MINKOWSKI'S INEQUALITY

Consider $f, g \in L^p(X)$, $1 \le p \le \infty$. Then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p \qquad (M)$$

Explicitly:

1) $1 \le p < \infty$ $\quad \left(\int_X |f(x)+g(x)|^p \, d\mu\right)^{1/p} \le \left(\int_X |f(x)|^p \, d\mu\right)^{1/p} + \left(\int_X |g(x)|^p \, d\mu\right)^{1/p}$

2) $p = \infty$ $\quad \operatorname{ess\,sup}|f+g| \le \operatorname{ess\,sup}|f| + \operatorname{ess\,sup}|g|$

Proof. 2) $p = \infty$ $\quad |f(x)+g(x)| \le |f(x)| + |g(x)|, \ \forall x$

$$\le \operatorname{ess\,sup}|f| + \operatorname{ess\,sup}|g|,$$
for a.e. $x \in X$

$$\Rightarrow \operatorname{ess\,sup}|f+g| \le \operatorname{ess\,sup}|f| + \operatorname{ess\,sup}|g|$$

1) $p = 1$ $\quad |f(x)+g(x)| \le |f(x)| + |g(x)| \quad \forall x$

$$\int_X |f(x)+g(x)| \, d\mu \le \int_X |f(x)| \, d\mu + \int_X |g(x)| \, d\mu$$

$$\Leftrightarrow \|f+g\|_1 \le \|f\|_1 + \|g\|_1$$

$1 < p < \infty$

$$\underbrace{\int_X |f(x)+g(x)|^p \, d\mu}_{= \|f+g\|_p^p} = \int_X |f(x)+g(x)|^{p-1} \cdot \overbrace{|f(x)+g(x)|}^{\le |f(x)| + |g(x)|} \, d\mu$$

$$\le \int_X |f(x)+g(x)|^{p-1} \cdot |f(x)| \, d\mu + \int_X |f(x)+g(x)|^{p-1} \cdot |g(x)| \, d\mu$$

We use Hölder's inequality for $f, g \in L^p(X)$

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$$

$$q = \frac{p}{p-1}$$

$$\|f+g\|_p^p \le \left(\int_X |f(x)+g(x)|^{(p-1)q}\,d\mu\right)^{1/q}\left(\int_X |f(x)|^p\,d\mu\right)^{1/p} +$$

$$\left(\int_X |f(x)+g(x)|^{(p-1)q}\right)^{1/q}\left(\int_X |g(x)|^p\,d\mu\right)^{1/p}$$

$$= \Bigg(\underbrace{\int_X |f(x)+g(x)|^p\,d\mu}_{<\infty}\Bigg)^{\frac{p-1}{p}}\Big(\|f\|_p + \|g\|_p\Big)$$

$< \infty$ since $f+g \in L^p(X)$ because $L^p(X)$ is a vector space (quotient space $L^p(X) = \mathcal{L}^p(X)/\sim$).

$$= \|f+g\|_p^{p-1}\left(\|f\|_p + \|g\|_p\right)$$

Hence

$$\|f+g\|_p^p \le \|f+g\|_p^{p-1}\left(\|f\|_p + \|g\|_p\right)$$

if $\|f+g\|_p \neq 0 \Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p$

otherwise, if $\|f+g\|_p = 0$ then

$0 \le \underbrace{\|f\|_p + \|g\|_p}_{\ge 0}$ the inequality is trivially satisfied. ∎

Property. For $1 \le p \le \infty$ the $\|\cdot\|_p$ is a norm.

Proof. 1) $\|f\|_p \ge 0$, $\forall f \in L^p$

2) $\|f\|_p = 0 \iff f = 0$ a.e. $x \in X$

3) $\|\alpha f\|_p \overset{p<\infty}{=} \left(\int_X |\alpha f(x)|^p\,d\mu\right)^{1/p} = |\alpha|\left(\int_X |f(x)|^p\,d\mu\right)^{1/p}$

$= |\alpha|\,\|f\|_p$

for $p=\infty$ $\quad \|\alpha f\|_\infty = \operatorname{ess\,sup}|\alpha f|$

$= |\alpha|\operatorname{ess\,sup}|f| = |\alpha|\,\|f\|_\infty$

4) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ Minkowski's inequality

$\forall f, g \in L^p(X), \quad \forall \alpha \in \mathbb{F}$. ▥

Exercises (including Ex 3 HMW1)

