separated by any function $f \in F$) must be first identified.

be an abstract algebra of bounded functions.

Recall that the pairing $\langle\cdot \mid \cdot\rangle$ on $F \times X$ is linear only in the first argument since X is simply a pointset. Let $\ell^1_X := \ell^1(X) \subseteq \mathbb{R}^X$ be the space of elements $r = (r_x)_{x\in X}$ with $\|r\|_1 := \sum_{x\in X} |r_x| < \infty$, regarded as a normed space $\underline{\ell}^1_X = \langle \ell^1_X, \dots, \|\cdot\|_1\rangle$. Then $\langle\cdot \mid \cdot\rangle$ extends to a bilinear real predicate $\langle\cdot \mid \cdot\rangle' : F \times \ell^1_X \to \mathbb{R}$.

Part I

Show that one obtains a normed dual pair $\langle (F, \ell^1_X), \dots, (\|\cdot\|_\infty, \|\cdot\|_1, \langle\cdot \mid \cdot\rangle')\rangle$ in this manner.

Part II

Elements $\mu \in \ell^1_X$ give *linear* maps $\langle\cdot \mid \mu\rangle : F \to \mathbb{R}$. Show that such maps need not respect the product of F (i.e., need *not* be algebra homomorphisms).

Characterize elements $\mu \in \ell^1_X$ yielding algebra homomorphisms $\langle\cdot \mid \mu\rangle$.

5.3 $L^{1,\infty}$-pairs

There are lessons to learn from Stone's Theorem. To begin, a sizable number of ingredients enter the definition of unital vector lattice-algebra. However, structurally speaking, the "underlying pointset" X is, in effect, structureless: it serves only to index a collection of homomorphisms $F \to \mathbb{R}$. Furthermore, despite the two-sort setting implied and the background role played by ℓ^∞, the "concrete" realization $\check{F} \subseteq \mathcal{C}_b(\check{X})$ only endows $\check{X}$ with a topology and not with a linear structure. To readers familiar with measure theory and the "capital 'L' " spaces $\mathcal{L}^\infty(Y)$ on a measure space $(Y, \mathcal{A}, \mu)$, this should indicate that "little 'ℓ' " spaces $\ell^\infty(X)$ —even when endowed with the sizable structure of a unital vector lattice-algebra— fall short of capturing structural features of their Big Brother $\mathcal{L}^\infty(Y)$.[3]

Vocabulary for $L^{1,\infty}$-pairs The vocabulary $V_{L^{1,\infty}}$ is obtained from the vocabulary $V^\vee_{\langle\cdot|\cdot\rangle}$ for dual lattice pairs as follows. The sort descriptors $\mathtt{X}, \mathtt{X}'$ of $V^\vee_{\langle\cdot|\cdot\rangle}$ are replaced by descriptors $\mathtt{L}^\infty$, $\mathtt{L}^1$ naming the "L^∞-sort" and the "L^1-sort", respectively. In addition, $V_{L^{1,\infty}}$ contains:

- a nullary symbol $\mathtt{1}$ of sort $\mathtt{L}^\infty$;
- a binary function symbol $\cdot : \mathtt{L}^\infty \times \mathtt{L}^\infty \to \mathtt{L}^\infty$ (for the *algebra multiplication on sort* $\mathtt{L}^\infty$);
- another binary function symbol $*\colon \mathtt{L}^\infty \times \mathtt{L}^1 \to \mathtt{L}^1$ (for the *module product of* $\mathtt{L}^\infty$ *on* $\mathtt{L}^1$), and
- a predicate symbol $\mathtt{I} : \mathtt{L}^1 \to \mathbb{R}$ for the *integration functional*.

[3] As an example of efforts directed at "capturing" the essence of spaces $\mathcal{L}^\infty(Y)$, we mention the notion of "M-space": a Banach lattice-algebra whose norm satisfies $\|x + y\| = \max\{\|x\|, \|y\|\}$ when $x \wedge y = 0$. By a classical result of Kakutani, an M-space admits a concrete realization as a sublattice of $\mathcal{C}_b(\check{X})$ for some compact space $\check{X}$. However, an M-space need not be isomorphic to a space $L^\infty(Y)$. In a sense, M-spaces are akin to spaces ℓ^∞, but distinct from spaces L^∞. By contrast (also by results of Kakutani), for $1 \le p < \infty$, spaces $L^p(Y)$ are characterized by natural axioms in the vocabulary of vector lattice-algebras. Our approach captures L^∞ spaces faithfully, essentially by adapting Kakutani's treatment of M-spaces, but in a richer vocabulary of dual "$L^{1,\infty}$-pairs".

