# Distribution theory,

# Fourier and Laplace transform

## Analysis (Course A)

## M.Sc. in Stochastics and Data Science

# 1 Distributions on open subsets of $\mathbb{R}^n$

**What's it about?** Distributions are continuous functionals on certain function spaces. They generalize the concept of function in the sense that any locally integrable function can be identified with a distribution and that many standard operations on functions extend to distributions. For this reason, distributions are also called *generalized functions*. In a certain sense, distributions even behave better than functions; for example, any distribution can be differentiated as many times as one wishes and the order of derivatives does not play any role. This makes distributions a natural environment for the investigation of *partial differential equations*.

In the following, $\Omega \subseteq \mathbb{R}^n$ is an open set. We write $K \subset\subset \Omega$, if $K$ is compact and $K \subset \Omega$.

## 1.1 Test functions – the space $\mathscr{D}(\Omega)$

The support of a continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is the set

$$\operatorname{supp} f := \text{closure of the set } \{x \in \mathbb{R}^n \mid f(x) \neq 0\}.$$

**1.1 Example** *a)* $f(x) = \max\{1 - x^2, 0\} \Rightarrow \operatorname{supp} f = [-1, 1]$.
*b)* $f(x) = \sin x \Rightarrow \operatorname{supp} f = \mathbb{R}$.

**1.2 Definition** (Space of test-functions) *We define*

$$\mathscr{D}(\Omega) = \left\{\phi \in \mathscr{C}^\infty(\mathbb{R}^n) \mid \operatorname{supp}\phi \subset\subset \Omega\right\}.$$

$\mathscr{D}(\Omega)$ is a subspace of the vector space of all functions $\mathbb{R}^n \to \mathbb{C}$, because

$$\operatorname{supp}(\phi + \psi) \subseteq \operatorname{supp}\phi \cup \operatorname{supp}\psi, \qquad \operatorname{supp}(\lambda\phi) \subseteq \operatorname{supp}\phi \quad (\lambda \in \mathbb{C}).$$

**1.3 Example** *Let* $\rho(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & : |x| < 1 \\ 0 & : |x| \geqslant 1 \end{cases}.$

*Then* $\rho \in \mathscr{D}(\mathbb{R}^n)$ *with* $\operatorname{supp}\rho = \{x \mid |x| \leqslant 1\}$.
*If* $x_0 \in \Omega$ *and* $r > 0$ *such that* $\overline{B_r(x_0)} \subseteq \Omega$, *then* $\phi(x) := \rho((x - x_0)/r)$ *belongs to* $\mathscr{D}(\Omega)$ *with* $\operatorname{supp}\phi = \overline{B_r(x_0)}$. *In particular,* $\mathscr{D}(\Omega) \neq \{0\}$.

For the following definition recall the multi-index notation for partial derivatives: If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ then

$$|\alpha| = |\alpha_1| + \ldots + |\alpha_n|, \qquad \partial_x^\alpha \phi(x) = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} \phi(x).$$

Note that for $C^\infty$-functions the order of application of the partial derivatives is irrelevant, due to the theorem of Schwarz.

**1.4 Definition** *On $\mathscr{D}(\Omega)$ define the norms $\|\cdot\|_j$, $j = 0, 1, 2, \ldots$ by*

$$\|\phi\|_j := \max_{x\in\mathbb{R}^n,\, |\alpha|\leqslant j} |\partial_x^\alpha \phi(x)|.$$

**1.5 Definition** *We say that a sequence $(\phi_k)_k \subseteq \mathscr{D}(\Omega)$ converges to $\phi \in \mathscr{D}(\Omega)$ if*

i) *there exists a set $K \subset\subset \Omega$ such that $\operatorname{supp}\phi_k \subseteq K$ for every $k$,*

ii) $\|\phi_k - \phi\|_j \xrightarrow{k\to+\infty} 0$ *for all $j = 0, 1, 2, \ldots$*

*In this case we write $\phi_k \xrightarrow{k\to+\infty} \phi$ or $\lim\limits_{k\to+\infty} \phi_k = \phi$. Note that then $\operatorname{supp}\phi \subseteq K$, too.*

Note that ii) is equivalent to the uniform convergence in $\mathbb{R}^n$ of $\partial_x^\alpha \phi_j$ to $\partial_x^\alpha \phi$ for every $\alpha \in \mathbb{N}_0^n$.

**1.6 Lemma** *Let $\beta \in \mathbb{N}_0^n$. If $\phi_k \xrightarrow{k\to+\infty} \phi$ then $\partial^\beta \phi_k \xrightarrow{k\to+\infty} \partial^\beta\phi$.*

PROOF: Use notation of Definition 1.5. Let $L = |\beta|$.

i) $\phi_k = 0$ on $\Omega\backslash K \Rightarrow \partial^\beta\phi_k = 0$ on $\Omega\backslash K \Rightarrow \operatorname{supp}\partial^\beta\phi_k \subseteq K$ for all $k$.

ii) $\|\partial^\beta\phi_k - \partial^\beta\phi\|_j \leqslant \|\phi_k - \phi\|_{j+L} \xrightarrow{k\to+\infty} 0$ for all $j$.

This finishes the proof. ∎

## 1.2 Distributions – the space $\mathscr{D}'(\Omega)$

**1.7 Definition** *A distribution (on $\Omega$) is any linear map $T : \mathscr{D}(\Omega) \to \mathbb{C}$ which is continuous in the following sense: For every convergent sequence $(\phi_k)_k \subseteq \mathscr{D}(\Omega)$ holds*

$$\lim_{k\to+\infty} T(\phi_k) = T\Big(\lim_{k\to+\infty}\phi_k\Big)$$

*(in short: $\phi_k \to \phi$ in $\mathscr{D}(\Omega)$ implies $T(\phi_k) \to T(\phi)$). The set of all distributions on $\Omega$ is denoted by $\mathscr{D}'(\Omega)$.*

$\mathscr{D}'(\Omega)$ is a subspace of the vector space of all linear maps $\mathscr{D}(\Omega) \to \mathbb{C}$.

**1.8 Theorem** (control estimates) *For a linear $T : \mathscr{D}(\Omega) \to \mathbb{C}$ the following are equivalent:*

*a)* $T \in \mathscr{D}'(\Omega)$

*b) For every* $K \subset\subset \Omega$ *there exist a* $C = C(K) \geqslant 0$ *and a* $j = j(K) \in \mathbb{N}$ *such that*

$$|T(\phi)| \leqslant C\|\phi\|_j = C \max_{x\in\mathbb{R}^n,\, |\alpha|\leqslant j} |\partial^\alpha \phi(x)| \qquad \forall \, {\phi\in\mathscr{D}(\Omega), \atop \operatorname{supp}\phi\subseteq K}.$$

Proof: b) ⇒ *a*): If $\phi_k \to \phi$ in $\mathscr{D}(\Omega)$ as in Definition 1.5. Then

$$|T(\phi_k) - T(\phi)| = |T(\phi_k - \phi)| \leqslant C\|\phi_k - \phi\|_j \xrightarrow{k\to+\infty} 0.$$

a) ⇒ *b*): Assume that b) does not hold. Thus there is a $K \subset\subset \Omega$ and a sequence $(\phi_k)_k$ with $\operatorname{supp}\phi_k \subseteq K$ such that

$$|T(\phi_k)| > k\|\phi_k\|_k \qquad \forall\, k = 1, 2, \ldots$$

Without loss of generality $T(\phi_k) = 1$ for every $k$ (otherwise substitute $\phi_k$ by $\psi_k := \phi_k/T(\phi_k)$). Then, given an arbitrary $j \in \mathbb{N}$,

$$\|\phi_k\|_j \overset{k\geqslant j}{\leqslant} \|\phi_k\|_k < \frac{1}{k} \xrightarrow{k\to+\infty} 0.$$

Hence $\phi_k \to 0$ in $\mathscr{D}(\Omega)$, but $T(\phi_k) = 1$. This is a contradiction. ■

**Remark** (and definition) If $j$ in Theorem 1.8.b) can be taken independent of $K$, we say that $T$ has finite order. Then the smallest such $j$ is called the order of $T$.

**1.9 Example** (Functions as distributions) *Write* $u \in L^1_{\mathrm{loc}}(\Omega)$ *if* $u$ *is measurable on* $\Omega$ *and*

$$\int_K |u(x)|\,dx < +\infty \qquad \forall\, K \subset\subset \Omega.$$

*For such an* $u$ *define*

$$T_u(\phi) := \int_\Omega u(x)\phi(x)\,dx, \qquad \phi \in \mathscr{D}(\Omega).$$

*This defines a distribution* $T_u \in \mathscr{D}'(\Omega)$*. In fact, for* $K \subset\subset \Omega$*,*

$$|T_u(\phi)| \leqslant \int_K |u(x)||\phi(x)|\,dx \leqslant \max_{x\in\mathbb{R}^n} |\phi(x)| \int_K |u(x)|\,dx = C_K\|\phi\|_0 \qquad \forall\, {\phi\in\mathscr{D}(\Omega), \atop \operatorname{supp}\phi\subseteq K}$$

*(in particular,* $T_u$ *has order* 0*). Distributions of this form are called regular distributions, the function* $u$ *is called the density of* $T_u$*. One can show that*

$$T_u = T_v \iff u = v \text{ almost everywhere in } \Omega,$$

*i.e., the density of a regular distribution is uniquely determined (almost everywhere).*

Since the map $u \mapsto T_u$ gives a bijective correspondence between densities and regular distributions, often one writes simply $u$ instead of $T_u$.

**1.10 Example** ($\delta$-distribution) *Define* $\delta : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ *by*

$$\boxed{\delta(\phi) = \phi(0), \qquad \phi \in \mathscr{D}(\mathbb{R}^n).}$$

*Then* $\delta$ *is a distribution of order* 0*, since*

$$|\delta(\phi)| = |\phi(0)| \leqslant \|\phi\|_0 \qquad \forall\, \phi \in \mathscr{D}(\mathbb{R}^n).$$

$\delta$ *is not a regular distribution: Assume there would exist a* $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ *with* $\delta = T_u$*, i.e.,*

$$\phi(0) = \int_{\mathbb{R}^n} u(x)\phi(x)\,dx \qquad \forall\, \phi \in \mathscr{D}(\mathbb{R}^n).$$

*Let* $\psi = e\rho$ *with* $\rho$ *as in Example 1.3. Then* $\psi \in \mathscr{D}(\mathbb{R}^n)$ *with* $\psi(0) = 1$ *and* $\operatorname{supp}\psi \subseteq B$ *with some ball* $B$ *centered in* 0*. Define* $\phi_k \in \mathscr{D}(\mathbb{R}^n)$ *by* $\phi_k(x) = \psi(kx)$*. Then*

$$1 = \psi(0) = \phi_k(0) = \int_{\mathbb{R}^n} u(x)\phi_k(x)\,dx = \int_B u(x)\phi_k(x)\,dx \xrightarrow{k\to+\infty} 0$$

*due to the dominated convergence theorem (note that* $u(x)\phi_k(x) \xrightarrow{k\to+\infty} 0$ *for each* $x \neq 0$ *and that* $|u(x)\phi_k(x)| \leqslant \|\phi\|_\infty |u(x)| \in L^1(B)$*). Hence* $1 = 0$*, which is a contradiction.*

Similarly, given $x_0 \in \mathbb{R}^n$, one defines the delta-distribution $\delta_{x_0}$ centered in $x_0$ by

$$\delta_{x_0}(\phi) := \phi(x_0), \qquad \phi \in \mathscr{D}(\mathbb{R}^n);$$

again it is a distribution of order 0 which is not regular.