$c_0 := \{ (a_n)_n \mid \lim_{n\to\infty} a_n = 0 \}$

$c_{00} := \{ (a_n)_n \mid \exists n_0, \forall n \geq n_0 : a_n = 0 \}$

Prove the following statements:

1) c_0, c_{00} are subspaces of ℓ^∞ and $(c_0, \|\ \|_\infty)$, $(c_{00}, \|\ \|_\infty)$ are normed spaces.

2) $c_{00} \underset{\neq}{\subseteq} c_0 \underset{\neq}{\subseteq} \ell^\infty$ strict inclusions

3) Consider $(a_n) \mid a_n = \frac{1}{\sqrt{n}}$, show that $(a_n) \in c_0$

For which $p < \infty$, $(a_n) \in \ell^p$?

4) $c_{00} \subset \ell^p$, $1 \leq p < \infty$ and $(c_{00}, \|\ \|_p)$ is a normed space.

Solution.

1) $c_0 \subset \ell^\infty$ because $\forall (a_n) \in c_0$, $\lim_{n\to\infty} a_n = 0$

and any convergent sequence is bounded $\Rightarrow$

$\sup_n |a_n| < \infty \Rightarrow (a_n) \in \ell^\infty$

c_0 is a subspace: $\forall \alpha, \beta \in \mathbb{F}$, $\forall (a_n), (b_n) \in c_0$

$$\lim_{n\to\infty} (\alpha a_n + \beta b_n) \overset{\text{linearity of limit}}{=} \alpha \underbrace{\lim_{n\to\infty} a_n}_{=0} + \beta \underbrace{\lim_{n\to\infty} b_n}_{=0} = 0$$

$\Rightarrow (\alpha a_n + \beta b_n)_n \in c_0$

Hence c_0 is a subspace of ℓ^∞ and $(c_0, \|\ \|_\infty)$ is a normed space by EX 1, HMW 1

c_{00} is a subset of ℓ^∞ since $\forall (a_n) \in c_{00}$,

$\exists n_0 \in \mathbb{N} : \quad a_n = 0, \quad \forall n \geq n_0 \implies \sup_n |a_n| = \max_{0 \leq n \leq n_0 - 1} |a_n| < \infty$

c_{00} is a subspace of ℓ^∞: $\forall \alpha, \beta \in \mathbb{F}, \forall (a_n), (b_n) \in c_{00}$

$(a_n) \in c_{00} \quad \exists n_0 \mid \quad a_n = 0, \ \forall n \geq n_0$
$(b_n) \in c_{00} \quad \exists n_0' \mid \quad b_n = 0, \ \forall n \geq n_0'$
$\alpha a_n + \beta b_n = 0 \quad \forall n \geq \max\{n_0, n_0'\}$

$\implies (\alpha a_n + \beta b_n)_n \in c_{00}$

$\implies$ by EX1 HMW1 $(c_{00}, \|\ \|_\infty)$ is a normed space

2) $c_{00} \subsetneq c_0$ consider $a_n = \frac{1}{n}$, $n \in \mathbb{N}_+$

$a_n > 0, \ \forall n \in \mathbb{N}_+ \implies (a_n) \notin c_{00}$

$\lim_{n \to \infty} \frac{1}{n} = 0 \implies (a_n) \in c_0$

$c_0 \subsetneq \ell^\infty$ consider $a_n = 1, \ \forall n$

$\lim_{n \to \infty} a_n = 1 \neq 0 \implies (a_n) \notin c_0$

but $\sup_n |a_n| = 1 < \infty \implies (a_n) \in \ell^\infty$

3) $(a_n) \in c_0$ because $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

$p < \infty$

$(a_n) \in \ell^p \iff \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty \iff \frac{p}{2} > 1$

$\iff p > 2$.

