Master Degree in
Stochastics and Data Science

Analysis (A)

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Examination Program

The examination tests concern all the topics, examples and exercises treated during the lectures, and the contents of the teaching material available on the web page of the course, at the url

https://math.i-learn.unito.it/enrol/index.php?id=1542

To help in the preparation of the exam, we list below the requested topics, with some links to the chapters of the reference book and notes, namely

[1] B.P. Rynne, M.A. Youngson, *Linear functional analysis*, Springer-Verlag London, 2008.

[2] J. Seiler, *Distribution theory, Fourier and Laplace transform*
(lecture notes available on the web page of the course)

[3] J. Seiler, *The Banach fixed point theorem*
(lecture notes available on the web page of the course)

Preliminaries ([1], Ch. 1)

Basic concepts from linear algebra. Linear (vector) spaces and linear subspaces, linear operators (transformations) T and their main properties, in particular the kernel of T and the image of T, the rank of T. Eigenvalues and Eigenvectors. The *span* of a set A, the dimension of a linear space. The concept of basis.

Metric spaces. The concept of metric (distance). Main examples. Convergence of a sequence. The topology of a metric space. Continuity of a function between metric spaces. Baire's Category Theorem. The concept of completeness. Bolzano-Weierstrass Theorem. The Heine-Cantor Theorem. The density of a set. The Weierstrass Approximation Theorem. The Stone-Weierstrass Theorem. A countable set. Separable metric spaces, examples. The space of continuous functions on a closed and bounded interval is separable.

Lebesgue measure and integration. Definition/characterization of the Lebesgue measure. Basic elements of the construction of the Lebesgue integral. Properties of the Lebesgue integral. The Lebesgue spaces.

Normed and Banach Spaces ([1], Ch. 2)

Normed-spaces. Basic examples: the space of continuous functions on a closed and bounded interval $\mathcal{C}([a,b])$, the Lebesgue spaces $L^p(X)$, $1 \leq p \leq \infty$, the Lebesgue spaces of sequences ℓ^p, $1 \leq p \leq \infty$. Hölder and Minkowski inequalities. A normed space is a metric space with the induced distance. Finite-dimensional normed spaces. Equivalent norms and related properties. Banach spaces, examples. Relation between completeness and closure of linear subspaces of metric/Banach spaces. The closure of a linear space is a linear space. Riesz' Lemma. Non-compactness of the unit disc and of the unit circle in an infinite-dimensional normed space. Types of convergences of series in normed spaces. Characterization of Banach spaces by means of series.

Inner Product Spaces, Hilbert Spaces ([1], Ch. 3)

Inner Products. Basic examples. The Cauchy-Schwarz inequality. The norm induced by the inner product. Parallelogram rule and polarization identity. Orthogonality. Orthonormal sets and bases. Generalization of Pythagoras' Theorem for a k-dimensional inner product space. Any finite-dimensional vector space admits an orthonormal basis. Hilbert spaces. Basic examples. Orthogonal complements. Characterization of the orthogonal complement for linear subspaces. The Projection Theorem. The Orthogonal Decomposition Theorem and its corollaries. Orthonormal sequences. Bessel's Inequality. Orthonormal bases in infinite dimensions. Fourier Series for $L^2([-\pi, \pi])$.

Linear Operators ([1], Ch. 4)

Characterization of continuous linear operators. The space $B(X, Y)$ of continuous linear operators between the normed space X and Y. Equivalent norms. Basic Examples. If T is in $B(X, Y)$, then Ker T and the graph of T are closed. A matrix A is a linear, bounded operator. The Density Principle (Theorem 4.19). Definition of isometry, examples. Norm of an isometry. Definition of isometric isomorphism. Theorem of Riesz-Fischer. If X is a normed space and Y a Banach space, then $B(X, Y)$ is a Banach space. The dual space X' of a normed space X (Corollary 4.29, Lemma 4.30). $B(X)$ is an algebra with identity (Lemmas 4.32 and 4.33). The inverse of a linear bounded operator (Lemma 4.35). Definition of isomorphic normed spaces. Properties of isomorphic nomed linear spaces (Lemma 4.38). The invertibility of $I - T$ for T in $B(X)$, X Banach space, with $||T|| < 1$. The Neumann series. The Open Mapping Theorem. The Banach Isomorphism Theorem. The Closed Graph Theorem. Characterization of invertible operators (Lemmas 4.46 and 4.47, Theorem 4.48). Examples 4.50 and 4.51. The Uniform Boundedness Principle (Theorem 4.52).

