M.Sc. in Stochastics and Data Science
Analysis Course A (Advanced)

Homework 1

Exercise 1. Let $(X, \|\cdot\|)$ be a normed vector space and let S be a subspace of X. Let $\|\cdot\|_S$ be the restriction of $\|\cdot\|$ to S. Show that $\|\cdot\|_S$ is a norm on S.

Exercise 2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces over $\mathbb{F}$ and let $Z = X \times Y$ be the Cartesian product of X and Y. This is a vector space.

(i) Show that $\|(x, y)\|_Z = \|x\|_X + \|y\|_Y$ defines a norm on Z.

(ii) Show that a sequence $\{(x_n, y_n)\} \subset Z$ converges to (x, y) in Z if and only if $\{x_n\}$ converges to x in X and $\{y_n\}$ converges to y in Y.

(iii) Show that a sequence $\{(x_n, y_n)\} \subset Z$ is a Cauchy sequence in Z if and only if $\{x_n\}$ is a Cauchy sequence in X and $\{y_n\}$ is a Cauchy sequence in Y.

Exercise 3. Consider the space of sequences

$$c_{00} = \{(a_n)_n \ : \exists n_0 \in \mathbb{N} : \forall n \geq n_0, a_n = 0\}.$$

(i) Show that $c_{00} \subset \ell^\infty$ and that it is a normed space with the norm $\|(a_n)\|_\infty = \sup_n |a_n|$.
Hint: Use Exercise 1.

(ii) Similarly, show that $c_{00} \subset \ell^p$, for every $1 \leq p < \infty$, and that it is a normed space with the norm $\|(a_n)\|_p = (\sum_n |a_n|^p)^{\frac{1}{p}}$.

Exercise 4. Let V be a non-empty set. Consider the function $d : V \times V \to \mathbb{R}$, defined by

$$d(x, y) = \begin{cases} 0 & : x = y \\ 1 & : x \neq y \end{cases}.$$

Show that d is a metric.

Exercise 5. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on a vector space X. Consider the induced metrics $d_1(x, y) = \|x - y\|_1$ and $d_2(x, y) = \|x - y\|_2$. Show that a sequence $\{x_n\}$ on X is a Cauchy sequence in the metric space (X, d_1) if and only if it is a Cauchy sequence in the metric space (X, d_2).

Exercise 6. Recall that $\mathcal{C}([-1,1])$ equipped with the norm $\|f\|_\infty = \sup_{x\in[-1,1]} |f(x)|$ is a Banach space.

(i) Prove that $|||f||| := \sup_{x\in[-1,1]} |e^{2x} f(x)|$ defines a norm on $\mathcal{C}([-1,1])$.

(ii) Show that $(\mathcal{C}([-1,1]), |||\cdot|||)$ is a Banach space.
Hint: Show that $|||\cdot|||$ and $\|\cdot\|_\infty$ are equivalent norms.

Exercise 7. Consider on $\mathcal{C}([0,1])$ the following two norms:

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \qquad \|f\|_1 = \int_0^1 |f(x)|\, dx.$$

Show that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent norms.
Hint: Consider the sequence $f_n(x) = x^n$, $n \in \mathbb{N}$, and show that the equivalence of norms fails on this sequence.

Exercise 8. Let $(X, \|\cdot\|)$ be a normed space, $x_0 \in X$ and $r > 0$. Define

$$D = \{x \in X : \|x - x_0\| \le r\} \quad \text{and} \quad B = \{x \in X : \|x - x_0\| < r\}.$$

(i) Show that D is closed and B is open in X.

(ii) Fix $y \in D$ and consider the sequence

$$y_n = \left(1 - \frac{1}{n}\right) y + \frac{x_0}{n}, \qquad n \in \mathbb{N}.$$

Show that $\lim_{n\to\infty} y_n = y$.

(iii) Use the previous step and show that $\overline{B} = D$.