HEINE-CANTOR THEOREM

Let M be a compact metric space with metric d.
If f is continuous on M then it is uniformly continuous on M.

WEIERSTRASS THEOREM

(M,d) compact metric space. $f: M \to \mathbb{R}$ continuous on M. Then there exist $x_m, x_M \in M$ such that

$$f(x_m) = \min_{x\in M} f(x), \qquad f(x_M) = \max_{x \in M} f(x).$$

WEIERSTRASS APPROXIMATION THEOREM (WAT)

Let $f:[a,b] \to \mathbb{R}$ be continuous. Then there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[a,b]$.

Metric formulation of WAT

$M = \mathcal{C}_{\mathbb{R}}([a,b])$ with the "uniform metric"

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$$

$$\mathcal{P}_{\mathbb{R}} = \{ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \ n \in \mathbb{N}, \ x \in \mathbb{R}, \ a_j \in \mathbb{R} \}$$

Observe that $\mathcal{P}_{\mathbb{R}}$ can be viewed as a subspace of $\mathcal{C}_{\mathbb{R}}([a,b])$ simply by considering $p(x): [a,b] \to \mathbb{R}$

WAT : $\mathcal{P}_{\mathbb{R}} \subset M$ is dense in M, i.e.,

$$\forall f \in M \quad \forall \varepsilon > 0 \quad \exists p \in \mathcal{P}_{\mathbb{R}} \ / \ d(f,p) = \max_{x\in[a,b]} |f(x) - p(x)| < \varepsilon$$

Note: WAT is not true for $M = C_{\mathbb{R}}(\mathbb{R})$. In fact consider $f(x) = e^x \in M$. By contradiction, take $\varepsilon = 1$ and assume there exists a polynomial

$p(x)$ / $|f(x)-p(x)|<1, \quad \forall x \in \mathbb{R}$

$$|e^x - p(x)| = \underbrace{\underset{\downarrow +\infty}{e^x}\left|1-\underset{\substack{\downarrow\, x\to+\infty \\ 0}}{\frac{p(x)}{e^x}}\right|}_{\to +\infty,\ x\to+\infty} < 1, \quad \forall x\in\mathbb{R}$$

take $x \to +\infty$

This is a contradiction!

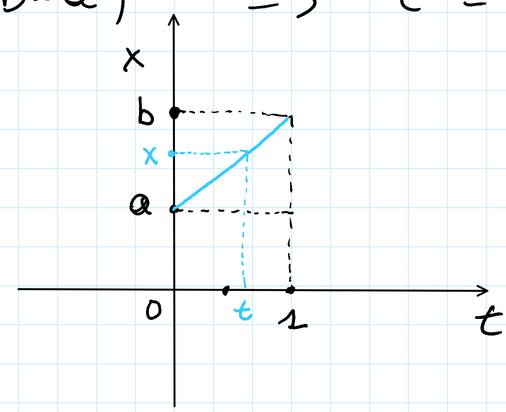
WAT - SKETCH OF PROOF

First we show that it is enough to prove WAT for functions f on the interval $[0,1]$, f continuous such $f(0)=f(1)=0$.

Step 1. Assume that WAT holds for $[0,1]$.

$f \in C_{\mathbb{R}}([a,b])$ $\qquad \tilde{f}(t) = f(a+t(b-a))$

$x = a+t(b-a) \implies t = \dfrac{x-a}{b-a} \quad (b>a)$



$\tilde{f} \in C_{\mathbb{R}}([0,1])$

Let $\varepsilon>0$ be given. WAT $\implies \exists\, \tilde{p} \in \mathcal{P}_{\mathbb{R}}$ such that $\max\limits_{t\in[0,1]} |\tilde{f}(t) - \tilde{p}(t)| < \varepsilon$

$p(x) := \tilde{p}\left(\dfrac{x-a}{b-a}\right) \in \mathcal{P}_{\mathbb{R}}$

$$\max_{x\in[a,b]} |f(x)-p(x)| = \max_{t\in[0,1]} |f(a+t(b-a))-p(a+t(b-a))|$$
$$= \max_{t\in[0,1]} |\tilde{f}(t)-\tilde{p}(t)| < \varepsilon$$

$\Rightarrow$ WAT holds true for any interval $[a,b]$.