An $L^{1,\infty}$-*pair* is a $V_{L^{1,\infty}}$-structure

$$\mathcal{L} = \langle (\mathcal{L}^\infty, \mathcal{L}^1), (0, 0', +, +', (r\cdot, r\cdot')_{r\in\mathbb{R}}, \vee, \vee', \wedge, \wedge', \mathbb{1}, \cdot, *), (\|\cdot\|_\infty, \|\cdot\|_1, \langle \cdot \mid \cdot \rangle, \mathcal{I}) \rangle$$

such that

- the $V^\vee_{\langle\cdot|\cdot\rangle}$-reduct of $\mathcal{L}$ is a dual Banach lattice pair;
- the V^1_{alg}-reduct of $\mathcal{L}$ (with sort $\mathcal{L}^\infty$ as underlying set) is a commutative unital Banach algebra;
- the operation $*\colon \mathcal{L}^\infty \times \mathcal{L}^1 \to \mathcal{L}^1$ endows the Banach space $\mathcal{L}^1$ with the structure of an $\mathcal{L}^\infty$-module (i.e., $*$ is bilinear and $\mathcal{L}^\infty$-homogeneous: $\mathbb{1} * \mu = \mu$ and $(f \cdot g) * \mu = f * (g * \mu)$ hold for $f, g \in \mathcal{L}^\infty$ and $\mu \in \mathcal{L}^1$);
- $\mathcal{I}$ is a positive[4] linear functional on $\mathcal{L}^1$;
- $|\mathcal{I}(\mu)| \le \mathcal{I}(|\mu|) = \|\mu\|_1$ for all $\mu \in \mathcal{L}^1$ (where $|\mu| = (-\mu) \vee' \mu$ as usual);
- $\langle f \mid \mu \rangle = \mathcal{I}(f * \mu)$ for all $f \in \mathcal{L}^\infty$ and $\mu \in \mathcal{L}^1$;
- [*Faithfulness* (also called *sufficiency*)] $\|f\|_\infty = \sup_{\mu\in\mathcal{L}^1[1]} |\langle f \mid \mu \rangle|$ for all $f \in \mathcal{L}^\infty$;
- [*Axioms of idempotents*]
  - For every $\mu \in \mathcal{L}^1$ there is an idempotent $g \in \mathcal{L}^\infty$ (i.e., an element satisfying $g = g^2$ $(= g \cdot g)$) such that $g * \mu = \mu^+$.
  - for every $f \in \mathcal{L}^\infty$ there is an idempotent $g \in \mathcal{L}^\infty$ such that $g \cdot f = f^+$.

59 Remarks. For reasons of clarity, we have used symbols $\cdot$ and $*$ for the operations of multiplication $(f, g) \mapsto f \cdot g$ and $(f, \mu) \mapsto f * \mu$. Per the usual convention, we will typically omit the operation symbols and denote those products simply fg, $f\mu$.

The structural ingredients and axioms for an $L^{1,\infty}$-pair are redundant. For instance, one may dispense with the pairing $\langle \cdot \mid \cdot \rangle$ and define $\langle f \mid \mu \rangle := \mathcal{I}(f\mu)$. One could also dispense with, say, the join operation(s), defining them as the negative of the meet of the negatives.

The motivation for the definition of $L^{1,\infty}$-pair should be clear: while abstracting away the operations of evaluation-at-a-point, we seek to structurally capture other properties of the dual pair (ℓ^1, ℓ^∞) of Banach lattices (more generally, in pairs of lattices $\ell^1_{(X)}$, $\ell^\infty_{(X)}$), including those of the algebra product (pointwise multiplication) of ℓ^∞, and its action on ℓ^1 by pointwise multiplication. While points of X are done away with, their presence is "felt" in the axiom of idempotents which, as we shall see, also implies that $\mathcal{L}^\infty$ is the dual of $\mathcal{L}^1$.

5.4 Representing $L^{1,\infty}$-pairs as lattices of functions and measures

60 Theorem (Representation of $L^{1,\infty}$-pairs *à la* Riesz)**.** *Given an $L^{1,\infty}$-pair $\mathcal{L} = \langle (\mathcal{L}^\infty, \mathcal{L}^1), \dots, (\|\cdot\|_\infty, \|\cdot\|_1, \mathcal{I}) \rangle$ there exist*

[4] I.e., satisfying $\mathcal{I}(\mu^+) \ge 0$ for all $\mu \in \mathcal{L}^\infty$.

