Lemma If $\{e_n\}$ is an o.n.b. for $L^2([a,b])$ then $\{e_n \otimes e_m(x,y)\}_{n,m}$ is an o.n.b. for $L^2([a,b]\times[a,b])$.

Proof $\{e_n \otimes e_m\}$ is an o.n.s. Indeed,

$$(e_n\otimes e_m, e_{n'}\otimes e_{m'})_{L^2([a,b]\times[a,b])} = (e_n, e_{n'})_{L^2([a,b])} (e_m, e_{m'})_{L^2([a,b])}$$

$$= \delta_{nn'}\delta_{mm'}$$

$\{e_n\otimes e_m\}_{n,m\in\mathbb{N}}$ is a complete sequence.

Equivalently, $\forall F \in L^2([a,b]\times[a,b])$,

$$\|F\|_2 = \sum_{n,m} |\langle F, e_n\otimes e_m\rangle|^2.$$

$$\int_a^b\int_a^b |F(x,y)|^2 dx dy = \int_a^b \left(\int_a^b |F(x,y)|^2 dx\right) dy$$

$$\overset{\text{Parseval Thrm}}{=} \int_a^b \sum_{n\in\mathbb{N}} |\langle F(\cdot,y), e_n\rangle|^2 dy$$

$$\overset{\text{Tonelli}}{=} \sum_{n\in\mathbb{N}} \int_a^b |\langle F(\cdot,y), e_n\rangle|^2 dy$$

$G_n(y) := \langle F(\cdot,y), e_n\rangle \in L^2([a,b])$:

$$\int_a^b |G_n(y)|^2 dy = \int_a^b |\langle F(\cdot,y), e_n\rangle|^2 dy$$

$$= \int_a^b \left| \int_a^b F(x,y) e_n(x)\, dx \right|^2 dy$$

$$\overset{C-S}{\leq} \int_a^b \|F(\cdot, y)\|^2_{L^2([a,b])} \|e_n\|^2_{L^2([a,b])}\, dy$$

$$= \int_a^b \int_a^b |F(x,y)|^2\, dx\, dy$$

$$= \|F\|_2 < \infty$$

Hence

$$\|F\|_2^2 = \sum_n \|G_n\|^2 \overset{\text{Parseval's Thm}}{=} \sum_n \sum_m |\langle G_n, e_m \rangle|^2$$

$$= \sum_{n,m} \left| \langle \langle F(\cdot, y), e_n \rangle_{L^2([a,b])}, e_m \rangle \right|^2$$

$$= \sum_{n,m} |\langle F, e_n \otimes e_m \rangle|^2$$