Lecture 2

martedì 21 settembre 2021 15:39

METRIC SPACES

Def. Consider $M \neq \emptyset$ a set. A map $d: M \times M \to \mathbb{R}$ is called a **metric (or distance)** if for all $x, y, z \in M$:

1) $d(x,y) \geq 0$
2) $d(x,y) = 0 \iff x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,z) \leq d(x,y) + d(y,z)$ "triangle inequality"

(M, d) is called a **metric space**.

Examples. 1) $M = \mathbb{R}^n$ "standard Euclidean metric":

$$d(x,y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} \qquad x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n)$$

$M = \mathbb{F}^n$ with $d_p(x,y) = \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{1/p}$

for $p = 2$ we come back to the Euclidean distance, $1 \leq p < \infty$

$p = \infty$ $\quad d_\infty(x,y) = \max_{1 \leq j \leq n} |x_j - y_j|$

2) $C_{\mathbb{F}}([a,b])$ with $d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$

3) (M, d) metric space

$N \subset M$ subset, define the "induced metric"

$d_N(x,y) = d(x,y), \quad \forall x, y \in N$

(N, d_N) is a metric space

Homework 1, Exercise 4

SEQUENCES AND SUBSEQUENCES

(M, d) metric space

A sequence in M is a map $a : \mathbb{N} \to M$;

write $a_n := a(n)$

Standard notations: (a_n), $(a_n)_n$, $(a_n)_{n\in\mathbb{N}}$

same notations with $\{\ \}$ instead of $(\ )$

We will use (a_n) for numerical sequences and $\{a_n\}$ for the other sequences.

If $v : \mathbb{N} \to \mathbb{N}$ is strictly increasing

then if $a : \mathbb{N} \to M$ is a sequence, the map $a \circ v$ is called subsequence of a

$n_k := v(k) \in \mathbb{N}$ $\qquad a_{n_k} := a \circ v(k)$

Notation: (a_{n_k}), $(a_{n_k})_k$, $(a_{n_k})_{k\in\mathbb{N}}$

similarly with $\{\ \}$ in place of $(\ )$

Example $\quad a_n = \dfrac{1}{2^n} \quad n \in \mathbb{N}$

$k \mapsto 2k$

$a_{n_k} = \dfrac{1}{2^{2k}}$ $\quad (2^{-2k})$ is a subsequence of (2^{-n})

CONVERGENCE AND COMPLETENESS

(M, d) metric space and a sequence $\{x_n\} \subset M$.

1) $\{x_n\}$ converges to $x \in M \iff$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \quad \forall n \geq N \quad d(x_n, x) < \varepsilon$

Write $x_n \to x$ or $x = \lim\limits_{n\to\infty} x_n$

2) $\{x_n\}$ is called a Cauchy sequence $\iff$
$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \quad \forall n, m \geq N \quad d(x_n, x_m) < \varepsilon$

Note: $x_n \to x$ in $M \iff \lim\limits_{n\to\infty} d(x_n, x) = 0$

Thrm. (M, d) metric space. $\{x_n\} \subset M$ sequence.
Then if $x_n \to x$ we have:

1) the limit x is unique.
2) Any subsequence $\{x_{n_k}\}$ is convergent to the same limit x
3) $\{x_n\}$ is a Cauchy sequence.

Def. (M, d) is called complete if every Cauchy sequence in M is convergent in M.

Examples. 1) The following spaces are complete:

- $\mathbb{R}^n$, $\mathbb{C}^n$ are complete with the standard metric or d_p or d_∞
- $C_{\mathbb{F}}([a,b])$ with $d(f,g) = \max\limits_{x\in[a,b]} |f(x) - g(x)|$ is complete.

Note: $f_n \to f$ in $C_{\mathbb{F}}([a,b])$
$\iff f_n \to f$ uniformly on $[a,b]$

2) $M = (0, 1)$ with $d(x,y) = |x - y|$ is not complete

$\left(\frac{1}{n}\right)_{n\in\mathbb{N}_+} \subset M$

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

$\left(\frac{1}{n}\right)$ is a convergent sequence in $\mathbb{R}$

$\Rightarrow \left(\frac{1}{n}\right)$ is a Cauchy sequence in $\mathbb{R}$

$\Rightarrow \left(\frac{1}{n}\right)$ is a Cauchy sequence in M

which does not converge in M

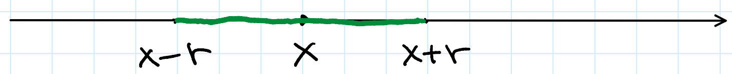
because $0 \notin M$.