- *a compact Haudorff topological space* Ω*;*
- *a unital Banach lattice-algebra* $\mathcal{F} \subseteq \mathcal{C}_{\mathrm{b}}(\Omega)$ *of (bounded) Borel-measurable functions on* Ω*,*
- *a Banach lattice* $\mathcal{M}$ *of regular finite measures on* Ω *(under the norm* $|||\cdot|||$ *of total variation),*
- *a* $V_{L^{1,\infty}}$*-isomorphism* $\mathcal{L}^\infty \times \mathcal{L}^1 \to \mathcal{F} \times \mathcal{M} : (f, \mu) \mapsto (\tilde{f}, \tilde{\mu})$ *(i.e., a bijection preserving all operations and predicates of the vocabulary* $V_{L^{1,\infty}}$*), where*
  - *the operation* $\mathrm{I}^{\mathcal{M}}$ *is classical integration* $\tilde{\mu} \mapsto \int \mathrm{d}\tilde{\mu}$ *(*$= \int 1 \,\mathrm{d}\tilde{\mu}$*);*
  - *the bilinear pairing on* $\mathcal{F} \times \mathcal{M}$ *is* $(\tilde{f}, \tilde{\mu}) \mapsto \int \tilde{f} \,\mathrm{d}\tilde{\mu}$ *(integration of functions with respect to* μ*);*
  - *the* $\mathcal{F}$*-module operation* $(\tilde{f}, \tilde{\mu}) \mapsto \tilde{f} * \tilde{\mu}$ *on* $\mathcal{M}$ *is adjoint to the multiplication of* $\mathcal{F}$ *on itself, i.e.,* $\tilde{f} * \tilde{\mu}$ *is the (regular) measure with associated integration functional* $\tilde{g} \mapsto \int (\tilde{f}\tilde{g}) \,\mathrm{d}\tilde{\mu}$*.*

Proof outline. Let $\mathcal{L}^{\infty+} \subseteq \mathcal{L}^\infty$ be the "positive cone" of elements $f \geq 0$. By the Axiom of Idempotents, every ideal $J \subseteq \mathcal{L}^\infty$ is generated by its subset $J^+ := J \cap \mathcal{L}^{\infty+}$ of positive elements, and is therefore closed under lattice operations. Since $\mathcal{L}^{\infty+}$ is closed under products, it follows that the "lattice completion" $J^\wedge := \{f \in \mathcal{L}^\infty : |f| \leq g \text{ for some } g \in J\}$ is also an ideal; moreover, $J^\wedge \ni \mathbb{1}$ only if $\mathbb{1} \in J$ (again, by the Axiom of Idempotents). It follows that a maximal ideal $J \subseteq \mathcal{L}^\infty$ is a (non-unital) sub-lattice algebra of $\mathcal{L}^\infty$, monotone in the sense that $J^\wedge = J$, whose quotient field $\mathbb{F} := \mathcal{L}^\infty / J \supseteq \mathbb{R}\mathbb{1} \simeq \mathbb{R}$ is itself a lattice under induced operations, and therefore $\mathbb{F} = \mathbb{R}\mathbb{1} \simeq \mathbb{R}$.

Thus, every maximal ideal $J \subseteq \mathcal{L}^\infty$ is of the form $J = \ker(\xi)$ where $\xi : \mathcal{L}^\infty \to \mathbb{R}$ is a homomorphism of real unital Banach lattice-algebras. Conversely, the kernel of every such homomorphism is a maximal ideal (hence a monotone sub-lattice-algebra). Let $\Omega \subseteq (\mathcal{L}^\infty)^*$ be the set of unital lattice-algebra homomorphisms $\mathcal{L}^\infty \to \mathbb{R}$. Since each $\xi \in \Omega$ preserves lattice operations, if $f \geq 0$, then $\xi(f) \geq 0$, so Ω is included in the positive cone of $(\mathcal{L}^\infty)^*$ (recall that $(\mathcal{L}^\infty)^*$ is itself a Banach lattice with operations induced from those of $\mathcal{L}^\infty$). Since $|f| \leq \|f\|_\infty \cdot \mathbb{1}$ for all $f \in \mathcal{L}^\infty$ and $\xi \in \Omega$ is a unital monotone homomorphism, we see that $|\xi(f)| \leq \xi(|f|) \leq \|f\|_\infty$, so $\|\xi\|^* \leq 1 = \xi(\mathbb{1})$, so $\|\xi\|^* = 1$; in particular, Ω is included in the "unit sphere" $S_{(\mathcal{L}^\infty)^*} := \{\xi \in (\mathcal{L}^\infty)^* : \|\xi\| = 1\} \subseteq (\mathcal{L}^\infty)^*$; more precisely, Ω is included in the positive sector $S^+_{(\mathcal{L}^\infty)^*} := S_{(\mathcal{L}^\infty)^*} \cap ((\mathcal{L}^\infty)^*)^+$.