Dual spaces ([1], Ch. 5)

Definition of the dual of a normed space. The dual space for a finite dimensional vector space. The Riesz-Fréchet Theorem. Consequences: the characterization of the dual of a Hilbert space. Examples of dual spaces. The case of ℓ^p, $L^p(X)$, for $1 \leq p < \infty$. The Hahn-Banach Theorem in normed spaces (Theorem 5.19). Consequences of the Hahn-Banach Theorem: the norm of a vector x in X expressed by means of the functionals $f \in X'$ (Corollary 5.22). Theorem 5.24: If X' is separable then so is X. The space ℓ^p is separable for $p < \infty$. ℓ^∞ is not separable. Definition of the bidual X'' of a normed space X. The canonical isometry J, from X to X''. Definition of reflexive spaces. Examples of reflexive and non-reflexive spaces (see the notes). Projections and complementary subspaces. Weak converge in X, Banach space. Examples. Main properties. Characterization of weak convergence. Convergence in the dual X': the norm-convergence, the weak convergence, the weak-* convergence. Relations among different types of convergence. Any norm-convergent sequence $\{x_n\}$ is weak convergent. Theorem of characterization of weak convergence. The Banach-Alaoglu Theorem and its consequences (in particular, Corollary 5.72).

Linear operators on Hilbert spaces ([1], Ch. 6)

Existence and uniqueness of the adjoint of a linear bounded operator on a Hilbert space. Comparison with the finite-dimensional case. Adjoint of the identity. Antilinearity and continuity of $^* : \mathcal{B}(\mathcal{H}, \mathcal{K}) \to \mathcal{B}(\mathcal{K}, \mathcal{H})$, adjoint of the composition, $(T^*)^* = T$, $\|T^*\| = \|T\|$, $\|T^* T\| = \|T\|^2$. Orthogonality relations $\operatorname{Ker} T = (\operatorname{Im} T^*)^\perp$, $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$, $(\operatorname{Ker} T)^\perp = \overline{\operatorname{Im} T^*}$, $(\operatorname{Ker} T^*)^\perp = \overline{\operatorname{Im} T}$. Invertibility criteria. For T invertible, $(T^*)^{-1} = (T^{-1})^*$. Normal, self-adjoint and unitary operators. Positive operators and projections.

Compact operators ([1], Ch. 7)

Equivalent definitions of compact operators between Banach spaces. Finite rank operators and finite-dimensional cases. Compactness of compositions of compact and bounded operators, as well as of linear combinations of compact operators. Non-compactness of the identity on an infinite-dimensional Hilbert space. Non-invertibility of compact operators on an infinite-dimensional Hilbert space. Compactness of the limits of Cauchy sequences of compact operators (that is, $K(X, Y)$ is closed in $B(X, Y)$). Compact operators on a Hilbert space as limits of finite rank operators. For $\mathcal{H}$ a Hilbert space, $T \in K(\mathcal{H}) \Leftrightarrow T^* \in K(\mathcal{H})$. Hilbert-Schmidt operators and their properties. Compact perturbations of the identity. Fredholm alternative.

Spectral theory ([1], Ch. 6, 7)

The spectrum $\sigma(T)$ and the resolvent set $\rho(T)$ of a bounded operator T on a Hilbert space. Continuous, point and residual spectrum; eigenvalues and eigenvectors. $\sigma(\mu I) = \{\mu\}$. $\sigma(T)$ is a closed set, $|\lambda| > \|T\| \Rightarrow \lambda \in \rho(T)$. $\sigma(T^*)$; $\sigma(p(T))$ for p polynomial; for T invertible, $\sigma(T^{-1})$. Spectral radius and numerical range of $T \in B(\mathcal{H})$. Properties of $\sigma(T)$ for $T = T^*$. Spectrum of a compact operator on a Hilbert space. Spectrum of a self-adjoint compact operator on a Hilbert space. Eigenvectors of a self-adjoint compact operator. Orthonormal bases of eigenvectors of a self-adjoint compact operator.

