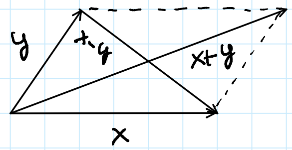
Theorem. Let X be a inner product space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.

Then we have:

1) Parallelogram rule (PL): $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \forall x,y$



2) If X is real: $(x,y) = \dfrac{\|x+y\|^2 - \|x-y\|^2}{4}$

3) If X is complex: $(x,y) = \sum_{k=0}^{3} \dfrac{i^k \|x+i^k y\|^2}{4}$

Proof: check as exercise writing $\| \ \|^2 = (\cdot,\cdot)$

Ex. 2 HMW2.

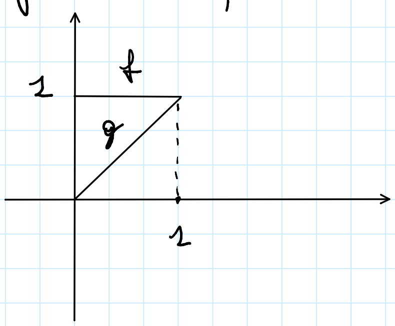
Note: One way to show that a given norm is not induced by a inner product is to show that it does not satisfy the parallelogram rule.

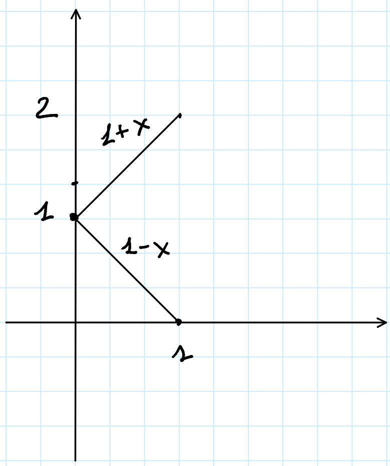
Example. $X = \mathcal{C}([0,1])$ $\|f\| = \sup_{x\in[0,1]} |f(x)|$

Let us show that this norm is not induced by a inner product.

$f(x) = 1, \quad \forall x \in [0,1]$

$g(x) = x, \quad \forall x \in [0,1]$





$\|f+g\|^2 + \|f-g\|^2 = 2^2 + 1^2 = 5$

$$2(\|f\|^2 + \|g\|^2) = 2(1^2 + 1^2) = 4$$

hence $5 \neq 4 \Rightarrow$ the parallelogram rule fails!

Example. ℓ^p, $p \neq 2$, is not an inner product space

$x = (1, 1, 0, 0, \dots)$ $\quad y = (1, -1, 0, 0, \dots)$

$x + y = (2, 0, 0, 0, \dots)$ $\quad x - y = (0, 2, 0, 0, 0, \dots)$

$p < \infty$. $\quad \|x+y\|_{\ell^p} = 2, \quad \|x-y\|_{\ell^p} = 2$

$$\|x\|_{\ell^p} = \left(\sum_{j=1}^{2} 1^p\right)^{1/p} = 2^{1/p}$$

$$\|y\|_{\ell^p} = 2^{1/p}$$

(PL) $\quad \|x+y\|_{\ell^p}^2 + \|x-y\|_{\ell^p}^2 = 2\left(\|x\|_{\ell^p}^2 + \|y\|_{\ell^p}^2\right)$

$$2^2 + 2^2 = 2\left(2^{2/p} + 2^{2/p}\right)$$

$$2^3 = 2^{\frac{2}{p}+2} \iff 3 = \frac{2}{p} + 2$$

$$\iff 1 = \frac{2}{p} \iff \boxed{p = 2}$$

$p = \infty$ $\quad \|x+y\|_{\ell^\infty} = 2 \quad \|x-y\|_{\ell^\infty} = 2$

$\|x\|_{\ell^\infty} = 1 \quad \|y\|_{\ell^\infty} = 1$

$$\|x+y\|_{\ell^2}^2 + \|x-y\|_{\ell^2}^2 = 4 + 4 = 8$$

$$2\left(\|x\|_{\ell^\infty}^2 + \|y\|_{\ell^\infty}^2\right) = 2(1+1) = 4$$

$\Rightarrow 8 \neq 4$ the PL fails!

