M.Sc. in Stochastics and Data Science
Analysis Course A (Advanced)
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Homework 3

Exercise 1. Given a sequence $(x_n) \in \ell^2$, consider the operator

$$T(x_1, x_2, x_3, \dots, x_n, \dots) = ((1+1)x_1, (1+\frac{1}{4})x_2, (1+\frac{1}{9})x_3, \dots, (1+\frac{1}{n^2})x_n, \dots).$$

(i) Show that $T : \ell^2 \to \ell^2$ is well-defined, linear and bounded.

(ii) Show that T is invertible. (*Hint:* Show that T is one-to-one and onto, then apply the Banach Isomorphism Theorem.)

Exercise 2. Let H be a complex Hilbert space and let M be a closed linear subspace of H. If $f \in M'$, show that there exists a $g \in H'$ such that

$$g(x) = f(x), \quad \forall x \in M, \text{ and } \quad \|f\| = \|g\|.$$

(*Hint:* Apply first The Riesz-Fréchet Theorem to the Hilbert space M and represent accordingly any element in M'. Then construct an extension of $f \in M'$ to the whole space H'.)

Exercise 3. Let X be a normed space and P, Q be projections on X. Show that:

(i) If $PQ = QP$ then PQ is a projection.

(ii) We have
$$QP = P \Leftrightarrow \mathrm{Im}P \subseteq \mathrm{Im}Q.$$

Exercise 4. Consider $H = \mathbb{R}^2$ and the subspace

$$W = \{(0, x_2), \quad x_2 \in \mathbb{R}\}.$$

Consider $f_W : W \to \mathbb{R}$, defined by

$$f(0, x_2) = f(x_2) = \sqrt{7}x_2, \quad x_2 \in \mathbb{R}.$$

Show that:

(i) $f_W \in W'$ and compute $\|f_W\|_{W'}$.

(ii) Construct an extension $\tilde{f}$ of f_W such that $\tilde{f} \in H'$, $\tilde{f}(w) = f_W(w)$, for every $w \in W$, and $\|\tilde{f}\|_{H'} = \|f_W\|_{W'}$.