Applications ([1], Ch. 8)
Well-posedness of equations of the form $Ru = f$ for a linear operator R. Integral equations of Fredholm type. Compactness of integral operators with continuous kernel on $\mathcal{C}([a,b])$. Ascoli-Arzelá theorem (characterization of compact subsets of $\mathcal{C}([a,b])$). Compactness of integral operators with kernel in $L^2([a,b] \times [a,b])$ on $L^2[a,b]$. Adjoint of an integral operator on $L^2[a,b]$. Integral operators with Hermitian/symmetric kernels. Boundary value problems for $L = \dfrac{d^2}{dx^2}$ on $[a,b]$: unboundedness, bijectivity, and Green function/operator; equivalent integral equation and solvability; eigenvalues and eigenfunctions. Sturm-Liouville problems (hints).

Banach Fixed point theorem (lecture notes [3])
Strict contraction maps on metric spaces. The Banach fixed point theorem. Application to initial value problems for ordinary differential equations.

Elements of distribution theory (lecture notes [2])
The test-function space $\mathscr{D}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open subset. Existence of non-trivial functions in $\mathscr{D}(\Omega)$. Convergent sequences in $\mathscr{D}(\Omega)$. The space $L^1_{\text{loc}}(\Omega)$. The space $\mathscr{D}'(\Omega)$ of distributions. Characterization of distributions by norm estimates. Examples: Regular distributions and their density, Dirac distributions, principal value of $1/x$. The distribution δ is not a regular distribution. Operations with distributions: partial derivative, multiplication by smooth functions. Derivative of regular distributions with piecewise $\mathcal{C}^1$ density. Examples. Convolution of distributions and test-functions. Application to linear partial differential equations. Fundamental solution. Theorem of Malgrange-Ehrenpreis. Examples. The space of rapidly decreasing functions $\mathscr{S}(\mathbb{R}^n)$. Inclusions $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. The space of temperate distributions $\mathscr{S}'(\mathbb{R}^n)$. Inclusion $\mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$. Examples.

The Fourier transform (lecture notes [2])
Fourier transform on $L^1(\mathbb{R}^n)$: elementary properties and inversion theorem. Fourier transform on $L^2(\mathbb{R}^n)$: Parseval's formula and Parseval-Plancherel's Theorem. Fourier transform on $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$: continuity and bijectivity. Fourier transform of $\partial^\alpha v$ and $x^\alpha v$, for $v \in \mathscr{S}(\mathbb{R}^n)$ and $v \in \mathscr{S}'(\mathbb{R}^n)$. Examples. Fourier transform of convolutions. The Heisenberg uncertainty principle.

Main results

(with requested proof (unmarked) and non-requested proof (marked with *))

1. Riesz' Lemma ([1], Theorem 2.25).
2. If X is an infinite-dimensional normed space, then neither the unit disc nor the unit circle is compact ([1], Theorem 2.26).
3. Parallelogram rule and polarization identity ([1], Theorem 3.15).

*4. The Gram-Schmidt orthogonalization procedure ([1], Lemma 3.20 (b)).

5. Generalization of Pythagoras' Theorem for a k-dimensional inner product space ([1], Theorem 3.22).

*6. Basic properties of the orthogonal complements ([1], Lemma 3.29).

*7. Characterization of the orthogonal complement for linear subspaces ([1], Lemma 3.30).

*8. The Projection Theorem (Theorem 3.32).

9. The Orthogonal Decomposition Theorem ([1], Theorem 3.34).

*10. Corollaries of the Orthogonal Decomposition Theorem ([1], Corollaries 3.35 and 3.36)

*11. An infinite-dimensional inner product space X admits an orthonormal sequence ([1], Theorem 3.40).

12. Bessel's Inequality ([1], Lemma 3.41).

*13. Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}$ be an orthonormal sequence in $\mathcal{H}$. Let $\{\alpha_n\}$ be a sequence of scalars. Then the series $\sum_{n=1}^{\infty} \alpha_n e_n$ converges if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. ([1], Theorem 3.42).