Lemma. $(\cdot, \cdot) : X \times X \to \mathbb{F}$ is continuous.

If $x_n \to x$, $y_n \to y$ in X then $(x_n, y_n) \to (x, y)$.

Proof. $|(x_n, y_n) - (x, y)| = |(x_n, y_n) - (x_n, y) + (x_n, y) - (x, y)|$

$$\leq |(x_n, y_n) - (x_n, y)| + |(x_n, y) - (x, y)|$$

$$\overset{\text{lin}(\cdot,\cdot)}{=} |(x_n, y_n - y)| + |(x_n - x, y)|$$

$$\overset{(CS)}{\leq} \|x_n\| \, \|y_n - y\| + \|x_n - x\| \, \|y\|$$

$$\big| \|x_n\| - \|x\| \big| \leq \|x_n - x\| \to 0 \quad \text{as } n \to \infty$$

$$\big|\, \|x_n\| - \|x\| \,\big| \leq \|x_n - x\| \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \|x_n\| \to \|x\| \quad \text{as } n \to \infty$$

in particular $\|x_n\|$ is bounded

taking $n \to \infty$ $\quad \|x_n\| \, \|y_n - y\| \to 0$

Hence by the comparison theorem

$$|(x_n, y_n) - (x, y)| \longrightarrow 0, \quad n \to \infty .$$

Orthogonality

Def. $(X, (\cdot,\cdot))$ inner product space. $x, y \in X$ are said to be **orthogonal** if $(x,y) = 0$

The set $\{e_1, \dots, e_n\} \subset X$ is said to be **orthonormal** if $\|e_j\| = 1$, $\forall j = 1, \dots, n$ and $(e_j, e_k) = \delta_{jk}$, $j,k = 1, \dots, n$.

Theorem. An orthonormal set $\{e_1, \dots, e_n\}$ is linearly independent. In particular if $\dim X = n$ then $\{e_1, \dots, e_n\}$ is a basis for X and every vector $x \in X$ can be written as

$$x = \sum_{j=1}^{n} \underbrace{(x, e_j)}_{\text{"components" of } x} e_j \qquad (*)$$

$\{e_1, \dots, e_n\}$ is called **orthonormal basis (o.n.b.)**

Moreover, if $\{v_1, \dots, v_n\}$ is a linearly independent set of X, define $S = Sp\{v_1, \dots, v_n\}$, then there exists an o.n.b. for S.

Proof. We show: $\{e_1, \dots, e_n\}$ is linearly independent:

Assume $\sum_{j=1}^{n} \alpha_j e_j = 0$ now take $m \in \{1, \dots, n\}$

$$0 = \Big(\underbrace{\sum_{j=1}^{n} \alpha_j e_j}_{=0}, e_m \Big) = \sum_{j=1}^{n} \alpha_j \underbrace{(e_j, e_m)}_{= \delta_{jm}} = \alpha_m$$

$$\Rightarrow \alpha_m = 0, \quad \forall m \in \{1, \dots, n\}$$

So $\{e_1, \dots, e_n\}$ is lin. independent. If $\dim X = n$

$\Rightarrow$ $\{e_1, \dots, e_n\}$ is a basis for X.

Let us show $(*)$

$\forall x \in X$ since $\{e_1, \dots, e_n\}$ is a basis $\exists\, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $x = \sum_{j=1}^{n} \alpha_j e_j$ unique decomposition!

$m \in \{1, \dots, n\}$

$$(x, e_m) = \left(\sum_{j=1}^{n} \alpha_j e_j, e_m \right) = \sum_{j=1}^{n} \alpha_j \underbrace{(e_j, e_m)}_{= \delta_{jm}} = \alpha_m$$

$\Rightarrow$ $\alpha_m = (x, e_m)$, $\forall m \in \{1, \dots, n\}$.

The second part is proved by induction.