Step 2. Assume WAT holds for continuous functions f on $[0,1]$ such that $f(0)=f(1)=0$

Consider $f\in C_{\mathbb{R}}([0,1])$, define $\tilde{f}(x)=f(x)-(\alpha+\beta x)$ with α,β to be determined such that $\tilde{f}(0)=\tilde{f}(1)=0$

$$\tilde{f}(0)=0 \Rightarrow \alpha=f(0)$$
$$\tilde{f}(1)=0 \Rightarrow \beta=f(1)-f(0)$$

Let $\varepsilon>0$ be given. By WAT there exists a $\tilde{p}\in P_{\mathbb{R}}$: $\max_{x\in[0,1]} |\tilde{f}(x)-\tilde{p}(x)| < \varepsilon$

$p(x) := \tilde{p}(x)+(\alpha+\beta x)$ with α,β determined above

$$\max_{x\in[0,1]} |f(x)-p(x)| = \max_{x\in[0,1]} |\underbrace{f(x)-(\alpha+\beta x)}_{=\tilde{f}(x)}+\underbrace{(\alpha+\beta x)-p(x)}_{=-\tilde{p}(x)}| < \varepsilon$$

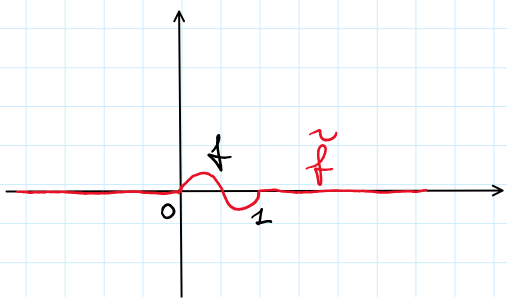
$\Rightarrow$ WAT is true for a general $f\in C_{\mathbb{R}}([0,1])$.

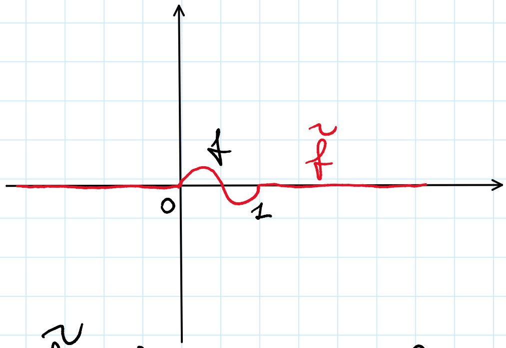
Step 3. Consider $f\in C_{\mathbb{R}}([0,1])$ such $f(0)=f(1)=0$

By Weierstrass Theorem $\Rightarrow$ f is bounded on $[0,1]$

By Heine-Cantor Theorem $\Rightarrow$ f is uniformly continuous on $[0,1]$

$$\tilde{f}(x) := \begin{cases} f(x), & x\in[0,1] \\ 0 & \text{otherwise} \end{cases}$$



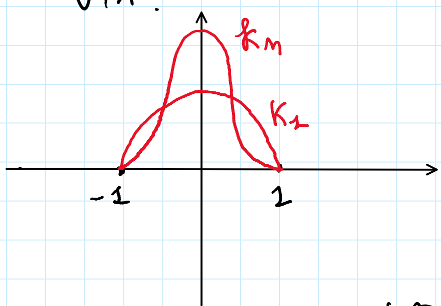


Observe that $\tilde{f}$ is uniformly continuous on $\mathbb{R}$ and bounded

$$k_n(x) = \begin{cases} c_n (1-x^2)^n, & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad n \in \mathbb{N}$$

with $\displaystyle\int_{\mathbb{R}} k_n(x)\,dx = 1 \quad \Rightarrow \quad c_n = \left(\int_{-1}^{1} (1-x^2)^n\,dx\right)^{-1},$

