M.Sc. in Stochastics and Data Science
Analysis Course A (Advanced)
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Homework 2

Exercise 1. Let X and Y be inner product spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ respectively, and let $Z = X \times Y$ be the Cartesian product space. Prove that the function $(\cdot,\cdot)_Z : Z \times Z \to \mathbb{F}$ defined by

$$((x_1, y_1), (x_2, y_2))_Z = (x_1, x_2)_X + (y_1, y_2)_Y$$

is an inner product on Z.

Exercise 2. Let X be an inner product space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$. For every $x, y \in X$, prove the following equalities

(i)
$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$
(the parallelogram rule);

(ii) If X is real then
$$(x,y) = \frac{\|x+y\|^2 - \|x-y\|^2}{4};$$

(iii) If X is complex then
$$(x,y) = \frac{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2}{4}$$
(the polarization identity).

Exercise 3. Let X and Y be linear subspaces of a Hilbert space H. Recall that

$$X + Y = \{x + y : x \in X, y \in Y\}.$$

Prove that $(X+Y)^\perp = X^\perp \cap Y^\perp$.

Exercise 4. Let H be a Hilbert space and let $A \subseteq H$ be a non-empty set. Show that:

(i) $A^{\perp\perp} = \overline{\mathrm{Sp}}A$;

(ii) $A^{\perp\perp\perp} = A^\perp$ (where $A^{\perp\perp\perp} := ((A^\perp)^\perp)^\perp$).

Exercise 5. Let H be a Hilbert space and let $\{e_n\}$ be an orthonormal sequence in H. Determine whether the following series converge in H:

$$(a) \quad \sum_{n=1}^{\infty} \frac{e_n}{n}, \qquad (b) \quad \sum_{n}^{\infty} \frac{e_n}{n^{\frac{1}{2}}}$$

Exercise 6. Consider the function $f \in L^2([-\pi, \pi])$ given by

$$f(x) = |x|, \quad x \in [-\pi, \pi).$$

Compute the Fourier series of f. *Hint:* Observe that f is even.

Exercise 7. Let $T : \mathcal{C}_{\mathbb{R}}([0,1]) \to \mathbb{R}$ be the transformation defined by

$$T(f) = \int_0^1 f(x)dx.$$

(i) Show that T is linear and bounded;

(ii) Show that $\|T\| \leq 1$;

(iii) Show that $\|T\| = 1$. *Hint:* Evaluate $T(g)$ where g is the function in $\mathcal{C}_{\mathbb{R}}([0,1])$ such that $g(x) = 1$ for every $x \in [0,1]$.

Exercise 8. Consider the space

$$\mathcal{C}'_{\mathbb{C}}([a,b]) = \{f : [a,b] \to \mathbb{C} : \, f \text{ is differentiable and } f' \text{ is continuous on } [a,b]\}$$

(i) Show that

$$\|f\|_{\mathcal{C}'} = \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)| = \|f\|_\infty + \|f'\|_\infty$$

is a norm on $\mathcal{C}'_{\mathbb{C}}([a,b])$.

(ii) For $f \in \mathcal{C}'_{\mathbb{C}}([a,b])$, consider the operator $T : \mathcal{C}'_{\mathbb{C}}([a,b]) \to \mathbb{C}$ defined by

$$T(f) = f'(a).$$

Show that T is linear and bounded.

Exercise 9. Given a sequence $(x_n) \in \ell^2$, consider the operator

$$T(x_1, x_2, x_3, x_4, \dots) = (0, 4x_1, x_2, 4x_3, x_4, \dots).$$

(i) Show that $T : \ell^2 \to \ell^2$ is linear and bounded.

(ii) Compute $\|T\|$. *Hint:* Consider the sequence $e_1 = (1, 0, 0, 0, \dots) \in \ell^2$ and compute Te_1.