**1.11 Example** *The function* $x \mapsto \frac{1}{x}$ *does not belong to* $L^1_{\text{loc}}(\mathbb{R})$*, hence does not define a regular distribution on* $\mathbb{R}$*. However,*

$$\boxed{T(\phi) := \lim_{\varepsilon\to 0+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{\phi(x)}{x}\,dx, \qquad \phi \in \mathscr{D}(\mathbb{R})}$$

*defines* $T \in \mathscr{D}'(\mathbb{R})$*. One writes also* $\text{pv-}\dfrac{1}{x} := T$ *(pv stands for principal value).*

*In fact, by Taylor expansion,* $\phi(x) = \phi(0) + x r_\phi(x)$ *with* $r_\phi \in \mathscr{C}^\infty(\mathbb{R})$*. Then*

$$T(\phi) = \int_{\mathbb{R}\setminus[-1,1]} \frac{\phi(x)}{x}\,dx + \lim_{\varepsilon\to 0+} \int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{\phi(x)}{x}\,dx = T_u(\phi) + \int_{-1}^{1} r_\phi(x)\,dx$$

*since*

$$\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{\phi(0)}{x}\,dx = \phi(0)\Big(\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx\Big) = 0,$$

*and where*

$$u(x) := \begin{cases} 0 & : |x| \leqslant 1 \\ 1/x & : |x| > 1 \end{cases} \in L^1_{\text{loc}}(\mathbb{R})$$

*defines a regular distribution. Moreover,*

$$|r_\phi(x)| = \left|\frac{\phi(x)-\phi(0)}{x}\right| = |\phi'(\xi_x)| \leqslant \max_{\xi\in\mathbb{R}^n} |\phi'(\xi)| \leqslant \|\phi\|_1, \qquad \phi \in \mathscr{D}(\mathbb{R});$$

*hence* $S \in \mathscr{D}'(\mathbb{R})$ *con* $|S(\phi)| \leqslant \|\phi\|_1$ *for all* $\phi$*. Thus* $T = T_u + S$ *is a distribution of order* $1$*.*

## 1.3 Multiplication of distributions with smooth functions

Let $u \in L^1_{loc}(\Omega)$ and $a \in \mathscr{C}^\infty(\Omega)$. Then $T_u$ and $T_{au}$ are regular distributions and

$$T_{au}(\phi) = \int_\Omega a(x)u(x)\phi(x)\,dx = T_u(a\phi).$$

Note that on the right-hand side we can substitute $T_u$ by an arbitrary distribution $T$.

**1.12 Theorem** (and definition) *Let* $a \in \mathscr{C}^\infty(\Omega)$ *and* $T \in \mathscr{D}'(\Omega)$*. Then*

$$(aT)(\phi) = T(a\phi), \qquad \phi \in \mathscr{D}(\Omega),$$

*defines a distribution* $aT \in \mathscr{D}'(\Omega)$*.*

For this result one needs to verify that $\phi_k \to \phi$ in $\mathscr{D}(\Omega)$ implies $a\phi_k \to a\phi$ in $\mathscr{D}(\Omega)$. The proof is based on Theorem 1.7 in combination with the product rule for the derivatives $\partial^\alpha(a\phi)$; we skip the details.

## 1.4 Differentiation of distributions

Let $u \in \mathscr{C}^1(\mathbb{R})$. Then $T_u$ and $T_{u'}$ are regular distributions. It is natural to call $T_{u'}$ the derivative of $T_u$. Observe that

$$\begin{aligned} T_{u'}(\phi) &= \int_{-\infty}^{+\infty} u'(x)\phi(x)\,dx \\ &= \underbrace{u(x)\phi(x)\Big|_{x=-\infty}^{x=+\infty}}_{=0} - \int_{-\infty}^{+\infty} u(x)\phi'(x)\,dx = -T_u(\phi') \qquad \forall\ \phi \in \mathscr{D}(\mathbb{R}). \end{aligned}$$

While $T_{u'}(\phi)$ makes sense only if $u$ is differentiable, the expression $-T_u(\phi')$ makes sense for any function $u$ and, much more, we can substitute $T_u$ by an arbitrary distribution $T$. This leads us to define $T' : \mathscr{D}(\mathbb{R}) \to \mathbb{C}$ by

$$T'(\phi) = -T(\phi'), \qquad \phi \in \mathscr{D}(\mathbb{R}).$$

This idea extends to partial derivatives and distributions on $\Omega$ in the following way:

**1.13 Theorem** (and definition) *Let $T \in \mathscr{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. Then*

$$(\partial^\alpha T)(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi), \qquad \phi \in \mathscr{D}(\Omega)$$

*defines the distribution $\partial^\alpha T \in \mathscr{D}'(\Omega)$.*

The proof that $\partial^\alpha T$ is a distribution is one of the homeworks.

Thus every distribution has derivatives of arbitrary order. Note that if $T = T_u$ is a regular distribution with $u \in C^N(\Omega)$ then $\partial^\alpha T_u = T_{\partial^\alpha u}$ for every $\alpha$ with $|\alpha| \leqslant N$.

**1.14 Example** *Let $h(x) = \begin{cases} 0 & : x < 0 \\ 1 & : x > 0 \end{cases}$ be the so-called Heavyside-function. It defines the regular distribution $T_h$. Then, for every $\phi \in \mathscr{D}(\mathbb{R})$,*

$$\begin{aligned} (T_h)'(\phi) &= -T_h(\phi') = -\int_{-\infty}^{\infty} h(x)\phi'(x)\,dx \\ &= -\int_0^\infty \phi'(x)\,dx = -\phi(x)\Big|_{x=0}^{x=+\infty} = \phi(0) = \delta(\phi). \end{aligned}$$

*Thus the derivative of $T_h$ coincides with the $\delta$-distribution.*

**1.15 Theorem** *Let $u \in L^1_{\text{loc}}(\mathbb{R})$ be of the form*

$$u(x) = \begin{cases} v(x) & : x > x_0 \\ w(x) & : x < x_0 \end{cases}, \qquad v \in \mathscr{C}^1([x_0, +\infty)), \quad w \in \mathscr{C}^1((-\infty, x_0]),$$

*(it does not matter how $u$ is defined in $x = x_0$). Then*

$$(T_u)' = T_{\widetilde{u}} + \big(v(x_0) - w(x_0)\big)\delta_{x_0}, \qquad \widetilde{u}(x) := \begin{cases} v'(x) & : x > x_0 \\ w'(x) & : x < x_0 \end{cases}.$$

*Note that*

$$v(x_0) - w(x_0) = u(x_0^+) - u(x_0^-) = \lim_{x \to x_0^+} u(x) - \lim_{x \to x_0^-} u(x)$$

*is the "height of the jump" that $u$ makes in $x_0$.*

The proof of the previous theorem is one of the homeworks. The theorem easily extends to functions with more than one point of discontinuity:

**1.16 Example** *Let $u(x) = |x^2 - 1| = \begin{cases} 1 - x^2 & : -1 < x < 1 \\ x^2 - 1 & : |x| > 1 \end{cases}$.*

*First derivative:* $(T_u)' = T_{\widetilde{u}}, \qquad \widetilde{u}(x) = \begin{cases} -2x & : -1 < x < 1 \\ 2x & : |x| > 1 \end{cases}$

*Second derivative:* $(T_u)'' = (T_{\tilde{u}})' = T_{\tilde{\tilde{u}}} + 4\delta_1 + 4\delta_{-1}$, $\qquad \tilde{\tilde{u}}(x) = \begin{cases} -2 & : -1 < x < 1 \\ 2 & : |x| > 1 \end{cases}$

*Third derivative:* $(T_u)''' = (T_{\tilde{\tilde{u}}})' + 4\delta_1' - 4\delta_{-1}' = 4\delta_1 - 4\delta_{-1} + 4\delta_1' + 4\delta_{-1}'$.

## 1.5 Convolution

For (suitable) functions $f, g : \mathbb{R}^n \to \mathbb{C}$ define the convolution

$$f * g : \mathbb{R}^n \longrightarrow \mathbb{C}, \qquad (f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy.$$

**1.17 Theorem** *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$. If $r$ satisfies $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ then $f * g \in L^r(\mathbb{R}^n)$,*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

*In particular: If $f, g \in L^1(\mathbb{R}^n)$ then $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_{L^1} \leqslant \|f\|_{L^1}\|g\|_{L^1}$.*

Proof: Let us only consider the case $p = q = r = 1$. Then

$$\begin{aligned}\int |(f * g)(x)|\,dx &= \int \Big| \int f(x-y)g(y)\,dy \Big|\,dx \leqslant \iint |f(x-y)||g(y)|\,dydx \\ &= \iint |f(x-y)||g(y)|\,dxdy = \int \Big( \int |f(x-y)|\,dx \Big) |g(y)|\,dy \\ &= \int \Big( \int |f(z)|\,dz \Big) |g(y)|\,dy = \|f\|_{L^1}\|g\|_{L^1},\end{aligned}$$

where interchanging the order of integration is justified by Fubini's theorem. ■

Let $T_u \in \mathscr{D}'(\mathbb{R}^n)$ be a regular distribution and $\phi \in \mathscr{D}(\mathbb{R}^n)$. Then

$$(u * \phi)(x) = \int_{\mathbb{R}^n} u(y)\phi(x-y)\,dy = T_u(\phi(x - \cdot)), \qquad x \in \mathbb{R}^n.$$

This observation leads to the following:

**1.18 Theorem** (and definition) *Let $T \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$. Then*

$$f(x) := T(\phi(x - \cdot)), \qquad x \in \mathbb{R}^n,$$

*defines a function $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ with*

$$\partial^\alpha f(x) = (\partial^\alpha T)(\phi(x - \cdot)) = T((\partial^\alpha \phi)(x - \cdot)).$$

*We write $T * \phi := f$ and call $T * \phi$ the convolution of $T$ with $\phi$.*

**1.19 Example** *$\delta * \phi = \phi$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$, since*

$$(\delta * \phi)(x) = \delta(\phi(x - \cdot)) = \phi(x - 0) = \phi(x) \qquad \forall\, x \in \mathbb{R}^n.$$

One can also define the convolution of two distributions, when at least one of them has compact support – we shall not enter into details here. You may try to guess the definition by rewriting $T_{u*v}(\phi)$ in terms of $T_u$ and $T_v$ and where $T_u$ and $T_v$ can be substituted by general distributions.