The "evaluation pairing" $\mathcal{L}^\infty \times (\mathcal{L}^\infty)^* \to \mathbb{R}$ will be denoted $\langle\!\langle f \mid \xi \rangle\!\rangle := \xi(f)$. We topologize $\Omega \subseteq (\mathcal{L}^\infty)^*$ in the weak-* topology. Much as in the proof of Alaoglu's Theorem, one sees that any accumulation point ξ of realized types $\langle\!\langle \cdot \mid \xi \rangle\!\rangle$ is also a realized type $\xi = \mathrm{tp}_{\langle \mathcal{L}^\infty |}(\xi')$ by some $\xi' \in \Omega$ (the additional ingredients are that ξ must preserve not just the linear, but also the unital lattice-algebra operations). Therefore, Ω *is weak-** *compact Hausdorff.*

The pairing $\langle \cdot \mid \cdot \rangle : \mathcal{L}^\infty \times \mathcal{L}^1 \to \mathbb{R}$ induces a Banach vector-lattice embedding $\mathcal{L}^1 \to (\mathcal{L}^\infty)^* : \mu \mapsto \langle \cdot \mid \mu \rangle$, which is isometric by the Axiom of Idempotents.

Every $f \in \mathcal{L}^\infty$ induces a functional $\langle\!\langle f \mid \cdot \rangle\!\rangle$ on $(\mathcal{L}^\infty)^*$ of norm $\|f\|_\infty$ and, by restriction, a continuous function on Ω pointwise bounded by $\|f\|_\infty$. Define $\|f\|_\Omega := \sup_{\xi \in \Omega} |\langle\!\langle f \mid \xi \rangle\!\rangle|$ ($\leq \|f\|_\infty$).

Claim: $\|f\|_\Omega = \|f\|_\infty$. By a standard reduction it suffices to consider $f \geq 0$ with, say, $\|f\|_\infty = 1$. For $0 < r < 1$, let $h := f - r\mathbb{1}$. If $g = g^2$ is an idempotent with $gh = h^+$, then $g \neq 0$ (since $r < 1 = \|f\|_\infty$); therefore, the ideal $J = (\mathbb{1} - g)\mathcal{L}^\infty$ is proper, and any $\xi \in \Omega$ whose kernel includes J satisfies $\langle\!\langle f \mid \xi \rangle\!\rangle \geq r$.

It follows from the Claim that the notions of convergence in $\mathcal{L}^\infty$ in $\|\cdot\|_\infty$-norm, and of uniform convergence (i.e., in $\|\cdot\|_\Omega$-norm) in $C_\mathrm{b}(\Omega)$ coincide. The topology on Ω is, by definition, initial under maps $\langle\!\langle f \mid \cdot\rangle\!\rangle$. Since $\mathcal{L}^\infty$ is a Banach lattice-algebra represented as an algebra of functions $\langle\!\langle f \mid \cdot\rangle\!\rangle \in C_\mathrm{b}(\Omega)$, it follows from the Stone-Weierstrass Theorem that $f \mapsto \tilde{f} := \langle\!\langle f \mid \cdot\rangle\!\rangle$ is a Banach lattice-algebra isometry $\mathcal{L}^\infty \simeq C_\mathrm{b}(\Omega)$. Henceforth, let $\mathcal{F} := C_\mathrm{b}(\Omega)$ be the classical Banach unital lattice-algebra of continuous (bounded) functions on Ω under the supremum norm (denoted $\|\cdot\|_\infty$ by an abuse of notation).

Each element $\mu \in \mathcal{L}^1$ gives a linear functional $\mu^* : \mathcal{F} \to \mathbb{R} : \tilde{f} \mapsto \langle f \mid \mu\rangle$ of norm $\|\mu^*\|^* = \|\mu\|_1$ (by Axiom of Idempotents, $g\mu = \mu^+$ for some idempotent g, so $\|\mu^*\|^* = \|(\mu^*)^+\|^* + \|(\mu^*)^-\|^* = \mu^*\big(\tilde{g}\tilde{\mathbb{1}} - (\tilde{\mathbb{1}} - \tilde{g})\big)$ where $\left\|\tilde{g}\tilde{\mathbb{1}} - (\tilde{\mathbb{1}} - \tilde{g})\right\|_\infty = 1$). The map $\mu \mapsto \mu^*$ is compatible with the lattice structure since $f \mapsto \tilde{f}$ and the pairing $\langle\cdot \mid \cdot\rangle$ are. By the classical Riesz Representation Theorem, μ^* is realized as the map $f \mapsto \int f\,\mathrm{d}\tilde{\mu}$ for some regular Borel measure $\tilde{\mu}$ on Ω of total variation $|||\tilde{\mu}||| = \|\mu^*\|^* = \|\mu\|_1$. The space $\mathcal{M} = \{\tilde{\mu} : \mu \in \mathcal{L}^1\}$ endowed with the natural Banach lattice-algebra structure and action of $\mathcal{F} = C_\mathrm{b}(\Omega)$ completes the list of ingredients. The preceding discussion shows that $(f, \mu) \mapsto (\tilde{f}, \tilde{\mu})$ is an isometric isomorphism. $\square$