*14. Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}_{n\geq 1} \subset \mathcal{H}$ an orthonormal sequence. Then, for any $x \in \mathcal{H}$, the series $\sum_{n=1}^{\infty}(x, e_n)e_n$ is convergent ([1], Corollary 3.44).

15. Orthonormal Bases in Infinite Dimensions ([1], Theorem 3.47).

16. Characterization of Continuous Linear Operators ([1], Lemma 4.1). Examples (all those mentioned in the lecture notes of Prof. Cordero).

*17. If T is in $B(X, Y)$, then Ker T and the Graph of T are closed ([1], Lemmas 4.11 and 4.13).

*18. The Density Principle ([1], Theorem 4.19).

*19. Let X and Y be normed spaces and $T \in L(X, Y)$. If T is an isometry, then $T \in B(X, Y)$ and $\|T\| = 1$ ([1], Lemma 4.23).

20. The Riesz-Fischer Theorem ([1], Theorem 4.25).

*21. Let X be a normed space, Y be a Banach space. Then, $B(X, Y)$ is a Banach space ([1], Theorem 4.27).

22. The Invertibility of $I - T$ for T in $B(X)$, X Banach space, with $||T|| < 1$ and the Neumann series ([1], Theorem 4.40 and Example 4.41).

*23. The Closed Graph Theorem ([1], Corollary 4.44)

*24. The Banach Isomorphism Theorem ([1], Corollary 4.45).

*25. Characterization of invertible operators ([1], Theorem 4.48).

26. The Riesz-Fréchet Theorem ([1], Theorem 5.2).

*27. The Characterization of the dual of a Hilbert space ([1], Theorem 5.3).

*28. The canonical isometry J, from X to X'' ([1], Lemmas 5.35 and 5.37).

*29. Any norm-convergent sequence $\{x_n\}$ is weak convergent (lecture notes).

*30. The Banach-Alaoglu Theorem ([1], Theorem 5.71).

31. Existence of the adjoint of an operator $T \in B(\mathcal{H}, \mathcal{K})$, $\mathcal{H}, \mathcal{K}$ complex Hilbert spaces ([1], Theorem 6.1).

32. Properties of the adjoint of an operator $T \in B(\mathcal{H}, \mathcal{K})$, $\mathcal{H}, \mathcal{K}$ complex Hilbert spaces ([1], Theorem 6.10).

33. Let $\mathcal{H}$ be a complex Hilbert space, $T \in B(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Definition of the spectrum of T: the point spectrum, the continuous spectrum, the residual spectrum (Cordero's lecture notes). Let $\mathcal{H}$ be a complex Hilbert space, $T \in B(\mathcal{H})$, and $\lambda \in \mathbb{C}$. Then, (a) $|\lambda| > \|T\| \Rightarrow \lambda \in \rho(T)$; (b) $\rho(T)$ is open ($\Leftrightarrow \sigma(T)$ is closed) ([1], Theorem 6.36. For the proof see Cordero's lecture notes).

*34. Properties of the spectrum of a self-adjoint operator on a complex Hilbert space ([1], Theorem 6.43).

35. Let $\mathcal{H}$ be a complex Hilbert space and let $T \in B(\mathcal{H})$ be self-adjoint. Then, eigenvectors corresponding to different eigenvalues of T are orthogonal ([1], Theorem 7.33 (only the last statement, for T self-adjoint, not necessarily compact)).

36. Let X, Y be normed spaces. Then, $K(X,Y) \subset B(X,Y)$ ([1], Theorem 7.2).

37. Let X, Y be normed spaces and $T \in B(X,Y)$.
(a) If T is of finite rank, then, $T \in K(X,Y)$.
(b) If either dim X or dim Y is finite then $T \in K(X,Y)$. ([1], Theorem 7.5).

