Equivalence of norms

X normed space, with $\|\cdot\|_1$, $\|\cdot\|_2$, two norms on X.
They are called **equivalent** if there exist $m, M > 0$ such that

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1, \quad \forall x \in X \qquad (E)$$

Note that $(E) \iff \frac{1}{M}\|x\|_2 \le \|x\|_1 \le \frac{1}{m}\|x\|_2, \forall x$

Property. The condition (E) is an equivalence relation on the sets of norms on X.

Property. X a normed space and $\|\cdot\|_1$, $\|\cdot\|_2$ equivalent norms. Then $\{x_n\} \subset X$ $x \in X$.

1) $x_n \to x$ w.r.to $\|\cdot\|_1$ $\iff$ $x_n \to x$ w.r.to $\|\cdot\|_2$
2) $\{x_n\}$ is a Cauchy seq. w.r.t. $\|\ \|_1$ $\iff$ $\{x_n\}$ is a Cauchy seq. w.r. to $\|\ \|_2$.

Sketch of proof.

1) $d_1(x_n, x) = \|x_n - x\|_1 \to 0$ as $n \to \infty$
Use condition (E) $\quad d_2(x_n, x) := \|x_n - x\|_2 \le M\|x_n - x\|_1$
By the comparison theorem $d_2(x_n, x) \to 0$, $n \to \infty$
The vice versa is analogous.

2) Assume $\{x_n\}$ Cauchy seq. w.r.t. $\|\ \|_1$: $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that $\forall n, m \ge n_0$ $\quad d_1(x_m, x_n) = \|x_m - x_n\|_1 < \frac{\varepsilon}{M}$

$\Rightarrow$ $\{x_n\}$ is a Cauchy seq. w.r.t. $\|\cdot\|_2$:

$$d_2(x_m, x_n) = \|x_m - x_n\|_2 \overset{(E)}{\le} M\|x_m - x_n\|_1 < M\frac{\varepsilon}{M} = \varepsilon$$

the vice versa is analogous.

Theorem. In a finite-dimensional vector space all the norms are equivalent.

Ex. $X=\mathbb{R}^n$, $1\le p_1,p_2\le\infty$ then $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$ are equivalent.

Direct computations for $\|\cdot\|_\infty$ and $\|\cdot\|_1$:

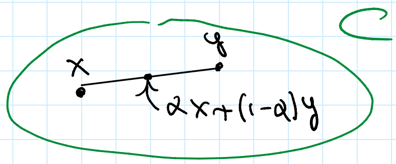
$$\|x\|_\infty = \max_{1\le j\le n}|x_j| \le \overset{\le \max_j |x_j|}{|x_1|} + |x_2| + \dots + |x_n| = \|x\|_1$$

$$\|x\|_1 \le \underbrace{\max_{1\le j\le n}|x_j| + \dots + \overbrace{\max_{1\le j\le n}|x_j|}^{=\|x\|_\infty}}_{n \text{ addends}} = n\|x\|_\infty, \ \forall x$$

$$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty, \quad \forall x\in\mathbb{R}^n.$$

Convex set: X vector space, $C\subseteq X$

C is convex $\iff \forall x,y\in C$, $\forall \alpha\in[0,1]$, $\alpha x+(1-\alpha)y\in C$



Example. $(X,\|\cdot\|)$ normed space, $\forall r>0$

$B_0(r)=\{x\in X : \|x\|<r\}$ is convex.

Proof. $\forall x,y\in B_0(r)$, $\forall\alpha\in[0,1]$

$$\|\alpha x+(1-\alpha)y\| \le \|\alpha x\|+\|(1-\alpha)y\| = \alpha\|x\|+(1-\alpha)\|y\|$$
$$< \alpha r+(1-\alpha)r = r$$

Check as exercise: $\overline{B_0(r)}=\{x\in X,\ \|x\|\le r\}$ is convex.

BANACH SPACES

Def. $(X,\|\cdot\|)$ normed space is called a **Banach space** if (X,d) with the induced metric $d(x,y)=\|x-y\|$ is complete; i.e., every Cauchy sequence in X is convergent in X.

Lemma. Consider $\|\cdot\|_1$ and $\|\cdot\|_2$ two equivalent norms on X. Then $(X, \|\ \|_1)$ is a Banach space $\iff (X, \|\ \|_2)$ is a Banach space.

Proof. It is a straightforward consequence of the property stated above.