4) $c_{00} \subset \ell^p$ $\forall (a_n) \in c_{00} \quad \exists n_0 : a_n = 0, \forall n \geq n_0$

$$\sum_{n=1}^{\infty} |a_n|^p = \sum_{n=1}^{n_0 - 1} |a_n|^p < \infty$$

c_{00} is a subspace $\forall \alpha, \beta \in \mathbb{F}, \ \forall (a_n), (b_n) \in c_{00}$

$a_n = 0, \ \forall n \geq n_0, \quad b_n = 0 \ \forall n \geq n_0' \quad N_0 = \max\{n_0, n_0'\}$

$$\sum_{n=1}^{\infty} |\alpha a_n + \beta b_n|^p = \sum_{n=1}^{N_0} |\alpha a_n + \beta b_n|^p$$

$$\leq |\alpha|^p \sum_{n=1}^{n_0 - 1} |a_n|^p + |\beta|^p \sum_{n=1}^{n_0' - 1} |b_n|^p < \infty$$

$\Rightarrow$ $(\alpha a_n + \beta b_n) \in \ell^p$

by Ex 1 HMW 1 $(c_{00}, \| \ \|_p)$ is a normed space ∎

Remark. $\mathbb{R}^n$ (or $\mathbb{C}^n$) can be seen as a subspace of c_{00} simply by using the bijection:

$$\varphi : \underset{\mathbb{R}^n}{\underset{\Uparrow}{x}} = (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, 0, 0, \dots 0)$$

Lemma. If $(X, \| \ \|)$ is a normed space then

$$d(x,y) := \|x - y\|, \qquad \forall x, y \in X$$

defines a metric d on X (the metric associated with or induced from the norm $\| \ \|$).

Proof.
1) $d(x,y) = \|x-y\| \geq 0$
2) $d(x,y) = 0 \iff \|x-y\| = 0 \iff x - y = 0 \iff x = y$
3) $d(y,x) = \|y - x\| = \|(-1)(x-y)\| = \underbrace{|-1|}_{=1} \|x-y\|$
$= d(x,y)$
4) $d(x,z) = \|x - z\| = \|x - y + y - z\| \leq \|x-y\| +$
$+ \|y - z\| = d(x,y) + d(y,z)$.

Example. $L^p(X)$, the map $d_p(f,g) = \|f-g\|_p$ is a metric on $L^p(X)$.

Inverse triangle inequality

$(X, \| \|)$ normed space. Then

$$\big| \|x\| - \|y\| \big| \le \|x-y\|, \quad \forall x,y \in X \quad (IT)$$

Proof. $\|x\| = \|x-y+y\| \le \|x-y\| + \|y\|, \quad \forall x, y$

$$\|x\| - \|y\| \le \|x-y\|$$

interchanging the role of x and y:

$$\|y\| - \|x\| \le \|y-x\| = \|x-y\|$$

$$-(\|x\| - \|y\|) \le \|x-y\|$$

$$\Longleftrightarrow \quad \|x\| - \|y\| \ge -\|x-y\|$$

so $\quad -\|x-y\| \le \|x\| - \|y\| \le \|x-y\|$. ∎

Corollary. $(X, \| \|)$ normed space. $F : X \to \mathbb{R}$ given by $F(x) = \|x\|$ then F is uniformly continuous on X.

Proof. For every $\varepsilon > 0$ choose $\delta = \varepsilon$ then

use in. in (IT)

$$| F(x) - F(y) | = \big| \|x\| - \|y\| \big| \le \|x-y\| < \varepsilon.$$

Lemma. $(X, \| \|)$ normed space. $\{x_n\}, \{y_n\} \subset X$ sequences, $(\alpha_n) \subset \mathbb{F}$ sequence. Then

1) $\alpha_n \to \alpha_n$, $x_n \to x \Rightarrow \alpha_n x_n \to \alpha x$

2) $x_n \to x$, $y_n \to y \Rightarrow x_n + y_n \to x + y$.

Proof. 1) $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x\|$

$$\le \|\alpha_n x_n - \alpha x_n\| + \|\alpha x_n - \alpha x\|$$

$$= \underbrace{|\alpha_n - \alpha|}_{\xrightarrow[n\to\infty]{} 0} \|x_n\| + |\alpha| \underbrace{\|x_n - x\|}_{\xrightarrow[n\to\infty]{}} \xrightarrow[n\to\infty]{} 0$$

$x_n \to x$, since it is a convergent sequence, it is bounded: $\exists M : \|x_n\| \le M, \forall n$.