## 1.6 Distributions and partial differential equations

A differential operator $A = \sum_{|\alpha| \leqslant m} a_\alpha \partial_x^\alpha$ with constant coefficients $a_\alpha \in \mathbb{C}$ induces a map

$$A : \mathscr{D}'(\mathbb{R}^n) \longrightarrow \mathscr{D}'(\mathbb{R}^n), \qquad AT = \sum_{|\alpha| \leqslant m} a_\alpha \partial^\alpha T.$$

Note that, by definition of the distributional derivative,

$$AT(\phi) = T(A^t \phi), \qquad A^t := \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} a_\alpha \partial_x^\alpha.$$

Given a distribution $S \in \mathscr{D}'(\mathbb{R}^n)$ we may ask whether there exists a solution $T \in \mathscr{D}'(\mathbb{R}^n)$ of the partial differential equation $AT = S$, i.e.,

$$T(A^t \phi) = S(\phi) \qquad \forall\, \phi \in \mathscr{D}(\mathbb{R}^n).$$

**1.20 Example** *The Laplacian or Laplace operator on $\mathbb{R}^n$ is $\Delta := \partial_{x_1}^2 + \ldots + \partial_{x_1}^2$. Note that $\Delta^t = \Delta$.*

**1.21 Definition** *A distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ is called a fundamental solution of $A$ if $AE = \delta$.*

The importance of the fundamental solution lies in the following: Given $\phi \in \mathscr{D}(\mathbb{R}^n)$, let $u := E * \phi$. Then, by Theorem 1.18 and Example 1.19, $u \in \mathscr{C}^\infty(\mathbb{R}^n)$ and

$$Au = A(E * \phi) = (AE) * \phi = \delta * \phi = \phi.$$

In other words,

$$u = E * \phi \text{ is a solution of the pde } Au = \phi.$$

**1.22 Theorem** (Malgrange-Ehrenpreis) *Every differential operator $A \neq 0$ with constant coefficients has a fundamental solution.*

**1.23 Example** *The Laplacian* $\Delta = \partial_1^2 + \ldots + \partial_n^2$ *has fundamental solution* $E = T_e$ *with*

$$e(x) = \frac{1}{2\pi} \ln|x| \quad (n = 2), \qquad e(x) = \frac{\Gamma(n/2)}{(2-n)2\pi^{n/2}} \frac{1}{|x|^{n-2}} \quad (n \geqslant 3).$$

*Then, for* $\phi \in \mathscr{D}(\mathbb{R}^n)$*, a solution of* $\Delta u = \phi$ *is the function* $u = e * \phi$*, i.e.,*

$$\boxed{u(x) = \frac{\Gamma(n/2)}{(2-n)2\pi^{n/2}} \int_{\mathbb{R}^n} \frac{\phi(x)}{|x-y|^{n-2}}\, dy \qquad (n \geqslant 3).}$$

**1.24 Example** *Let* $A = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$*,* $a \neq 0$*, be a second order differential operator on* $\mathbb{R}$*. Let* $v \in \mathscr{C}^\infty(\mathbb{R})$ *be the unique solution of the homogeneous initial value problem*

$$av'' + bv' + cv = 0, \qquad v(0) = 0, \quad v'(0) = 1/a.$$

*Define* $u$ *on* $\mathbb{R}$ *by* $u(x) = \begin{cases} v(x) & : x > 0 \\ 0 & : x < 0 \end{cases}$*. Then the regular distribution* $T_u$ *is a fundamental solution of* $A$*.*

Proof: Apply Theorem 1.15. Since $u$ is continuous in $x_0 = 0$,

$$(T_u)' = T_{\tilde{u}}, \qquad \tilde{u}(x) = \begin{cases} v'(x) & : x > 0 \\ 0 & : x < 0 \end{cases}.$$

Now $\tilde{u}$ has a jump of height $1/a$ in $x_0 = 0$, hence

$$(T_u)'' = (T_{\tilde{u}})' = T_{\tilde{\tilde{u}}} + \frac{1}{a}\delta, \qquad \tilde{\tilde{u}}(x) = \begin{cases} v''(x) & : x > 0 \\ 0 & : x < 0 \end{cases}.$$

Hence

$$AT_u = a(T_u)'' + b(T_u)' + cT_u = T_{a\tilde{\tilde{u}}} + b\tilde{u} + cu + \delta = \delta$$

since

$$a\tilde{\tilde{u}}(x) + b\tilde{u}(x) + cu(x) = \begin{cases} av''(x) + bv'(x) + cv(x) & : x > 0 \\ 0 & : x < 0 \end{cases} = 0.$$

This completes the proof. ∎

# 2 The Fourier transform and tempered distributions

**What's it about?** The Fourier transform is an important tool in the analysis of partial differential equations. First it is defined for integrable functions. To extend the Fourier transform from functions to distributions one needs to introduce a new class of distributions, the so-called tempered distributions. They are continuous functionals on a new space of functions, the so-called rapidly decreasing functions.

## 2.1 The Fourier transform on $L^1$-functions

For $f \in L^1(\mathbb{R}^n)$ define the Fourier transform of $f$ by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx, \qquad \xi \in \mathbb{R}^n,$$

where $x\xi = x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$ is the inner-product of $x$ with $\xi$.

**2.1 Lemma** *The following assertions are true:*

*a)* $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ *is linear and continuous with operator-norm* $\|\mathcal{F}\| \leqslant 1$.

*b) If* $f \in L^1(\mathbb{R}^n)$ *then* $\widehat{f} \in \mathscr{C}(\mathbb{R}^n)$ *and* $\lim_{|\xi|\to+\infty} \widehat{f}(\xi) = 0$.

b) is the so-called Theorem of Riemann-Lebesgue.

PROOF: Let $f \in L^1(\mathbb{R}^n)$. Then

$$|\mathcal{F}f(\xi)| = \left|\int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx\right| \leqslant \int_{\mathbb{R}^n} \underbrace{\left|e^{-ix\xi}\right|}_{=1} \cdot |f(x)|\,dx = \int_{\mathbb{R}^n} |f(x)|\,dx = \|f\|_{L^1},$$

i.e., $\|\mathcal{F}f\|_{L^\infty} \leqslant \|f\|_{L^1}$. This shows a). For b) let $\xi_k \to \xi$. Note that

$$\left|e^{-ix\xi_k} f(x)\right| \leqslant |f(x)| \in L^1(\mathbb{R}^n) \qquad \forall\, k.$$

Hence, by Lebesgue's dominated convergence theorem,

$$\widehat{f}(\xi_k) = \int_{\mathbb{R}^n} e^{-ix\xi_k} f(x)\,dx \xrightarrow{k\to+\infty} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx = \widehat{f}(\xi).$$

The proof on the limit will be omitted here. ∎

**2.2 Lemma** *Let* $f, g \in L^1(\mathbb{R}^n)$. *The following assertions are true:*

*a)* $\widehat{f * g} = \widehat{f}\,\widehat{g}$ *b)* $\int_{\mathbb{R}^n} \widehat{f}(\xi) g(\xi)\,d\xi = \int_{\mathbb{R}^n} f(\xi)\widehat{g}(\xi)\,d\xi$

PROOF: a) Note that $e^{-ix\xi} = e^{-i(x-y)\xi} e^{-iy\xi}$. By Fubini's theorem we thus obtain

$$\begin{aligned}\mathcal{F}(f * g)(\xi) &= \int e^{-ix\xi} \Big( \int f(x-y) g(y)\, dy \Big)\, dx \\ &= \int e^{-iy\xi} g(y) \Big( \int e^{-i(x-y)\xi} f(x-y)\, dx \Big)\, dy \\ &= \int e^{-iy\xi} g(y) \Big( \int e^{-iz\xi} f(z)\, dz \Big)\, dy = \widehat{f}(\xi) \cdot \widehat{g}(\xi).\end{aligned}$$

b) Again Fubini's theorem gives

$$\begin{aligned}\int \widehat{f}(\xi) g(\xi)\, d\xi &= \int \Big( \int e^{-ix\xi} f(x)\, dx \Big) g(\xi)\, d\xi \\ &= \int \Big( \int e^{-ix\xi} g(\xi)\, d\xi \Big) f(x)\, dx = \int \widehat{g}(x) f(x)\, dx\end{aligned}$$

This finishes the proof. ■

## 2.2 Rapidly decreasing functions – the space $\mathscr{S}(\mathbb{R}^n)$

**2.3 Definition** *Let $\mathscr{S}(\mathbb{R}^n)$ be the space of all functions $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ satisfying*

$$\|\varphi\|_{(N)} := \sup_{x \in \mathbb{R}^n,\, |\alpha|+|\beta| \leqslant N} \big| x^\beta \partial_x^\alpha \varphi(x) \big| < +\infty \qquad \forall\, N \in \mathbb{N}_0.$$

*A sequence $(\varphi_k)_k \subset \mathscr{S}(\mathbb{R}^n)$ is said to converge to $\varphi \in \mathscr{S}(\mathbb{R}^n)$ if*

$$\|\varphi_k - \varphi\|_{(N)} \xrightarrow{k \to +\infty} 0 \qquad \forall\, N \in \mathbb{N}_0.$$

**2.4 Example** *The Gaussian $\varphi(x) = \exp(-|x|^2)$ is rapidly decreasing.*

**2.5 Remark** *Let us define*

$$d(\varphi, \psi) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|\varphi - \psi\|_j}{1 + \|\varphi - \psi\|_j}, \qquad \varphi, \psi \in \mathscr{S}(\mathbb{R}^n).$$

*One can show that $d$ defines a metric on $\mathscr{S}(\mathbb{R}^n)$ and that $(\mathscr{S}(\mathbb{R}^n), d)$ is a complete metric space. Moreover, convergence of a sequence $(\varphi_k)_k$ in the metric is equivalent to convergence in the above defined sense.*

**2.6 Lemma** *$\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for every $1 \leqslant p \leqslant +\infty$.*

PROOF: If $\phi \in \mathscr{D}(\mathbb{R}^n)$ then $x^\beta \partial^\alpha \phi$ belongs to $\mathscr{D}(\mathbb{R}^n)$ and thus is bounded on $\mathbb{R}^n$.

Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then, for every $N \in \mathbb{N}$,

$$\|\varphi\|_{L^p}^p = \int_{|x|\leqslant 1} |\varphi(x)|^p \, dx + \int_{|x|\geqslant 1} |x|^{-2Np} ||x|^{2N}\varphi(x)|^p \, dx. \tag{2.1}$$

If $\tau_n$ is the measure of the unit-ball then

$$\int_{|x|\leqslant 1} |\varphi(x)|^p \, dx \leqslant \tau_n \max_{|x|\leqslant 1} |\varphi(x)|^p \leqslant \tau_n \|\varphi\|_{(0)}^p.$$

Since $|x|^{2N} = (x_1^2 + \ldots + x_n^2)^N = \sum_{|\beta|\leqslant 2N} c_{N,\beta} x^\beta$ with certain constants $c_{N,\beta}$, the second integral in (2.1) can be estimated by

$$c_N \int_{|x|\geqslant 1} |x|^{-2Np} \, dx \|\varphi\|_{(2N)}^p$$

with a certain costant $c_N$. Introducing polar coordinates,

$$\int_{|x|\geqslant 1} |x|^{-2Np} = n\tau_n \int_1^{+\infty} r^{-2Np} r^{n-1} \, dr.$$

This integral is finite if we choose $N \in \mathbb{N}$ such that $n - 1 - 2Np < -1$, i.e., $N > n/2p$. Summing up, there exist constants $C_N$ such that

$$\|\varphi\|_{L^p}^p \leqslant C_N \|\varphi\|_{(2N)} \qquad \forall\, \varphi \in \mathscr{S}(\mathbb{R}^n) \quad \forall\, N > \frac{n}{2p}. \tag{2.2}$$

This shows the claim. ∎

Note that (2.2) implies that the convergence $\varphi_k \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$ implies convergence of the sequence in $L^p(\mathbb{R}^n)$.

**2.7 Theorem** *If $1 \leqslant p < +\infty$ then $\mathscr{D}(\mathbb{R}^n)$ is a dense subset of $L^p(\mathbb{R}^n)$, i.e.,*

$$\forall\, f \in L^p(\mathbb{R}^n) \quad \exists\, (\varphi_k)_k \subset \mathscr{D}(\mathbb{R}^n) : \quad \|\varphi_k - f\|_{L^p} \xrightarrow{k\to+\infty} 0.$$

**2.8 Definition** *Let $X$ be a normed vector-space or $X = \mathscr{S}(\mathbb{R}^n)$. A linear map $T : \mathscr{S}(\mathbb{R}^n) \to X$ is said to be continuous if for every convergent sequence $(\varphi_k)_k \subset \mathscr{S}(\mathbb{R}^n)$ holds*

$$\lim_{k\to+\infty} T(\varphi_k) = T\Big(\lim_{k\to\infty} \varphi_k\Big).$$

**2.9 Theorem** *The following identities hold for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and all $\alpha \in \mathbb{N}_0^n$ :*

$$\widehat{\partial_x^\alpha \varphi}(\xi) = i^{|\alpha|} \xi^\alpha \widehat{\varphi}(\xi), \qquad (-i)^{|\alpha|} \widehat{x^\alpha \varphi}(\xi) = \partial_\xi^\alpha \widehat{\varphi}(\xi).$$

*The Fourier transform induces a continuous map $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.*

PROOF: By induction, it suffices to show both equations for $|\alpha| = 1$ only. Using integration by parts,

$$\widehat{\partial_{x_j}\varphi}(\xi) = \int e^{-ix\xi}(\partial_{x_j}\varphi)(x)\,dx = -\int \big(\partial_{x_j}e^{-ix\xi}\big)\varphi(x)\,dx = i\xi_j\int e^{-ix\xi}\varphi(x)\,dx = i\xi_j\widehat{\varphi}(\xi).$$

For the second equation note that

$$\partial_{\xi_j}\widehat{\varphi}(\xi) = \partial_{\xi_j}\int e^{-ix\xi}\varphi(x)\,dx = \int \big(\partial_{\xi_j}e^{-ix\xi}\big)\varphi(x)\,dx = -\int e^{-ix\xi}ix_j\varphi(x)\,dx = -i\,\widehat{x_j\varphi}(\xi).$$

For the continuity first note that (2.2) yields

$$|\widehat{\varphi}(\xi)| \leqslant \int_{\mathbb{R}^n}|\varphi(x)|\,dx = \|\varphi\|_{L^1} \leqslant C_0\|\varphi\|_{(n+2)}, \qquad \xi\in\mathbb{R}^n,$$

with a suitable constant $C_0$. Therefore

$$\|\widehat{\varphi}\|_{(0)} \leqslant C_0\|\varphi\|_{(n+2)}.$$

By induction, using the two rules, one then shows that

$$\|\widehat{\varphi}\|_{(N)} \leqslant C_N\|\varphi\|_{(N+n+2)} \qquad \forall\,\varphi\in\mathscr{S}(\mathbb{R}^n) \quad \forall\,N\in\mathbb{N}. \tag{2.3}$$

Hence $\varphi\in\mathscr{S}(\mathbb{R}^n)$ implies $\widehat{\varphi}\in\mathscr{S}(\mathbb{R}^n)$ and $\varphi_k\to\varphi$ in $\mathscr{S}(\mathbb{R}^n)$ implies $\widehat{\varphi}_k\to\widehat{\varphi}$ in $\mathscr{S}(\mathbb{R}^n)$. ∎

**2.10 Theorem** *$\mathscr{F}^2\varphi(x) = (2\pi)^n\varphi(-x)$ for every $\varphi\in\mathscr{S}(\mathbb{R}^n)$. In particular, the Fourier transform $\mathscr{F}:\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^n)$ is bijective with inverse $\mathscr{F}^{-1}$ given by*

$$(\mathscr{F}^{-1}\psi)(x) = \check{\psi}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}e^{ix\xi}\psi(\xi)\,d\xi.$$

PROOF: Let $f(x) := (2\pi)^{-n/2}e^{-\|x\|^2/2}$. Note that $\|f\|_{L^1(\mathbb{R}^n)} = 1$.

**Step1:** We show that $\widehat{f}(\xi) = (2\pi)^{n/2}f(\xi)$.

Proof for $n=1$ (the general case is a homework): Let $u(x) = e^{-x^2/2}$. Then

$$u'(x) = -xe^{-x^2/2} = -xu(x), \qquad u(0) = 1,$$

and

$$\widehat{u}'(\xi) \overset{2.9}{=} -i\,\widehat{xu}(\xi) = i\,\widehat{u'}(\xi) \overset{2.9}{=} i^2\xi\widehat{u}(\xi) = -\xi\widehat{u}(\xi),$$
$$\widehat{u}(0) = \int_{-\infty}^{+\infty}e^{-x^2/2}\,dx = \sqrt{2\pi}.$$

Thus both $u$ and $\widehat{u}/\sqrt{2\pi}$ are solutions of the initial value problem

$$y'(t) = -ty(t), \qquad y(0) = 1.$$

Since the solution of this initial problem is unique, we have $\widehat{u} = u/\sqrt{2\pi}$.

**2. Step:** Let $f_\varepsilon(x) = \varepsilon^{-n} f(x/\varepsilon)$. Then $\{f_\varepsilon\}_{\varepsilon>0}$ is an approximate identity (see the first part of the lecture) and

$$\widehat{f(\varepsilon\cdot)}(\xi) = (2\pi)^{n/2} f_\varepsilon(\xi).$$

Therefore,

$$\begin{aligned}(2\pi)^{-n/2}\mathscr{F}^2\varphi(y) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{-iy\xi} f(\varepsilon\xi)\widehat{\varphi}(\xi)\,d\xi = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} f(\varepsilon\xi)\mathscr{F}[\varphi(\cdot - y)](\xi)\,d\xi \\ &\overset{2.2.b)}{=} (2\pi)^{n/2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} f_\varepsilon(x)\varphi(x-y)\,dx = (2\pi)^{n/2}\varphi(-y),\end{aligned}$$

**3. Step:** For the second claim define the operator $R : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by $(R\varphi)(x) = (2\pi)^n\varphi(-x)$. Then $\mathscr{F}^2 = R$. Obviously, $R$ is bijective with inverse given by $(R^{-1}\varphi)(x) = (2\pi)^{-n}\varphi(-x)$. Hence $(R^{-1}\mathscr{F})\mathscr{F} = R^{-1}\mathscr{F}^2 = \mathrm{id} = \mathscr{F}^2R^{-1} = \mathscr{F}(\mathscr{F}R^{-1})$. This shows that $\mathscr{F}$ has a left- and a right-inverse, hence is bijective with $\mathscr{F}^{-1} = R^{-1}\mathscr{F}$. ■

**2.11 Theorem** (Parseval's formula) *For arbitrary $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$,*

$$(\widehat{\varphi}, \widehat{\psi})_{L^2(\mathbb{R}^n)} = (2\pi)^n(\varphi,\psi)_{L^2(\mathbb{R}^n)}, \qquad \|\widehat{\varphi}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2}\|\varphi\|_{L^2(\mathbb{R}^n)}.$$

Proof: First note that

$$\overline{\widehat{\psi}(\xi)} = \overline{\int e^{-ix\xi}\psi(x)\,dx} = \int e^{ix\xi}\overline{\psi(x)}\,dx = (2\pi)^n(\mathscr{F}^{-1}\overline{\psi})(\xi),$$

Thus, due to Lemma 2.2.b),

$$\begin{aligned}(\widehat{\varphi}, \widehat{\psi})_{L^2(\mathbb{R}^n)} &= \int \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi = \int \varphi(\xi)\mathscr{F}(\mathscr{F}^{-1}\overline{\psi})(\xi)\,d\xi \\ &= (2\pi)^n\int\varphi(\xi)\overline{\psi}(\xi)\,d\xi = (2\pi)^n(\varphi,\psi)_{L^2(\mathbb{R}^n)}.\end{aligned}$$

For the norm, apply this with $\psi = \varphi$. ■

**2.12 Theorem** (Plancherel's theorem) *Let $A := (2\pi)^{-n/2}\mathscr{F}$. Then $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ extends to a unitary isomorphism $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, i.e., $AA^* = A^*A = \mathrm{id}$ on $L^2(\mathbb{R}^n)$.*

Proof: By Parseval's formula,

$$(A\varphi, A\psi)_{L^2} = (2\pi)^{-n}(\widehat{\varphi},\widehat{\psi})_{L^2} = (\varphi,\psi)_{L^2} \qquad \forall\, \varphi,\psi \in \mathscr{S}(\mathbb{R}^n). \tag{2.4}$$

In particular,

$$\|A\varphi\|_{L^2} = \|\varphi\|_{L^2} \qquad \forall\, \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Let $f \in L^2(\mathbb{R}^n)$. Then there exists $(\varphi_k)_k \subset \mathscr{S}(\mathbb{R}^n)$ with $\varphi_k \to f$ in $L^2(\mathbb{R}^n)$. Then $(A\varphi_k)_k$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, since

$$\|A\varphi_k - A\varphi_\ell\|_{L^2} = \|A(\varphi_k - \varphi_\ell)\|_{L^2} = \|\varphi_k - \varphi_\ell\|_{L^2}.$$

Hence $(A\varphi_k)_k$ converges in $L^2(\mathbb{R}^n)$ and we define

$$Af := \lim_{k\to+\infty} A\varphi_k$$

(homework: $Af$ does not depend on the choice of the sequence $(\varphi_k)_k$). Note that

$$\|Af\|_{L^2} = \lim_{k\to+\infty} \|A\varphi_k\|_{L^2} = \lim_{k\to+\infty} \|\varphi_k\|_{L^2} = \|f\|_{L^2}.$$

Hence $A$ extends to a bounded operator in $L^2(\mathbb{R}^n)$ with operator-norm equal to 1.