Theorem. Every finite-dimensional vector space X is a Banach space.

Proof. Assume $\dim X = n$ and $\{e_1, \dots, e_n\}$ is a basis for X. $\forall x \in X$ $x = \sum_{j=1}^{n} \lambda_j e_j$ unique choice of scalars $(\lambda_1, \dots, \lambda_n) \in \mathbb{F}^n$. Recall that all norms are equivalent in a finite-dim. vector space so we focus on $\|(\lambda_1, \dots, \lambda_n)\|_2 = \sqrt{\sum_{j=1}^{n} |\lambda_j|^2}$

We define on X $\|x\|_2 := \sqrt{\sum_{j=1}^{n} |\lambda_j|^2}$ which is a norm on X since $\|(\lambda_1, \dots, \lambda_n)\|_2$ is a norm on $\mathbb{F}^n$. $\mathbb{F}^n$ is complete (Banach space) Every Cauchy seq. $\{x_k\} \subset X$

$x_k = (\lambda_1^k, \dots, \lambda_n^k)$ $\quad \left((\lambda_1^k, \dots, \lambda_n^k) \right)_{k \in \mathbb{N}}$ is a Cauchy seq. for $\mathbb{F}^n$ $\to$ it converges to $(\lambda_1, \dots, \lambda_n) \in \mathbb{F}$ $\Rightarrow$ $x := (\lambda_1, \dots, \lambda_n)$ then $x_k \to x$ in X because

$\|x_k - x\|_2 = \|(\lambda_1^k - \lambda_1, \dots, \lambda_n^k - \lambda_n)\|_2 \xrightarrow{k \to \infty} 0$ ☒

Theorem. 1) $(L^p(X), \|\ \|_p)$ is a Banach space, $1 \le p \le \infty$.

2) K compact metric space, then $(C_{\mathbb{F}}(K), \|\ \|_\infty)$ is a Banach space.

Theorem. $(X, \|\ \|)$ normed space. $Y \subset X$ subspace of X.
1) If Y is complete (w.r.to induced norm) then Y is closed.
2) If X is a Banach space then Y is complete $\iff$ Y is closed.

Proof. 1) Consider any seq. $\{x_n\} \subset Y$ | $x_n \to x \in X$. Goal: $x \in Y$. Since $x_n \to x$ in X then $\{x_n\}$ is a Cauchy seq. in X: $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$: $\forall n, m \geq n_0$ $\|x_n - x_m\| < \varepsilon$ $\Rightarrow$ $\{x_n\}$ is a Cauchy seq. in Y, since Y is complete $x_n \to x \in Y$.
2) Assume Y is closed. Take a Cauchy seq. $\{x_n\} \subset Y \subset X$, X is a Banach space $\Rightarrow$ $x_n \to x \in X$ then $x \in Y$ because Y is closed. The vice versa is proved in 1).

Example. $(\mathcal{C}_{\mathbb{R}}([a,b]), \|\ \|_\infty)$ Banach space.

$\mathcal{P}_{\mathbb{R}} \subset \mathcal{C}_{\mathbb{R}}([a,b])$ $\quad$ $\mathcal{P}_{\mathbb{R}}$ subspace of $\mathcal{C}_{\mathbb{R}}([a,b])$

and so $(\mathcal{P}_{\mathbb{R}}, \|\ \|_\infty)$ is a normed space.
By WAT $\overline{\mathcal{P}_{\mathbb{R}}}^{\|\ \|_\infty} = \mathcal{C}_{\mathbb{R}}([a,b])$ and take for example $f(x) = e^x \in \mathcal{C}_{\mathbb{R}}([a,b]) \setminus \mathcal{P}_{\mathbb{R}}$ so $\mathcal{P}_{\mathbb{R}}$ is not closed! By the previous theorem $\mathcal{P}_{\mathbb{R}}$ is not complete.