$n = 1$ $\{v_1\}$ take $e_1 = \dfrac{v_1}{\|v_1\|}$ $\Rightarrow$ $\|e_1\| = 1$

$\{e_1\}$ o.n.b. for $S = Sp\{v_1\}$.

Assume that the result holds true for an integer $n \geq 1$

Consider $\{v_1, \dots, v_n, v_{n+1}\}$ a set of linearly independent vectors and $S = Sp\{v_1, \dots, v_{n+1}\}$.

By the inductive assumption there exists an o.n.b. $\{e_1, \dots, e_n\}$ for $Sp\{v_1, \dots, v_n\}$.

$v_{n+1} \notin Sp\{v_1, \dots, v_n\} = Sp\{e_1, \dots, e_n\}$

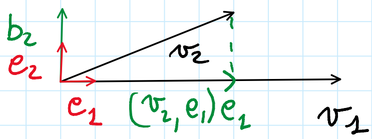
Define $$b_{n+1} := v_{n+1} - \underbrace{\sum_{j=1}^{n} (v_{n+1}, e_j) e_j}_{\in Sp\{e_1, \dots, e_n\}} \neq 0$$

$$(b_{n+1}, e_m) = (v_{n+1}, e_m) - \sum_{j=1}^{n} (v_{n+1}, e_j) \underbrace{(e_j, e_m)}_{= \delta_{jm}}$$
$$= (v_{n+1}, e_m) - (v_{n+1}, e_m) = 0,$$

$\forall m = 1, \dots, n$

$e_{n+1} = \dfrac{b_{n+1}}{\|b_{n+1}\|} \Rightarrow \|e_{n+1}\| = 1$ and $(e_{n+1}, e_m) = 0$, $\forall m = 1, \dots, n$. Then $\{e_1, \dots, e_{n+1}\}$ is a orthonormal set contained in $S = Sp\{v_1, \dots, v_{n+1}\} \Rightarrow$ $\{e_1, \dots, e_{n+1}\}$ is an o.n.b. for S.

Note: this procedure is known as "Gram-Schmidt algorithm"



$e_1 = \dfrac{v_1}{\|v_1\|}$, $b_2 = v_2 - (v_2, e_1)e_1$

$e_2 = \dfrac{b_2}{\|b_2\|}$

Theorem. (Generalization of Pythagoras' Theorem)

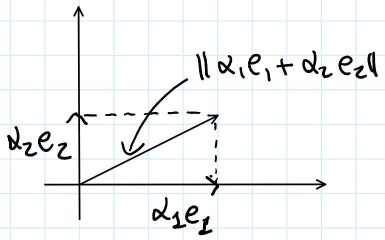
Let $(X, (\cdot,\cdot))$ be an n-dimensional inner product space and $\{e_1, \dots, e_n\}$ be an o.n.b. for X.

Then $\forall \alpha_j \in \mathbb{F}$, $j = 1, \dots, n$,

$$\left\| \sum_{j=1}^{n} \alpha_j e_j \right\|^2 = \sum_{j=1}^{n} |\alpha_j|^2 \qquad \text{(GPT)}.$$

ex. $X = \mathbb{R}^2$

$\alpha_1, \alpha_2 \in \mathbb{R}_+$



$\|\alpha_1 e_1 + \alpha_2 e_2\|^2 = \alpha_1^2 + \alpha_2^2$

classical Pythagoras' Thrm

Proof.

$$\left\| \sum_{j=1}^{n} \alpha_j e_j \right\|^2 = \left(\sum_{j=1}^{n} \alpha_j e_j, \sum_{k=1}^{n} \alpha_k e_k \right)$$

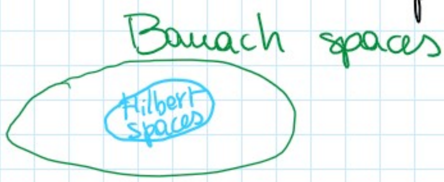
$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \overline{\alpha_k} \underbrace{(e_j, e_k)}_{= \delta_{jk}}$$