38. If X is an infinite-dimensional normed spaces, then the identity operator I on X is not compact ([1], Theorem 7.6).

39. If X is an infinite-dimensional normed spaces and $T \in K(X)$, then T is not invertible ([1], Corollary 7.7).

*40. Let X be a normed space, Y be a Banach space, $T \in B(X,Y)$ and $\{T_n\} \subset K(X,Y)$ be such that $T_n \to T$ for $n \to +\infty$ in $B(X,Y)$. Then, $T \in K(X,Y)$ ([1], Theorem 7.9).

*41. Properties of Hilbert-Schmidt operators ([1], Theorem 7.16).

42. Spectrum of a compact operator on an infinite-dimensional space ([1], Theorem 7.18 (only the first statement) and Theorem 7.19).

*43. Let $\mathcal{H}$ be a complex Hilbert space, $T \in K(\mathcal{H})$ and $T = T^*$. Then, at least one of the numbers $\|T\|$, $-\|T\|$ is an eigenvalue of T ([1], Theorem 7.32).

44. Banach fixed point theorem: Any strict contraction on a complete metric space has a fixed point; a-priori and a-posteriori estimate for approximating sequence ([3]).

45. If $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$ then $\partial^\alpha \phi_k \to \partial^\alpha \phi$ in $\mathcal{D}(\Omega)$ ([2], Lemma 1.6).

*46. Characterization of distributions by norm-estimates ([2], Theorem 1.8).

47. Regular distributions associated with locally integrable functions ([2], Example 1.9).

48. The δ-distribution ([2], Example 1.10).

*49. If $T \in \mathcal{D}'(\Omega)$ then $\partial^\alpha T \in \mathcal{D}'(\Omega)$ ([2], Theorem 1.13).

50. The derivative of a distribution with piecewise $\mathcal{C}^1$-density. ([2], Theorem 1.15, Examples 1.14, 1.16).

*51. Theorem of Malgrange-Ehrenpreis ([2], Theorem 1.22).

52. The Fourier transform $\mathscr{F}$ is a linear and bounded map from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ ([2], Lemma 2.1).

53. The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is a continuous function ([2], Lemma 2.1).

54. $\mathscr{F}(f * g) = \mathscr{F}f \cdot \mathscr{F}g$ and $\int \mathscr{F}f \cdot g = \int f \cdot \mathscr{F}g$, for $f, g \in L^1(\mathbb{R}^n)$ ([2], Lemma 2.2).

*55. $\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ ([2], Lemma 2.6).

*56. The Fourier transform of $\partial^\alpha \varphi$ and $x^\alpha \varphi$, for $\varphi \in \mathscr{S}(\mathbb{R}^n)$ ([2], Theorem 2.9).

*57. Continuity and bijectivity of $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ ([2], Theorems 2.9 and 2.10).

58. Parseval's formula ([2], Theorem 2.11).

*59. Parseval-Plancherel's theorem ([2], Corollary 2.12).

60. The Fourier transform $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ and its inverse ([2], Section 2.3 and Theorem 2.15).

61. The Heisenberg uncertainty principle ([2], Theorem 2.22) only for $x_0 = \xi_0 = 0$.

Important! The following notions are considered fundamental. It is necessary to know them, as well as **to be able to state the corresponding definitions in a precise and rigorous way**.

1. Definitions of complete metric space, of normed space, of Banach space, and of Hilbert space.
2. Definition of linear bounded operator among Banach spaces and of the related operator norm.
3. Definition of $L^p(A)$ space, $1 \leq p \leq \infty$, $A \subseteq \mathbb{R}^n$ measurable set, and of the corresponding norms.
4. Definition of convolution of two functions.
5. Definition of the adjoint of a linear bounded operator on a Hilbert space.
6. Definition of a compact operator.
7. Definition of spectrum and of resolvent set of a linear bounded operator on a Hilbert space.
8. Definition of the function spaces $\mathscr{D}(\Omega)$ and $\mathscr{S}(\mathbb{R}^n)$, and convergence of sequences in these spaces.
9. Definition of the distribution spaces $\mathscr{D}'(\Omega)$ and $\mathscr{S}'(\mathbb{R}^n)$.
10. Definition of partial derivatives of a distribution.
11. Definition of the Fourier transform on $L^1(\mathbb{R}^n)$, $L^2(\mathbb{R}^n)$, $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$.