2) $\|x_n + y_n - (x+y)\| = \|x_n - x + y_n - y\|$

$$\leq \underbrace{\|x_n - x\|}_{\substack{\to 0 \\ n\to\infty}} + \underbrace{\|y_n - y\|}_{\substack{\to 0 \\ n\to\infty}} \xrightarrow[n\to\infty]{} 0$$

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Ex. 2 HMW 1

Theorem. Every finite dimensional vector space X can be endowed with a norm.

Proof. Assume $\dim X = n$ and $\{e_1, \dots, e_n\} \subset X$ basis for X. This means that $\forall x \in X$,

$x = \sum_{j=1}^{n} \alpha_j e_j$ for a unique choice of scalars $\alpha_j \in \mathbb{F}$.

Define $\tilde{x} := (\alpha_1, \dots, \alpha_n, 0, 0, \dots) \in c_{00} \subset \ell^2$

$S := \{\tilde{x} \mid x \in X\} \subset c_{00}$ subspace of ℓ^2

$(S, \|\ \|_{\ell^2})$ is a normed space

Put on X the norm: $\|x\|_2 := \|\tilde{x}\|_{\ell^2} = \left(\sum_{j=1}^{n} |\alpha_j|^2\right)^{1/2}$

is a norm on X.

Consequence: $\mathbb{R}^n$, $\mathbb{C}^n$ are normed spaces with the standard norm $\|\cdot\|_2$.

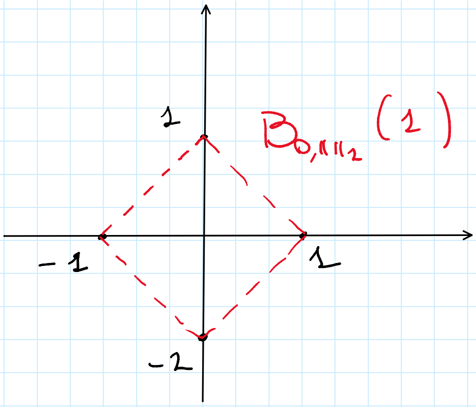
There are many different norms:

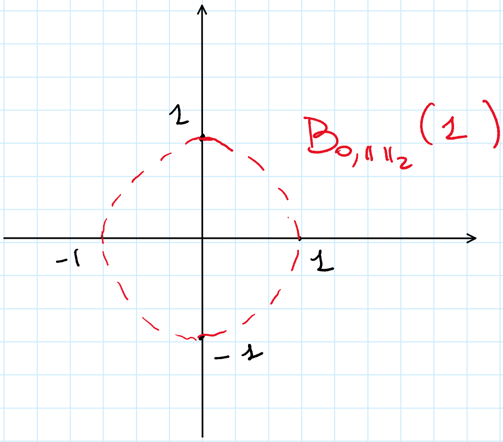
Example. in $\mathbb{R}^2$, $x = (x_1, x_2)$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$

$\|x\|_1 = |x_1| + |x_2|$, $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$

$\|x\|_\infty = \max\{|x_1|, |x_2|\}$

In this case, $B_{0,\|\|_1}(1) = \{x \in \mathbb{R}^2 \mid \|x\|_1 < 1\}$


$B_{0,\|\|_1}(1)$
1
-1
1
-1


$B_{0,\|\|_2}(1)$
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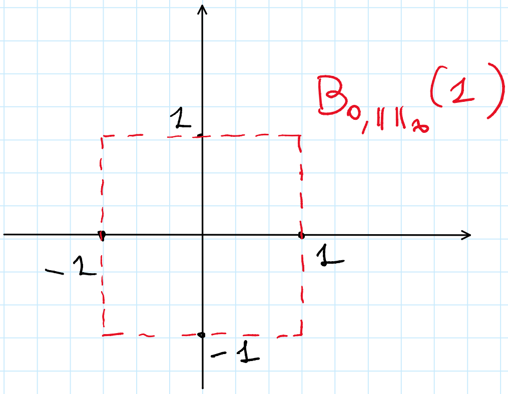
-1
1
-1


$B_{0,\|\|_\infty}(1)$
1
-1
1
-1