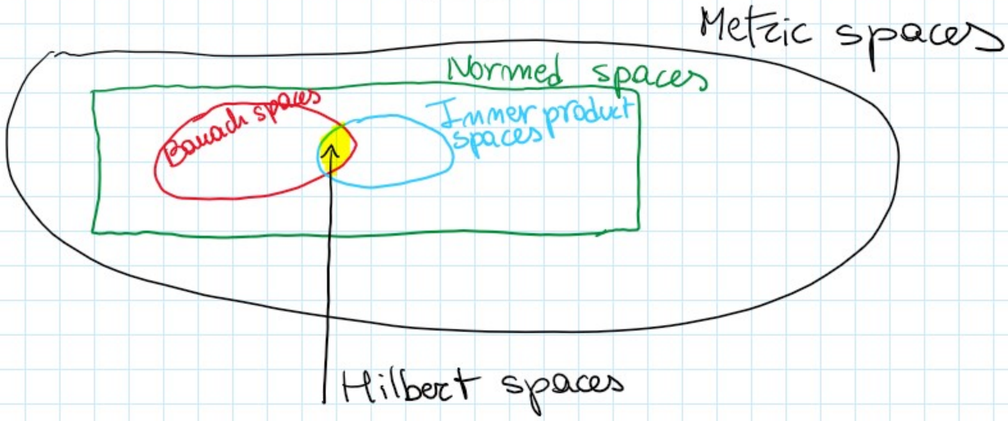
$$= \sum_{j=1}^{n} \alpha_j \overline{\alpha_j} = \sum_{j=1}^{n} |\alpha_j|^2 . \quad \blacksquare$$

Def. An inner product space $(X, (\cdot,\cdot))$ which is complete with respect to $\|x\| = (\cdot,\cdot)^{1/2}$ is called a Hilbert space.

Examples. Every finite-dimensional inner product space is a Hilbert space. E.g. $X = \mathbb{R}^n, \mathbb{C}^n$

ℓ^2, $L^2(X)$ are Hilbert spaces.





Lemma. H Hilber space. If $Y \subset H$ is a subspace then Y is a Hilbert space $\iff$ Y is closed.

Proof. It follows from the property that a subspace $Y \subset H$ of a Banach space H is complete $\iff$ Y is closed.

ORTHOGONAL COMPLEMENT

Def. $(X, (\cdot,\cdot))$ inner product space and $A \subseteq X$, A subset of X. The orthogonal complement of A

$$A^{\perp} = \{ x \in X : (x,a) = 0, \ \forall a \in A \}$$

$A^\perp$ consists of the vectors of X which are orthogonal to every vector a in A.

Example. $X = \mathbb{R}^3$ $\{e_1, e_2, e_3\}$ o.n.b.

$$A = Sp\{e_1, e_2\} = \{(a_1, a_2, 0), \quad a_1, a_2 \in \mathbb{R}\}$$

$$A^\perp = Sp\{e_3\} = \{(0, 0, a_3), \quad a_3 \in \mathbb{R}\}$$

$\forall x \in X$ $\quad x = (x_1, x_2, x_3)$, $\quad x \in A^\perp \iff$

$$((x_1, x_2, x_3), (a_1, a_2, 0)) = 0, \quad \forall a_1, a_2 \in \mathbb{R}$$

$$\Rightarrow \quad a_1 x_1 + a_2 x_2 = 0, \quad \forall a_1, a_2 \in \mathbb{R}$$

choose $a_2 = 0 \Rightarrow a_1 x_1 = 0 \quad \forall a_1 \in \mathbb{R}$

$\Rightarrow x_1 = 0$

similarly, choose $a_1 = 0 \Rightarrow a_2 x_2 = 0, \ \forall a_2$

$\Rightarrow x_2 = 0 \quad \Rightarrow \quad x = (0, 0, x_3), \ x_3 \in \mathbb{R}$

More generally, if X is an n-dimensional vector space and $\{e_1, \ldots, e_n\}$ is an o.n.b. then take if $A = Sp\{e_1, \ldots, e_k\}$ $k < n \Rightarrow$

$$A^\perp = Sp\{e_{k+1}, \ldots, e_n\}.$$