Now let $B = (2\pi)^{n/2}\mathscr{F}^{-1} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. Then $B$ is the inverse of $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. Hence substituting $\varphi$ and $\psi$ in (2.4) by $B\varphi$ and $B\psi$, respectively, we find

$$(B\varphi, B\psi)_{L^2} = (\varphi, \psi)_{L^2} \qquad \forall\, \varphi, \psi \in \mathscr{S}(\mathbb{R}^n).$$

Repeating the above argument, $B$ extends to a bounded operator $B : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

If $(\varphi_k)_k \subset \mathscr{S}(\mathbb{R}^n)$ converges in $L^2(\mathbb{R}^n)$ to $f$, then $(B\varphi_k)_k \subset \mathscr{S}(\mathbb{R}^n)$ converges in $L^2(\mathbb{R}^n)$ to $Bf$. It follows

$$ABf = \lim_{k\to+\infty} AB\varphi_k = \lim_{k\to+\infty} \varphi_k = f$$

and analogously $BAf = f$. This means that $B = A^{-1}$ as operators $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Next let $\varphi, \psi \in L^2(\mathbb{R}^n)$ and $(\varphi_k)_k, (\psi_k)_k \subset \mathscr{S}(\mathbb{R}^n)$ with $\varphi_k \to f$ and $\psi_k \to g$ in $L^2(\mathbb{R}^n)$. Then

$$(A\varphi, A\psi)_{L^2} = \lim_{k\to+\infty} (A\varphi_k, A\psi_k)_{L^2} = \lim_{k\to+\infty} (\varphi_k, \psi_k)_{L^2} = (\varphi, \psi)_{L^2}.$$

This implies $A^*A = 1$. Since $A$ is invertible it follows $A^* = A^{-1}$, i.e., $A$ is unitary. ∎

**2.13 Remark** *Let $X, Y$ be a Banach spaces and $D \subset X$ be a dense subspace. Assume that $T : D \to Y$ is linear and*

$$\|Tx\|_Y \leqslant M\|x\|_X \qquad \forall\, x \in D$$

*with some constant $M \geqslant 0$. Then there exists a unique $\widetilde{T} \in \mathscr{L}(X, Y)$ such that $\widetilde{T}x = Tx$ for all $x \in D$. In fact,*

$$\widetilde{T}x := \lim_{k\to+\infty} Tx_k,$$

*where $(x_k) \subset D$ is an arbitrary sequence converging to $x$.*

## 2.3 Tempered distributions – the space $\mathscr{S}'(\mathbb{R}^n)$

We define the space of tempered distributions

$$\mathscr{S}'(\mathbb{R}^n) := \Big\{T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C} \mid T \text{ linear and continuous}\Big\},$$

where continuity refers to Definition 2.8, i.e., $\varphi_k \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$ implies $T(\varphi_k) \to T(\varphi)$.

**2.14 Theorem** *For a linear map $T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ the following are equivalent:*

*a)* $T \in \mathscr{S}'(\mathbb{R}^n)$

*b)* *There exist an $N \in \mathbb{N}_0$ and a $C \geqslant 0$ such that*

$$|T(\varphi)| \leqslant C\|\varphi\|_{(N)} = C \max_{x\in\mathbb{R}^n,\,|\alpha|+|\beta|\leqslant N} |x^\beta \partial^\alpha \varphi(x)| \qquad \forall\, \varphi \in \mathscr{S}(\mathbb{R}^n).$$

PROOF: Similar to the proof of Theorem 1.8. ■

The basic concepts of distributions seen before can be adapted to tempered distributions:

- Regular tempered distributions: Those $T \in \mathscr{S}'(\mathbb{R}^n)$ of the form

  $$T(\varphi) = \int_{\mathbb{R}^n} u(x)\varphi(x)\,dx, \qquad \varphi \in \mathscr{S}(\mathbb{R}^n),$$

  where $u \in L^1_{loc}(\mathbb{R}^n)$ is a function such that $\frac{u(x)}{1+|x|^N} \in L^1(\mathbb{R}^n)$ for some $N \geqslant 0$. We call $u$ the density of $T$ and write $T = T_u$.

  **Example:** If $u \in L^p(\mathbb{R}^n)$ for some $1 \leqslant p \leqslant +\infty$ then $u$ defines a regular tempered distribution. $u(x) = e^x$ does **not** define a regular tempered distribution.

- Differentiation: If $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ define $\partial^\alpha T \in \mathscr{S}'(\mathbb{R}^n)$ by

  $$\partial^\alpha T(\varphi) = (-1)^{|\alpha|} T(\partial^\alpha \varphi) \qquad \varphi \in \mathscr{S}(\mathbb{R}^n).$$

- Multiplication with functions: If $T \in \mathscr{S}'(\mathbb{R}^n)$ and $a \in \mathscr{C}^\infty(\mathbb{R}^n)$ one can define $aT \in \mathscr{S}'(\mathbb{R}^n)$ by

  $$aT(\varphi) = T(a\varphi), \qquad \varphi \in \mathscr{S}(\mathbb{R}^n),$$

  provided $a$ is of tempered growth, i.e.,

  $$\forall\, \alpha \in \mathbb{N}_0^n \quad \exists\, N = N(\alpha) \geqslant 0 : \quad \sup_{|x|\geqslant 1} |\partial^\alpha a(x)||x|^{-N} < +\infty.$$

  The condition of tempered growth is necessary to ensure that $a\varphi$ belongs to $\mathscr{S}(\mathbb{R}^n)$.

  **Example:** $a(x) = e^x$ is **not** of tempered growth. Instead, any polynomial is of tempered growth.

- Convolution: Let $T \in \mathscr{S}'(\mathbb{R}^n)$. If $\varphi \in \mathscr{S}(\mathbb{R}^n)$ then
$$(T * \varphi)(x) = T(\varphi(x - \cdot)), \qquad x \in \mathbb{R}^n,$$
defines a function $T * \varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ of tempered growth.

The main motivation for introducing tempered distributions is, however, the Fourier transform. Note that for $f \in L^1(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$
$$T_{\hat{f}}(\varphi) = \int \hat{f}(x)\varphi(x)\,dx = \int f(x)\hat{\varphi}(x)\,dx = T_f(\hat{\varphi}).$$
Therefore, for $T \in \mathscr{S}'(\mathbb{R}^n)$, we define $\mathscr{F}T = \widehat{T}$ by
$$\boxed{\widehat{T}(\varphi) := T(\hat{\varphi}), \qquad \varphi \in \mathscr{S}(\mathbb{R}^n).}$$
Since $\mathcal{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ and $T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ are continuous, $\widehat{T} = T \circ \mathcal{F} : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous too, i.e., $\widehat{T} \in \mathscr{S}'(\mathbb{R}^n)$. Analogously one defines $\mathcal{F}^{-1}T = \check{T} \in \mathscr{S}'(\mathbb{R}^n)$ by
$$\boxed{\check{T}(\varphi) := T(\check{\varphi}), \qquad \varphi \in \mathscr{S}(\mathbb{R}^n).}$$
By construction, the following result is obvious:

**2.15 Theorem** *The Fourier transform $\mathcal{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is bijective with inverse $\mathcal{F}^{-1} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$.*

**2.16 Example** *The Fourier transform $\delta^{(k)} \in \mathscr{S}'(\mathbb{R})$ is a regular distribution given by*
$$\widehat{\delta^{(k)}}(\varphi) = (-1)^k\delta(\widehat{\varphi^{(k)}}) = i^k\widehat{x^k\varphi}(0) = \int (ix)^k\varphi(x)\,dx = T_{p_k}(\varphi)$$
*with the polynomial $p_k(x) = (ix)^k$.*

**2.17 Theorem** *Let $f \in L^1(\mathbb{R}^n)$ with $\hat{f} \in L^1(\mathbb{R}^n)$. Then*
$$f(x) = (2\pi)^{-n}\int e^{ix\xi}\hat{f}(\xi)\,d\xi \qquad \textit{for almost all } x \in \mathbb{R}^n.$$

Proof: Let $g(x)$ be the expression on the right-hand side. Note that $|g(x)| \leqslant \|\hat{f}\|_{L^1}$ for every $x$, i.e., $g$ is bounded. Now
$$\begin{aligned} T_f(\varphi) = (\mathscr{F}^{-1}\widehat{T_f})(\varphi) = T_{\hat{f}}(\check{\varphi}) &= \int \hat{f}(\xi)\Big((2\pi)^{-n}\int e^{ix\xi}\varphi(x)\,dx\Big)\,d\xi \\ &= \int \Big((2\pi)^{-n}\int e^{ix\xi}\hat{f}(\xi)\,d\xi\Big)\varphi(x)\,dx = T_g(\varphi). \end{aligned}$$
In particular,
$$T_f(\phi) = T_g(\phi) \qquad \forall\, \phi \in \mathscr{D}(\mathbb{R}^n).$$
Hence $f(x) = g(x)$ almost everywhere by Example 1.9. ■

## 2.4 Distributions and partial differential equations (II)

**2.18 Lemma** *The rules in Theorem 2.9 do not only hold for $\varphi \in \mathscr{S}(\mathbb{R}^n)$ but for arbitrary tempered distributions.*

To proof this lemma is part of the homeworks.