$\forall n$.



$$p_n(x) := \underset{\substack{\uparrow \\ \text{convolution product}}}{k_n * \tilde{f}}\,(x) = \int_{-\infty}^{+\infty} k_n(x-y)\,\tilde{f}(y)\,dy$$

$$= \int_0^1 k_n(x-y) f(y)\,dy$$

Result (to be proven below):

$p_n \longrightarrow \tilde{f}$ uniformly on $\mathbb{R}$

$\Rightarrow p_n \longrightarrow f$ uniformly on $[0,1]$

$\forall x \in [0,1], \quad \forall y \in [0,1] \qquad x-y \in [-1, 1]$

$$k_n(x-y) = c_n \left(1-(x-y)^2\right)^n \qquad \text{expand using the multinomial formula}$$

$$= c_n \sum_{j,k=0}^{2n} a_{jk}\, x^j y^k$$

$$p_n(x) = \int_0^1 K_n(x-y) f(y)\,dy = c_n \underbrace{\sum_{j=0}^{2n} x^j \sum_{k=0}^{2n} a_{jk} \underbrace{\int_0^1 y^k f(y)\,dy}_{\beta_k}}_{=:\alpha_{jn}}$$

$$= c_n \sum_{j=0}^{2n} \alpha_{jn}\, x^j$$

$\uparrow$ it is a polynomial

$\Rightarrow$ WAT holds for f

COMPLETION OF THE PROOF OF WAT

$\mathcal{L}^1(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R}$ measurable and such that $\int_{\mathbb{R}} |f(x)|\,dx < \infty \}$

$f, g \in \mathcal{L}^1(\mathbb{R})$ $\quad f \sim g \iff f(x) = g(x)$ a.e.

that is, f and g are equal up to sets of measure zero

$\sim$ is an equivalence relation

$L^1(\mathbb{R}) = \mathcal{L}^1(\mathbb{R}) / \sim$

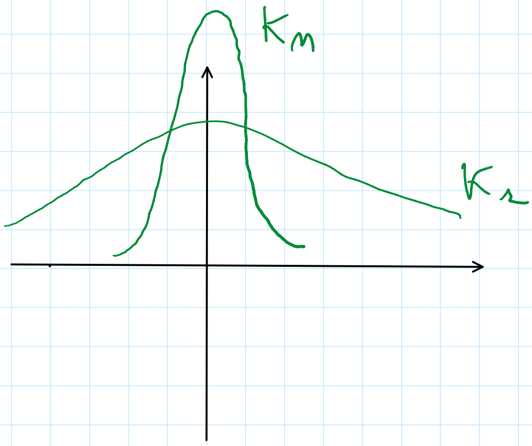
Approximate Identity (a.i.)

$\{K_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ such that

1) $K_n(x) \geq 0$ a.e.

2) $\int_{\mathbb{R}} K_n(x)\,dx = 1$, $\forall n$

3) $\lim_{n\to+\infty} \int_{|x|>\delta} K_n(x)\,dx = 0$, $\forall \delta > 0$



Examples. Take $K \in L^1(\mathbb{R})$, $K(x) \geq 0$ a.e. and $\int_{\mathbb{R}} K(x)dx = 1$

ex. $K(x) = e^{-\pi x^2}$ or $K(x) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x)$

↑ characteristic function

$n \in \mathbb{N}_+$, $K_n(x) := nK(nx) \Rightarrow \{K_n\}$ is an a.i.