Example. c_{00} is a subspace of ℓ^∞
$(\ell^\infty, \|\ \|_\infty)$ is a Banach space
$(c_{00}, \|\ \|_\infty)$ is a normed space
c_{00} is not closed:

Consider $x = (x_n)_n$ $\quad x_n = \frac{1}{n}$, $n \in \mathbb{N}_+$

$x \in \ell^\infty$ and note that $x \notin c_{00}$ because $x_n > 0$, $\forall n$

$x^1 = (1, 0, 0, 0, \ldots)$

$x^2 = (1, \frac{1}{2}, 0, 0, \ldots)$

$x^3 = (1, \frac{1}{2}, \frac{1}{3}, 0, 0, \ldots)$

$\vdots$

$x^k = (1, \frac{1}{2}, \ldots, \underset{\substack{\uparrow \\ k\text{-entry}}}{\frac{1}{k}}, 0, 0, \ldots)$

$\{x^k\} \subset c_{00}$

$$\|x^k - x\|_\infty = \|(0, 0, \ldots, 0, -\frac{1}{k+1}, -\frac{1}{k+2}, \ldots)\|_\infty$$
$$= \frac{1}{k+1} \xrightarrow[k\to\infty]{} 0$$

$x^k \to x$ as $k \to \infty$, hence c_{00} is not closed

$\Rightarrow$ c_{00} is not complete by the previous theorem.

Example. $c_0 \subset \ell^\infty$ is a Banach space w.r.t. $\|\cdot\|_\infty$.

It is enough to show that c_0 is closed.

Take any sequence $\{x^k\} \subset c_0$ | $x^k \to x \in \ell^\infty$

we have that $x \in c_0$. In fact

$x = (x_n)$, $\quad x^k = (x^k_n)$

$$|x_n| = |x_n - x^k_n + x^k_n| \le |x_n - x^k_n| + |x^k_n|$$
$$\le \sup_{n\in\mathbb{N}} |x_n - x^k_n| + |x^k_n|$$
$$= \|x - x^k\|_\infty + |x^k_n| \qquad (*)$$

$\forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} : \forall k \ge k_0 \quad \|x - x^k\|_\infty < \frac{\varepsilon}{2}$

fix $k \ge k_0$ then $\exists n_0 \in \mathbb{N} : \forall n \ge n_0$

$|x^k_n| < \frac{\varepsilon}{2}$. So, for $n \ge n_0$

$$\|x - x^k\|_\infty + |x^k_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

hence $|x_n| < \varepsilon$ for $n \ge n_0$.

Lemma. $(X, \|\ \|)$ normed space, $Y \subset X$ subspace then $\overline{Y}$ is subspace.

Lemma. $(X, \|\ \|)$ normed space, $A \subseteq X$ subset. Then

$$\text{Span}(A) = \bigcap \{ W \mid W \text{ subspace of } X \text{ and } A \subseteq W \}$$

$$\overline{\text{Span}(A)} = \bigcap \{ C \mid C \text{ closed subspace of } X,\ A \subseteq C \}$$

RIESZ' LEMMA

$(X, \|\ \|)$ normed space, Y closed subspace of X such that $Y \neq X$. Let $\alpha \in (0,1)$ be given. Then $\exists\, x_\alpha \in X$, $\|x_\alpha\| = 1$ such that

$$\|x_\alpha - y\| > \alpha, \quad \forall y \in Y.$$

Proof. Consider $x \in X \setminus Y$ (x exists because $Y \subsetneq X$)

Define $d = \inf \{ \|x - y\|,\ y \in Y \}$

Since Y is closed we have that $d > 0$. In fact, by contradiction, assume $d = 0 \Rightarrow \exists \{y_k\} \subset Y$ such that $\|x - y_k\| \to 0$, $k \to \infty$

$\Leftrightarrow y_k \to x$; since Y is closed we have $x \in Y$ that is a contradiction!

So $d > 0$. Since $0 < \alpha < 1$, $\frac{1}{\alpha} > 1$, so $\frac{d}{\alpha} > d$ by definition of infimum $\exists z \in Y$ such that $d \leq \|x - z\| < \frac{d}{\alpha}$. Define $x_\alpha := \frac{x - z}{\|x - z\|}$

so $\|x_\alpha\| = 1$

$$\|x_\alpha - y\| = \left\| \frac{x - z}{\|x - z\|} - y \right\| = \frac{1}{\|x - z\|} \|x - z - \|x - z\| y\|$$

$$= \frac{1}{\|x - z\|} \| x - \underbrace{(\overset{\in Y}{z} + \|x - z\| \overset{\in Y}{y})}_{\in Y \text{ subspace}} \| \geq \frac{1}{\|x - z\|} d$$

$$> \frac{\alpha}{\cancel{d}} \cancel{d} = \alpha, \quad \forall y \in Y. \blacksquare$$

Exercises 5, 6, 7, 8 MMW1