Let $A = \sum_{|\alpha| \leqslant m} a_\alpha \partial_x^\alpha$ be a differential operator with constant coefficients. Set

$$a(\xi) := \sum_{|\alpha| \leqslant m} a_\alpha (i\xi)^\alpha \qquad \text{(the so-called } \textit{symbol} \text{ of } A).$$

Note that $a$ is a polynomial, hence is of tempered growth. Then, for $T \in \mathscr{S}'(\mathbb{R}^n)$, we have

$$\widehat{AT} = \sum_{|\alpha| \leqslant m} a_\alpha \widehat{\partial^\alpha T} = \sum_{|\alpha| \leqslant m} a_\alpha i^{|\alpha|} \xi^\alpha \widehat{T} = a\widehat{T},$$

or, equivalently,

$$AT = \mathscr{F}^{-1}(a\widehat{T}).$$

Suppose now that $a$ has no zeros. Then $1/a$ is of tempered growth and

$$AT = S \iff T = \mathscr{F}^{-1}\Big(\frac{1}{a}\widehat{S}\Big)$$

gives a unique solution for the pde $AT = S$ for arbitrary $S \in \mathscr{S}'(\mathbb{R}^n)$.

**2.19 Example** *Let $\Delta = \partial_1^2 + \ldots + \partial_n^2$ be the Laplacian and $\lambda \in \mathbb{C} \backslash (-\infty, 0]$ arbitrary. The symbol of $A := \lambda - \Delta$ is*

$$a(\xi) = \lambda - (-\xi_1^2 - \ldots - \xi_n^2) = \lambda + |\xi|^2.$$

*In particular, $a(\xi) \neq 0$ for all $\xi$. Hence*

$$\lambda - \Delta : \mathscr{S}'(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n), \qquad \lambda \in \mathbb{C}\backslash(-\infty, 0],$$

*is bijective with inverse given by*

$$(\lambda - \Delta)^{-1} S = \mathscr{F}^{-1}\Big(\frac{1}{\lambda + |\cdot|^2}\widehat{S}\Big).$$

The idea of this approach is that properties of a differential operator $A$ can be read of from its symbol. This approach can be extended also to differential operators with variable coefficients, i.e., $A = \sum_{|\alpha| \leqslant m} a_\alpha(x) \partial_x^\alpha$, and leads to the theory of *pseudodifferential operators*.

**2.20 Remark** (Sobolev (or Bessel potential) spaces) *In this remark we write shortly $f = T_f$ for regular distributions. For $k \in \mathbb{N}_0$ we say*

$$f \in H_2^k(\mathbb{R}^n) :\iff \partial^\alpha f \in L^2(\mathbb{R}^n) \text{ for all } |\alpha| \leqslant k$$

*(where the derivatives are understood in the distributional sense, i.e., for every such $\alpha$ the distribution $\partial^\alpha T_f$ is a regular distribution whose density is an $L^2$-function). The spaces $H_2^k(\mathbb{R}^n)$ can be considered as a substitute of the classical spaces $\mathcal{C}^k(\mathbb{R}^n)$ of $k$-times continuously differentiable functions.*
*For $s \in \mathbb{R}$ we define*

$$H_2^s(\mathbb{R}^n) := \Big\{ T \in \mathcal{S}'(\mathbb{R}^n) \mid \mathscr{F}^{-1}\big((1+|\xi|^2)^{s/2}\widehat{T}\big) \in L^2(\mathbb{R}^n) \Big\}.$$

*Recalling Parseval's theorem ($f \in L^2(\mathbb{R}^n) \iff \widehat{f} \in L^2(\mathbb{R}^n)$) it is meaningful to define the norm*

$$\|T\|_{H_2^s(\mathbb{R}^n)} := \|\mathscr{F}^{-1}\big((1+|\xi|^2)^{s/2}\widehat{T}\big)\|_{L^2(\mathbb{R}^n)} \cong \|(1+|\xi|^2)^{s/2}\widehat{T}\|_{L^2(\mathbb{R}^n)}.$$

*This makes $H_2^s(\mathbb{R}^n)$ a Banach space (even a Hilbert space). One can show that in case $s = k$ the two definitions are equivalent.*
*If $A$ is a differential operator of order $m$ with constant coefficients, then*

$$A : H_2^s(\mathbb{R}^n) \longrightarrow H_2^{s-m}(\mathbb{R}^n), \qquad s \in \mathbb{R},$$

*is a continuous map; it is an isomorphism if the symbol $a(\xi)$ of $A$ has no zeros.*
*Similarly, substituting $L^2(\mathbb{R}^n)$ by $L^p(\mathbb{R}^n)$, $1 < p < +\infty$, one can define Banach spaces $H_p^k(\mathbb{R}^n)$ and $H_p^s(\mathbb{R}^n)$. The above statements remain valid.*

The following example shows an application of the Fourier transform to so-called boundary value problems.

**2.21 Example** *Let $\mathbb{R}_+^{n+1} = \{(x,t) \mid x \in \mathbb{R}^n,\ t > 0\}$ be a half-space. Given $f \in \mathscr{S}(\mathbb{R}^n)$ we want to find a function $u$ which satisfies*

$$\begin{aligned} (\Delta_x + \partial_t^2)u(x,t) &= 0 && \forall\, x \in \mathbb{R}^n \quad \forall\, t > 0,\\ u(x,0) &= f(x) && \forall\, x \in \mathbb{R}^n. \end{aligned}$$

*Let us proceed with some formal computations. Define*

$$v(\xi,t) := (\mathcal{F}_{x\to\xi}u)(\xi,t) = \int e^{-ix\xi} u(x,t)\, dx$$

*Applying the Fourier transform in $x$ to the above equations yields*

$$\begin{aligned} \partial_t^2 v(\xi,t) - |\xi|^2 v(\xi,t) &= 0 && \forall\, \xi \in \mathbb{R}^n \quad \forall\, t > 0,\\ v(\xi,0) &= \widehat{f}(\xi) && \forall\, \xi \in \mathbb{R}^n. \end{aligned}$$

*For each fixed $\xi \in \mathbb{R}^n$ this is an ODE having the two solutions*

$$v_+(\xi,t) = \hat{f}(\xi)e^{|\xi|t}, \quad v_-(\xi,t) = \hat{f}(\xi)e^{-|\xi|t}$$

*We choose $v(\xi,t) = v_-(\xi,t)$ so that we can apply the inverse Fourier transform. In fact, defining*

$$P_t(x) := \mathcal{F}^{-1}_{\xi\to x}(e^{-|\xi|t})(x) \qquad \text{(Poisson kernel)}$$

*we have*

$$(\mathcal{F}_{x\to\xi}u)(\xi,t) = v_-(\xi,t) = \hat{P}_t(\xi)\hat{f}(\xi) = \widehat{P_t * f}(\xi),$$

*and therefore $u(x,t) = (P_t * f)(x)$. One can show that*

$$P_t(x) = C_n t|(x,t)|^{-n-1}$$

*with some constant $C_n$. Hence we find the solution formula*

$$u(x,t) = (P_t * f)(x) = C_n \int_{\mathbb{R}^n} \frac{t}{|x-y,t|^{n+1}} f(y)\,dy.$$

## 2.5 The Heisenberg uncertainty principal

**2.22 Theorem** *Let $\varphi \in \mathscr{S}(\mathbb{R})$ and $x_0, \xi_0 \in \mathbb{R}$. Then*

$$\|\varphi\|^2_{L^2(\mathbb{R})} \leqslant \sqrt{\frac{2}{\pi}}\,\|(x-x_0)\varphi(x)\|_{L^2(\mathbb{R})}\|(\xi-\xi_0)\hat{\varphi}(\xi)\|_{L^2(\mathbb{R})}.$$

Proof: **1. Step:** Let first $x_0 = \xi_0 = 0$. Then integration by parts yields

$$\begin{aligned}\|\varphi\|^2_{L^2(\mathbb{R})} &= \int \varphi(x)\overline{\varphi(x)}\,dx = -\int x(\varphi\overline{\varphi})'(x)\,dx = -\int x(\varphi'\overline{\varphi} + \varphi\overline{\varphi'})(x)\,dx \\ &= -2\int x\mathrm{Re}\left(\varphi'(x)\overline{\varphi}(x)\right)dx\end{aligned}$$

Taking the modulus and applying Cauchy-Schwarz inequality yields

$$\|\varphi\|^2_{L^2(\mathbb{R})} \leqslant 2\int |x\varphi(x)||\varphi'(x)|\,dx \leqslant 2\|x\varphi(x)\|_{L^2(\mathbb{R})}\|\varphi'\|_{L^2(\mathbb{R})}.$$

The claim then follows from

$$\sqrt{2\pi}\|\varphi'\|_{L^2(\mathbb{R})} \overset{2.11}{=} \|\mathscr{F}\varphi'\|_{L^2(\mathbb{R})} \overset{2.9}{=} \|\xi\hat{\varphi}\|_{L^2(\mathbb{R})}.$$

**2. Step:** Define $\psi(x) = e^{-ix\xi_0}\varphi(x+x_0)$ and calculate that $\|\psi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$ and

$$\|x\psi(x)\|_{L^2(\mathbb{R})} = \|(x-x_0)\varphi(x)\|_{L^2(\mathbb{R})}, \qquad \|\xi\hat{\psi}(\xi)\|_{L^2(\mathbb{R})} = \|(\xi-\xi_0)\hat{\varphi}(\xi)\|_{L^2(\mathbb{R})}.$$

Apply the estimate of Step 1 to $\psi$. ∎

We conclude this section by stating the following result due to Amrein and Berthier:

**2.23 Theorem** *Let $E, F \subset \mathbb{R}$ be two sets of finite measure. Then there exists a constant $C \geqslant 0$ such that*

$$\|f\|_{L^2(\mathbb{R})} \leqslant C\big(\|f\|_{L^2(\mathbb{R}\setminus E)} + \|\widehat{f}\|_{L^2(\mathbb{R}\setminus F)}\big) \qquad \forall\, f \in L^2(\mathbb{R}).$$

In particular, if both $f$ and $\widehat{f}$ have compact support then $f = 0$.

# 3 The Laplace transform

**What's it about?** The Laplace transform is a useful tool in the analysis of ordinary differential equations. We discuss basic properties and applications to initial value problems of the form

$$y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+\ldots+a_1y'(x)+a_0y(x)=b(x), \qquad x>0, \tag{3.1}$$

with initial conditions

$$y(0)=y_0, \quad y'(0)=y_1, \quad \ldots \quad y^{(n-1)}(0)=y_{n-1}, \tag{3.2}$$

where $n\geqslant 1$. For simplicity we shall mainly focus on the case of order $n=2$.