Open ball centered at x with radius $r > 0$:

$$B_x(r) = B_r(x) = B(x,r) = \{ y \in M : d(y,x) < r \}$$

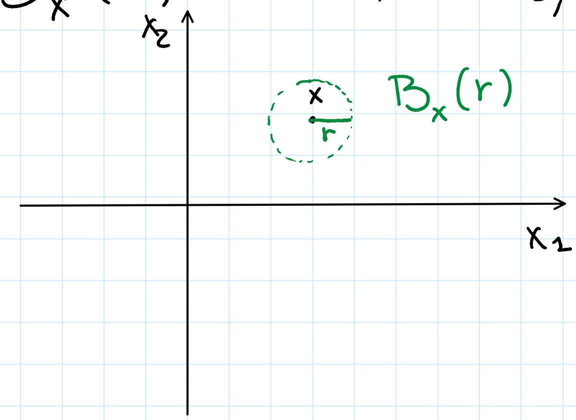
Examples. $\cdot\ M = \mathbb{R} \qquad d(x,y) = |x-y|$

$$B_x(r) = (x-r,\ x+r)$$

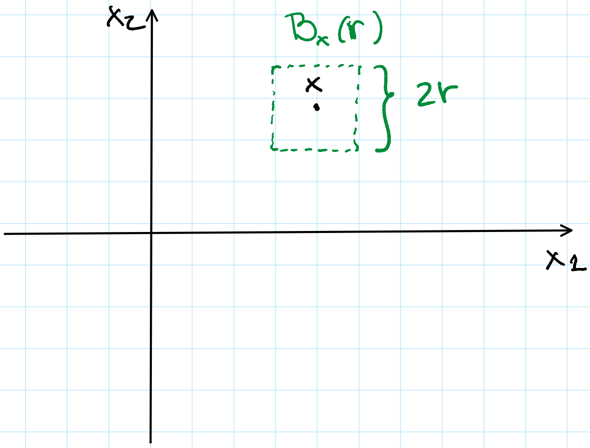


$\cdot\ M = \mathbb{R}^2 \qquad d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

$B_x(r) \qquad x = (x_1, x_2)$



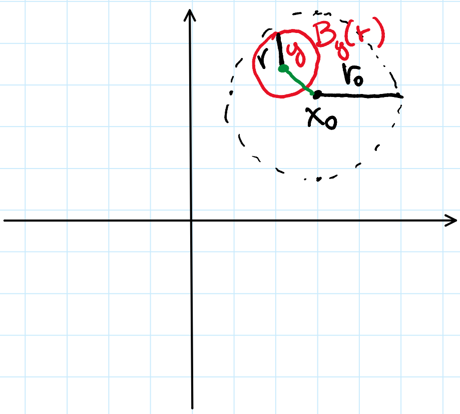
with the d_∞ distance:



Def. • A set $A \subseteq M$ is called open if $\forall x \in A \ \exists r > 0 : B_x(r) \subseteq A$

• A set $C \subseteq M$ is called closed if $M \setminus C$ is open.

Example. $A = B_{x_0}(r_0)$ is open



consider $y \in B_{x_0}(r_0)$ $\quad \tilde{r} := r_0 - d(y, x_0) > 0$

choose $r \mid \quad 0 < r \leq \tilde{r}$

$\forall x \in B_y(r) \quad d(x, x_0) \leq d(x, y) + d(y, x_0)$

$$< r + d(y, x_0)$$

$$\leq r_0 - \cancel{d(y, x_0)} + \cancel{d(y, x_0)}$$

$\Rightarrow x \in B_{x_0}(r_0) \quad \Rightarrow \quad B_{x_0}(r_0)$ is open

Thrm. Consider $C \subseteq M$. Then

C is closed $\iff$ for every sequence $\{x_n\} \subset C$ which converges in M, $\lim_{n\to\infty} x_n \in C$

Def. $A \subseteq M$ set. The closure of A is

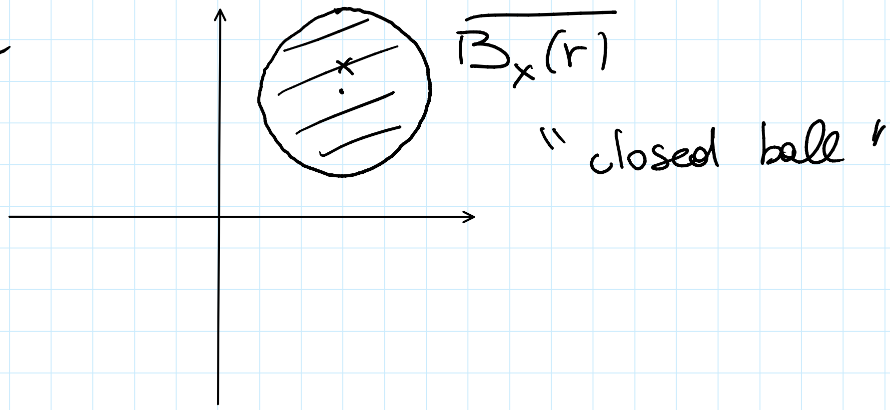
$$\bar{A} = \bigcap \{ C \text{ closed in } M : A \subseteq C \}$$