CONVOLUTION. $f, g \in L^1(\mathbb{R})$

the convolution product $f * g(x) = \int_{-\infty}^{+\infty} f(x-y)\, g(y)\, dy$

$$= \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy$$

$$= g * f(x)$$

Theorem. If $f, g \in L^1(\mathbb{R})$ then $f * g$ is well-defined a.e. and $f * g \in L^1(\mathbb{R})$.

Theorem. Let f be uniformly continuous and bounded. Consider $\{K_n\}$ a.i. Then

1) $K_n * f$ is uniformly continuous and bounded on $\mathbb{R}$

2) $K_n * f \longrightarrow f$ uniformly on $\mathbb{R}$ $\left(\Leftrightarrow \sup_{x \in \mathbb{R}} |K_n * f(x) - f(x)| \underset{n \to \infty}{\to} 0\right)$

PROOF.

1) By assumption $\exists M > 0 \mid |f(x)| \leq M, \forall x \in \mathbb{R}$

$$|K_n * f(x)| \leq \int_{\mathbb{R}} K_n(y)\,|f(x-y)|\,dy \leq M \underbrace{\int_{\mathbb{R}} K_n(y)dy}_{=1} = M$$

$K_n * f$ uniformly continuous: Let $\varepsilon > 0$ be given

$\exists\, \delta > 0 \mid \quad |f(z) - f(z')| < \varepsilon, \quad \forall z, z' \mid |z - z'| < \delta$

Consider $x, x' \in \mathbb{R} \mid \quad |x - x'| < \delta$, then

$$|(K_n * f)(x) - (K_n * f)(x')| \leq \int_{\mathbb{R}} K_n(y) \underbrace{|f(\underbrace{x-y}_{z}) - f(\underbrace{x'-y}_{z'})|}_{< \varepsilon \text{ since } |z-z'|<\delta} dy$$

$$< \varepsilon \underbrace{\int_{\mathbb{R}} K_n(y)dy}_{=1} = \varepsilon$$

$\Rightarrow K_n * f$ is uniformly continuous

2) $K_n * f \to f$ uniformly

Let $\varepsilon > 0$ be given.

$$|K_n * f(x) - \underbrace{\textstyle\int_{\mathbb{R}} K_n(y)dy = 1}\, f(x)| = \left| \int_{\mathbb{R}} K_n(y) f(x-y)dy - \int_{\mathbb{R}} K_n(y) f(x)dy \right|$$

$$\leq \int_{\mathbb{R}} K_n(y)\,|f(x-y) - f(x)|\,dy$$

$$= \underbrace{\int_{|y| \leq \delta} K_n(y)|f(x-y) - f(x)|dy}_{=A} + \underbrace{\int_{|y|>\delta} K_n(y)|f(x-y) - f(x)|dy}_{=B}$$

f is unif. cont.: given the $\varepsilon > 0$ above then there exists a $\delta > 0 \mid \quad \forall u, v \in \mathbb{R}, \; |u - v| < \delta \Rightarrow$

$$|f(u) - f(v)| < \frac{\varepsilon}{2}$$

$$A < \int_{|y| \leq \delta} K_n(y) \frac{\varepsilon}{2}\, dy \leq \frac{\varepsilon}{2} \overbrace{\int_{\mathbb{R}} K_n(y)dy}^{=1} = \frac{\varepsilon}{2}$$

Consider the integral B:

$$B \leq \int_{|y|>\delta} K_n(y)\,[\,\underbrace{|f(x-y)|}_{\leq M} + \underbrace{|f(x)|}_{\leq M}\,]\,dy$$

$$\leq \underbrace{2M \int_{|y|>\delta} K_n(y)\,dy}_{< \frac{\varepsilon}{2}}$$

choose $\tilde{\varepsilon} = \frac{\varepsilon}{4M}$, then $\exists N \in \mathbb{N} \,/\, \forall n \geq N$

$$\int_{|y|>\delta} K_n(y)\,dy < \tilde{\varepsilon}$$ by assumption 3 of a.i.

So $|K_n * f(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall x \in \mathbb{R}$

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