## 3.1 Definition and basic properties

**3.1 Definition** *A function $f:(0,\infty)\to\mathbb{C}$ is called $\mathcal{L}$-transformable if there exists a $\sigma\in\mathbb{R}$ such that*

$$x\mapsto e^{-\sigma x}f(x) \ \in\ L^1((0,+\infty)). \tag{3.3}$$

*In this case, we set*

$$\sigma_f := \inf\{\sigma\in\mathbb{R} \mid \sigma \text{ satisfies } (3.3)\}.$$

*The Laplace transform of $f$, denoted by $\mathcal{L}f$, is then defined as*

$$(\mathcal{L}f)(s)=(\mathcal{L}f(x))(s)=\int_0^{+\infty} e^{-sx}f(x)\,dx, \qquad \operatorname{Re}s>\sigma_f.$$

Note that $\mathcal{L}f$ is defined on the half-plane $\{s\in\mathbb{C}\mid \operatorname{Re}s>\sigma_f\}$, since $|e^{-sx}|=e^{-x\operatorname{Re}s}$.

**3.2 Example** *Let $a\in\mathbb{C}$. Then*

$$(\mathcal{L}e^{ax})(s)=\int_0^{+\infty}e^{(a-s)x}\,dx \overset{a\neq s}{=} \lim_{b\to+\infty}\frac{e^{(a-s)x}}{a-s}\Big|_{x=0}^{x=b}=\frac{1}{s-a}, \qquad \operatorname{Re}s>\operatorname{Re}a,$$

*since* $|e^{(a-s)x}|=e^{\operatorname{Re}(a-s)x}\xrightarrow{x\to+\infty}0$ *provided* $\operatorname{Re}s>\operatorname{Re}a$.

**3.3 Example** *Let $\omega\in\mathbb{R}$. Then*

$$\cos(\omega x)=\frac{1}{2}\Big(e^{i\omega x}+e^{-i\omega x}\Big), \qquad \sin(\omega x)=\frac{1}{2i}\Big(e^{i\omega x}-e^{-i\omega x}\Big).$$

*Using the previous example with $a=\pm i\omega$ we find*

$$(\mathcal{L}\cos(\omega x))(s)=\frac{s}{s^2+\omega^2}, \qquad (\mathcal{L}\sin(\omega x))(s)=\frac{\omega}{s^2+\omega^2}, \qquad \operatorname{Re}s>0.$$

**3.4 Example** *Let $k \in \mathbb{N}$. Then, for* $\operatorname{Re} s > 0$*,*

$$(\mathcal{L}x^k)(s) = \int_0^{+\infty} e^{-sx} x^k \, dx = \underbrace{-\frac{x^k}{s} e^{-sx} \Big|_{x=0}^{x=+\infty}}_{=0} + \int_0^{\infty} \frac{k}{s} e^{-sx} x^{k-1} \, dx = \frac{k}{s} (\mathcal{L}x^{k-1})(s).$$

*By iteration we conclude*

$$\boxed{(\mathcal{L}x^k)(s) = \frac{k!}{s^k} (\mathcal{L}1)(s) = \frac{k!}{s^{k+1}}, \qquad \operatorname{Re} s > 0.}$$

**3.5 Lemma** *Let $f$ and $g$ be $\mathcal{L}$-transformable and $a, b \in \mathbb{C}$. Then:*

*a)* $\mathcal{L}(af + bg) = a(\mathcal{L}f) + b(\mathcal{L}g)$ *on the half-plane* $\operatorname{Re} s > \max(\sigma_f, \sigma_g)$*,*

*b)* $(\mathcal{L}e^{ax} f(x))(s) = (\mathcal{L}f)(s - a)$ *on the half-plane* $\operatorname{Re} s > \sigma_f + \operatorname{Re} a$*,*

*c)* $(\mathcal{L}x^n f(x))(s) = (-1)^n \frac{d^n}{ds^n} (\mathcal{L}f)(s)$ *on the half-plane* $\operatorname{Re} s > \sigma_f$*.*

Proof: a) is obvious.

b) $(\mathcal{L}e^{ax} f(x))(s) = \int_0^{+\infty} e^{-sx} e^{ax} f(x) \, dx = \int_0^{+\infty} e^{-(s-a)x} f(x) \, dx = (\mathcal{L}f)(s - a)$

c) $\frac{d^n}{ds^n} (\mathcal{L}f)(s) = \frac{d^n}{ds^n} \int_0^{+\infty} e^{-sx} f(x) \, dx = \int_0^{+\infty} (-x)^n e^{-sx} f(x) \, dx = (-1)^n (\mathcal{L}x^n f(x))(s)$. ■

## 3.2 The inverse Laplace transform

Note that if $\sigma > \sigma_f$, then

$$\boxed{(\mathcal{L}f)(\sigma + i\tau) = \int_0^{+\infty} e^{-ix\tau} \big( e^{-\sigma x} f(x) \big) \, dx = [\mathscr{F}(e^{-\sigma \cdot} \tilde{f})](\tau),}$$

where $\tilde{f}$ is the function that is equal to 0 on $(-\infty, 0)$ and coincides with $f$ on $(0, +\infty)$. From this and the known injectivity of the Fourier transform, one obtains the injectivity of the Laplace transform: If $f$ and $g$ are $\mathcal{L}$-transformable and their Laplace transforms coincide on some vertical line in the complex plane, then $f = g$ almost everywhere. Moreover, if $\tau \mapsto (\mathcal{L}f)(\sigma + i\tau)$ belongs to $L^1(\mathbb{R})$, by Theorem 2.17 one recovers the function $f$:

$$f(x) = e^{\sigma x} \big( \mathscr{F}^{-1}_{\tau \to x} (\mathcal{L}f)(\sigma + i\tau) \big)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(\sigma + i\tau)x} (\mathcal{L}f)(\sigma + i\tau) \, d\tau, \qquad x > 0.$$

The previous result suggests to define, for suitable functions $F$ holomorphic for $\operatorname{Re} s \geqslant \sigma$,

$$(\mathcal{L}^{-1}F)(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx}F(s)\,ds.$$ [1]

The "art" in using the Laplace transform is to recognize whether a function $F$ belongs to the range of $\mathcal{L}$ and to calculate the inverse $\mathcal{L}^{-1}F$. To this scope one often uses techniques from complex analysis (like the residue theorem). Some examples can be also found by elementary calculations:

**3.6 Example** *Sei* $F(s) = \dfrac{1}{(s-a)^n}$ *with* $n \in \mathbb{N}$ *and* $a \in \mathbb{C}$. *Using* $(\mathcal{L}x^n)(s) = \dfrac{n!}{s^{n+1}}$, *we find* $(\mathcal{L}e^{ax}x^n)(s) = \dfrac{n!}{(s-a)^{n+1}}$. *Thus*

$$\left(\mathcal{L}^{-1}\frac{1}{(s-a)^n}\right)(x) = \frac{x^{n-1}}{(n-1)!}e^{ax}.$$

## 3.3 Laplace transform and convolution

The (Laplace-)convolution $f * g$ of suitable functions $f, g : (0, +\infty) \to \mathbb{C}$ is defined by

$$(f * g)(x) := \int_0^x f(x-y)g(y)\,dy, \qquad x > 0.$$

**3.7 Theorem** *If* $f$ *and* $g$ *are* $\mathcal{L}$*-transformable, then*

$$\mathcal{L}(f * g) = (\mathcal{L}f)(\mathcal{L}g)$$

(*on the half-plane* $\operatorname{Re} s > \max(\sigma_f, \sigma_g)$). *If* $F$ *and* $G$ *are Laplace transforms, then*

$$\mathcal{L}^{-1}(FG) = (\mathcal{L}^{-1}F) * (\mathcal{L}^{-1}G).$$

## 3.4 Application to initial value problems

The following theorem describes the key property of the Laplace transform which is used in the analysis of initial value problems.

[0] If $\gamma : I \to \mathbb{C}$ is a $C^1$-function on an interval $I$ and $f$ is defined on $\gamma(I)$, the integral of $f$ along $\gamma$ is

$$\int_\gamma f(s)\,ds := \int_I f(\gamma(\tau))\gamma'(\tau)\,d\tau.$$

Then $\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty} := \int_\gamma$ with $\gamma(\tau) = \sigma + i\tau$, $\tau \in \mathbb{R}$.

**3.8 Theorem** *Let $f \in \mathscr{C}^N([0,+\infty))$ such that $f$ and all its derivatives are $\mathcal{L}$-transformable. Then the following is true*:

$$(\mathcal{L}f^{(j)})(s) = s^j(\mathcal{L}f)(s) - \sum_{k=1}^{j} f^{(k-1)}(0)\, s^{j-k}, \qquad \operatorname{Re} s > \sigma_f.$$

Proof: First note that

$$(\mathcal{L}f')(s) = \int_0^\infty e^{-sx} f'(x)\, dx = e^{-sx} f(x)\Big|_{x=0}^{x=+\infty} + \int_0^\infty s e^{-sx} f(x)\, dx = s(\mathcal{L}f)(s) - f(0).$$

Then

$$(\mathcal{L}f'')(s) = s(\mathcal{L}f')(s) - f'(0) = s(s(\mathcal{L}f)(s) - f(0)) - f'(0) = s^2(\mathcal{L}f)(s) - f(0)s - f'(0).$$

This procedure can be iterated. ■

Now consider the initial value problem

$$\boxed{y''(x) + a_1 y'(x) + a_0 y(x) = b(x), \qquad y(0) = y_0, \quad y'(0) = y_1.} \tag{3.4}$$

Applying the Laplace transform, this is equivalent to

$$s^2(\mathcal{L}y)(s) - y(0)s - y'(0) + a_1\big(s(\mathcal{L}y)(s) - f(0)\big) + a_0(\mathcal{L}y)(s) = (\mathcal{L}b)(s)$$

hence to

$$\underbrace{(s^2 + a_1 s + a_0)}_{=:P(s)}(\mathcal{L}y)(s) - (\underbrace{y_0 s + y_1 + a_1 y_0}_{=:Q(s)}) = (\mathcal{L}b)(s).$$

$P$ is called the characteristic polynomial of the ode. Solving for $\mathcal{L}y$, we find

$$(\mathcal{L}y)(s) = \frac{Q(s) + (\mathcal{L}b)(s)}{P(s)} = \frac{Q(s)}{P(s)} + \frac{(\mathcal{L}b)(s)}{P(s)}. \tag{3.5}$$

This identity determines $y$ uniquely due to the injectivity of $\mathcal{L}$ mentioned above. By applying the inverse Laplace transform we obtain a formula for the solution of (3.4):

$$\boxed{y(x) = \Big(\mathcal{L}^{-1}\frac{Q + \mathcal{L}b}{P}\Big)(x) = \Big(\mathcal{L}^{-1}\frac{Q}{P}\Big)(x) + \Big(b * \mathcal{L}^{-1}\frac{1}{P}\Big)(x).} \tag{3.6}$$

Using a partial fraction decomposition (note that the degree of $Q$ is smaller than that of $P$), both inverse Laplace transforms can be calculated.