$\bar{A}$ is the smallest closed set containing A

Example. $M = \mathbb{R}^n$ standard metric

$$\overline{B_x(r)} = \{ y \in \mathbb{R}^n \mid d(y,x) \le r \}$$

es. $n = 2$



Attention: this is not true in a generic metric space (M, d)

Thrm. $A \subseteq M$ and $x \in M$. Then are equivalent:

1) $x \in \bar{A}$
2) $\forall \varepsilon > 0 \ \exists a \in A : d(x,a) < \varepsilon$
3) $\inf_{a \in A} d(x,a) = 0$
4) $\exists \{x_n\} \subset A : x_n \to x$.

Def. $A \subseteq M$ is called dense in M if $\bar{A} = M$

Examples. • $M = \mathbb{R}$ $\quad A = \mathbb{Q}$ or $A = \mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$ (with the standard metric)

- $(0,1)$ is dense in $[0,1]$

$\left(\frac{1}{n}\right) \subset (0,1)$ $\lim_{n\to\infty} \frac{1}{n} = 0$

$\left(1-\frac{1}{n}\right)_{n\geq 2} \subset (0,1)$ $\lim_{n\to\infty} 1-\frac{1}{n} = 1$

$\Rightarrow \overline{(0,1)} = [0,1]$

Def. $A \subseteq M$ is called **bounded** if $\exists\, x\in M,\ r>0 \mid A \subseteq B_x(r)$.

BAIRE CATEGORY THEOREM

Consider (M,d) metric space <u>complete</u>.

If $M = \bigcup_{j=1}^{\infty} C_j$ with $C_j \subseteq M$ closed then $\exists\, k\in\mathbb{N},\ x\in M,\ r>0$ such that $B_x(r) \subseteq C_k$

(i.e., at least one closed set C_j contains an open ball).

Def. $K \subseteq M$ is called

- **compact** if every sequence $\{x_n\} \subseteq K$ contains a convergent subsequence $\{x_{n_k}\}$, $x_{n_k} \to x \in K$ (the limit point x is in K).
- **relatively compact** if $\overline{K}$ is compact ($\Leftrightarrow$ every sequence $\{x_n\} \subset K$ admits a convergent subsequence)

Thrm. $K \subseteq M$ is compact $\Rightarrow$ K is closed and bounded

Observe that " $\Leftarrow$ " is not true in general

Bolzano - Weierstrass Theorem. Let $K \subseteq \mathbb{R}^n$ or $K \subseteq \mathbb{C}^n$. Then K is compact $\Leftrightarrow$ K is closed and bounded.

CONTINUITY IN METRIC SPACES

Consider (M, d_M), (N, d_N) metric spaces.

Consider $f : M \to N$.

Def. 1) f is continuous at $x \in M$ if

$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall y \in M \ d_M(y,x) < \delta \Rightarrow d_N(f(y), f(x)) < \varepsilon$

2) f is continuous on M if f is continuous at every $x \in M$

3) f is uniformly continuous on M if

$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall x, y \in M \ d_M(x,y) < \delta \Rightarrow$

$d_N(f(x), f(y)) < \varepsilon$

Examples. $M = N = \mathbb{R}$ $\quad f(x) = x$ uniformly continuous

$f(x) = x^2$ or $f(x) = e^x$ continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$ (troubles arise in a neighbourhood of $+\infty$)

Thrm. $f : M \to N$.

1) f is continuous at $x \in M \Leftrightarrow$ for every sequence $\{x_n\} \subseteq M$ with $x_n \to x$ holds $f(x_n) \to f(x)$

2) f is continuous on $M \Leftrightarrow f^{-1}(A) \subseteq M$ is open for every $A \subseteq N$ open

$\Leftrightarrow f^{-1}(C) \subseteq M$ is closed for every $C \subseteq N$ closed

$\Leftrightarrow$ $f(C) \subseteq M$ is closed for every $C \subseteq N$ closed.

T.A. Marco Morandotti: Wednesday 6^{th} October, 4.30-6.30pm