**3.9 Example** (Partial fraction decomposition) *Let $P(s) = (s-\lambda_0)(s-\lambda_1)$ with $\lambda_0 \neq \lambda_1$. Then*

$$\frac{Q(s)}{P(s)} = \frac{a}{s-\lambda_0} + \frac{b}{s-\lambda_1} = \frac{(a+b)s - a\lambda_1 - b\lambda_0}{(s-\lambda_0)(s-\lambda_1)}$$

*with coefficients*

$$a = \frac{Q(s)}{s-\lambda_1}\Big|_{s=\lambda_0} = \frac{Q(\lambda_0)}{\lambda_0-\lambda_1}, \qquad b = \frac{Q(s)}{s-\lambda_0}\Big|_{s=\lambda_1} = \frac{Q(\lambda_1)}{\lambda_1-\lambda_0};$$

*alternatively, $a$ and $b$ are the unique solutions of the linear system*

$$a+b = y_0, \qquad -a\lambda_1 - b\lambda_0 = y_1 + a_1 y_0.$$

**3.10 Example** *Consider the problem*

$$y''(x) - 2y'(x) - 8y(x) = 0, \qquad y(0)=1, \quad y'(0)=2$$

*(note that $b(x)=0$). In this case,*

$$P(s) = s^2-2s-8 = (s+2)(s-4), \qquad Q(s) = s.$$

*By partial fraction decomposition:*

$$\frac{Q(s)}{P(s)} = \frac{s}{(s+2)(s-4)} = \frac{a}{s+2} + \frac{b}{s-4}$$

*with*

$$a = \frac{s}{s-4}\Big|_{s=-2} = \frac{1}{3}, \qquad b = \frac{s}{s+2}\Big|_{s=4} = \frac{2}{3}.$$

*Therefore*

$$y(x) = \Big(\mathcal{L}^{-1}\frac{Q}{P}\Big)(x) = \frac{1}{3}\Big(\mathcal{L}^{-1}\frac{1}{s+2}\Big)(x) + \frac{2}{3}\Big(\mathcal{L}^{-1}\frac{1}{s-4}\Big)(x) = \frac{1}{3}e^{-2x} + \frac{2}{3}e^{4x}.$$

**3.11 Example** *Consider the problem*

$$y''(x) - 2y'(x) + 2y(x) = b(x), \qquad y(0)=2, \quad y'(0)=3.$$

*In this case,*

$$P(s) = s^2-2s+2 = (s-(1+i))(s-(1-i)), \qquad Q(s) = 2s-1.$$

*We decompose*

$$\frac{2s-1}{P(s)} = \frac{a}{s-(1+i)} + \frac{b}{s-(1-i)}$$

*with*

$$a = \frac{2s-1}{s-(1-i)}\Big|_{s=1+i} = \frac{1+2i}{2i} = 1-\frac{i}{2}, \qquad b = \frac{2s-1}{s-(1+i)}\Big|_{s=1-i} = \frac{1-2i}{-2i} = 1+\frac{i}{2}.$$

*Therefore,*

$$\begin{aligned}\Big(\mathcal{L}^{-1}\frac{Q}{P}\Big)(x) &= \Big(\mathcal{L}^{-1}\frac{1-\frac{i}{2}}{s-(1+i)}\Big) + \Big(\mathcal{L}^{-1}\frac{1+\frac{i}{2}}{s-(1-i)}\Big)\\ &= \Big(1-\frac{i}{2}\Big)e^{(1+i)x} + \Big(1+\frac{i}{2}\Big)e^{(1-i)x} = e^x\Big((e^{ix}+e^{-ix}) + \frac{i}{2}(e^{-ix}-e^{ix})\Big)\\ &= e^x\big(2\cos x + \sin x\big).\end{aligned}$$

*Note that this function solves the above problem in case* $b(x) = 0$. *Moreover*

$$\frac{1}{P(s)} = \frac{1}{2i}\Big(\frac{1}{s-(1+i)} - \frac{1}{s-(1-i)}\Big),$$
$$\Big(\mathcal{L}^{-1}\frac{1}{P}\Big)(x) = \frac{e^{(1+i)x} - e^{(1-i)x}}{2i} = e^x\frac{e^{ix} - e^{-ix}}{2i} = e^x \sin x.$$

*We obtain*

$$\Big(b * \mathcal{L}^{-1}\frac{1}{P}\Big)(x) = e^x \int_0^x b(y)e^{-y}\sin(x-y)\,dy.$$

*Summing up, the solution is*

$$y(x) = e^x \sin x + e^x \int_0^x b(y)e^{-y}\sin(x-y)\,dy.$$

In case the coefficients of the ode and all initial values are real numbers one may employ the real partial fraction composition:

**3.12 Example** *Consider the problem*

$$y''(x) + 2y'(x) + 5y(x) = e^{-x}\sin x, \qquad y(0) = 0, \quad y'(0) = 1.$$

*Applying the Laplace transform we find*

$$(s^2+2s+5)(\mathscr{L}y)(s) - 1 = \frac{1}{(s+1)^2+1}, \quad \text{i.e.} \quad (\mathscr{L}y)(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}.$$

*The identity*

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

*leads to the linear system*

$$\begin{aligned} A + C &= 0 \\ 2A + B + 2C + D &= 1 \\ 5A + 2B + 2C + 2D &= 2 \\ 5B + 2D &= 3, \end{aligned}$$

*which has the unique solution* $A = C = 0$, $B = 1/3$ *and* $D = 2/3$. *We find that*

$$\begin{aligned} y(x) &= \Big(\mathscr{L}^{-1}\frac{1/3}{s^2+2s+2}\Big)(x) + \Big(\mathscr{L}^{-1}\frac{2/3}{s^2+2s+5}\Big)(x) \\ &= \frac{1}{3}\Big(\mathscr{L}^{-1}\frac{1}{(s+1)^2+1}\Big)(x) + \frac{1}{3}\Big(\mathscr{L}^{-1}\frac{2}{(s+1)^2+4}\Big)(x) \\ &= \frac{1}{3}e^{-x}(\sin x + \sin(2x)), \end{aligned}$$

*where the last identity holds due to Example* 3.3 *and Lemma* 3.5.*b*).

The final example shows an application of the Laplace transform to systems of ode.

**3.13 Example** *We want to solve the (first order) system*

$$\begin{aligned} y_1'(x) + 2y_2'(x) - 3y_2(x) &= 12e^x \\ y_1'(x) - y_2'(x) - 6y_1(x) &= 6 \\ y_1(0) = y_2(0) &= 0. \end{aligned}$$

*We need to find the two functions $y_1$ and $y_2$. Applying $\mathcal{L}$ to both equations yields*

$$\begin{aligned} s(\mathcal{L}y_1)(s) + (2s-3)(\mathcal{L}y_2)(s) &= 12\frac{1}{s-1} \\ (s-6)(\mathcal{L}y_1)(s) - s(\mathcal{L}y_2)(s) &= 6\frac{1}{s}, \end{aligned}$$

*which in matrix notation becomes*

$$\underbrace{\begin{pmatrix} s & 2s-3 \\ s-6 & -s \end{pmatrix}}_{=:A(s)} \begin{pmatrix} (\mathcal{L}y_1)(s) \\ (\mathcal{L}y_2)(s) \end{pmatrix} = 6\begin{pmatrix} \frac{2}{s-1} \\ \frac{1}{s} \end{pmatrix}.$$

*The inverse of $A(s)$ is*

$$A(s)^{-1} = -\frac{1}{3(s-2)(s-3)} \begin{pmatrix} -s & 3-2s \\ 6-s & s \end{pmatrix}.$$

*Together with a partial fraction decomposition we find*

$$\begin{aligned} (\mathcal{L}y_1)(s) &= \frac{1}{2}\frac{1}{s-1} - 3\frac{1}{s-2} + \frac{7}{2}\frac{1}{s-3} - \frac{1}{s} \\ (\mathcal{L}y_2)(s) &= 6\frac{1}{s-2} - \frac{7}{2}\frac{1}{s-3} - \frac{5}{2}\frac{1}{s-1}. \end{aligned}$$

*Applying $\mathcal{L}^{-1}$ results in*

$$y_1(x) = \frac{1}{2}e^x - 3e^{2x} + \frac{7}{2}e^{3x} - 1, \qquad y_2(x) = 6e^{2x} - \frac{7}{2}e^{3x} - \frac{5}{2}e^x.$$

## 3.5 Higher order equations

The solution formula (3.6) remains valid for the general case, i.e., (3.1) and (3.2). In this case the characteristic polynomial is

$$P(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = s^n + \sum_{\ell=0}^{n-1} a_\ell s^\ell,$$

while

$$Q(s) = \sum_{\ell=0}^{n-1}\Big(\sum_{k=\ell}^{n-1} a_{k+1}y_{k-\ell}\Big)s^\ell.$$

Now assume that

$$P(s) = (s-\lambda_1)^{\ell_1} \cdot \ldots \cdot (s-\lambda_k)^{\ell_k}$$

with its pairwise different zeros $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and $\ell_1 + \dots + \ell_k = n$. If $q(s)$ is an arbitrary polynomial of degree at most $n-1$ (in particular, $q(s) = Q(s)$ or $q(s) = 1$), then

$$\frac{q(s)}{P(s)} = \sum_{i=1}^{k} \sum_{j=1}^{\ell_i} \frac{c_{ij}}{(s-\lambda_i)^j}$$

with suitable coefficients $c_{ij} \in \mathbb{C}$ (which are determined by a system of linear equations). The inverse Laplace transform can be found by using Example 3.6.

In case all coefficients $a_0, \dots, a_{n-1}$ and all initial values $y_0, \dots, y_{n-1}$ are real numbers one can use real partial fraction decomposition. In this case $q(s)/P(s)$ can be written in the form

$$\frac{q(s)}{P(s)} = \sum_{i=1}^{k} \sum_{j=1}^{\ell_i} \frac{c_{ij}}{(s-\sigma_i)^j} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{ij}s + b_{ij}}{(s^2 + p_{ij}s + q_{ij})^j},$$

where all involved coefficients are real numbers, and the involved second order polynomials do not have real zeros. Again, the inverse Laplace transform can be calculated explicitly.